On the pasture territories covering maximal grass

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Summary. In this paper we consider the so-called pasture territory problem, its basic elements, and some related extremal problems. We describe the pasture territory as a graph of a piecewise smooth and continuous function $f(x, y)$ defined on a closed, connected domain of a plane. Considering extremal problems are related with finding the location of the nomadic residence, when the exploiting pasture territory has maximum grass mass, and finding the bound of the territory, when the place of the residence is fixed[1,2,5].

Key words: Pasture territory, Herbage density, piecewise smoothness, non-negative measure, watering-place, closure of a set, upper semi-continuity, convexity.

1 Basic elements of the pasture surface and the main maximization problem linked with location of the nomad residence

Let $K \subseteq \Re^2$ be a closure of an open and connected set with a piecewise smooth boundary. Suppose that K consists of a union of a finite number of domains K_i with piecewise smooth boundaries. Then the pasture surface is defined as a graph of a continuous function $f : K \to R$ such that $f(x, y)$ is twice differentiable on the interior of K_i for any i.
We define the watering place for the herd as a closure of a set $W \subseteq f(K)$ with an empty interior. That means pasture surface does not contain the interior of the water resource[1,2].
We denote a closed set $Q \subseteq f(K)$ as the possible locations for the nomadic residence.

Theorem 1. *Between any two points in $f(K)$, there exists a minimal curve in $f(K)$ connecting them.*

Proof. Suppose $O_1, O_2 \in f(K)$, $O_1 \neq O_2$. Since the connectedness of K, it follows that the points $f^{-1}(O_1)$ and $f^{-1}(O_2)$ can be connected by a rectifiable planar curve l. Then $f(l)$ is also a rectifiable surface curve with a length d. Let us construct a planar disk $B(f^{-1}(O_1),d) := \{z \in \Re^2| \; \|z - f^{-1}(O_1)\|_2 \leq d\}$ with a center $f^{-1}(O_1)$ and a radius d. Then the graph $f(B(f^{-1}(O_1),d)\cap K)$ is a complete metric space with a surface metric. This space, evidently, contains the curve $f(l)$ and the point O_2. Hence, by the theorem 3 (P.112) of [3], there exists a minimal surface curve connecting O_1 and O_2.

If the nomadic residence is located at the point $O \in Q$, we define the maximal possible exploiting area $A_r(O,\bar{W}) \subseteq f(K)$ as the union of all points $M \in f(K)$ such that there exists a loop $l \subseteq f(K)$ of length no more than $2r$ passing through the points M, O and some point $N \in \overline{W}$. This means, for a day, while grazing and watering one's livestock, the herdsman must pass the distance no more than $2r$. The $r > 0$ is called the radius of grazing. It is clear that $A_r(O,\bar{W})$ is a connected compact set[1,2].
Pasture surface $f(K)$ is a complete metric space, where the distance $\rho_1(M,N)$ for the points $M,N \in f(K)$ is equal to the length of a minimal curve connecting them. This metric ρ_1 is called a surface metric. Any minimal curve consists of possible pieces of the boundary $\partial f(K)$ and some geodesics.

Surface ellipse $E_r(O_1,O_2)$ with focuses $O_1,O_2 \in f(K)$ is a compact set satisfying

$$\rho_1(O_1,M) + \rho_1(O_1,O_2) + \rho_1(M,O_2) \leq 2r, \;\; \forall M \in E_r(O_1,O_2).$$

Each isolated part of the boundary $\partial E_r(O_1,O_2)$ is a closed curve.

When $\rho_1(O_1,O_2) = r$, $\mathit{int}\; E_r(O_1,O_2) = \emptyset$.

When $\rho_1(O_1,O_2) < r$, $\mathit{int}\; E_r(O_1,O_2) \neq \emptyset$.

We denote by $W_r(O)$ the subset of $\overline{W}$ such that

$$W_r(O) := \{N \in \overline{W}| \;\; \rho_1(O,N) \leq r\}.$$

Assume that $\rho_1(O,N) < r$ for any $N \in W_r(O)$. Then the next theorem holds.

Theorem 2. *The boundary $\partial(\overline{\mathit{int}\; A_r(O,\bar{W})})$ is a union of a finite number of closed, rectifiable curves.*

Proof. Since

$$A_r(O,\bar{W}) = \bigcup_{N \in W_r(O)} E_r(O,N),$$

the boundary $\xi(0) = \partial A_r(O,\bar{W})$ consists of $\partial(\overline{\mathit{int}\; A_r(O,\bar{W})})$ and some possible isolated curves. It is clear that $\overline{\mathit{int} A_r(O,\bar{W})}$ is a union of a family of ellipses

$E_r(O,N),\ N \in W_r(O)$, where $\overline{intE_r(O,N)} = E_r(O,N)$. Since $\overline{intA_r(O,\bar{W})}$ is a compact set, we can choose some ellipses $E_r(O,N_1),...,E_r(O,N_k)$ covering $\overline{intA_r(O,\bar{W})}$ in union. As each $\partial E_r(O,N_i), i=\overline{1,k}$ is a union of a finite number of closed and rectifiable curves, $\partial(\overline{int\ A_r(O,\bar{W})})$ also is a union of a finite number of closed and rectifiable curves.

Corollary 1. *When there exists only a finite number of points $N_i \in W_r(O)$ satisfying $\rho_1(O,N_i) = r$ and total length of the isolated curves of $\partial A_r(O,\bar{W})$ is finite, the boundary $\xi(O) = \partial A_r(O,\bar{W})$ has a finite length.*

Herbage density is a non-negative measure $\mu(K)$ such that for any compact set $M \subseteq K$,

$$\mu(M) < \infty$$

and the charge $Z(A)$ generated by bounded function $g(x,y) = \sqrt{1+f_x^2+f_y^2}$:

$$Z(A) = \int\limits_A \sqrt{1+f_x^2+f_y^2}\ d\mu$$

is absolutely continuous, where $A \subseteq K$ is any measurable subset with respect to μ [3](p.331).

The main maximization problem for nomads is to find the best place for the residence, i.e.,

$$G(O) = \int\limits_{f^{-1}(A_r(O,\bar{W}))} \sqrt{1+f_x^2+f_y^2} d\mu = \int\limits_{A_r(O,\bar{W})} d\mu \to \max; \quad O \in Q. \quad (1.1)$$

This problem is considered very difficult because of defining the boundary of $A_r(O,\bar{W})$.

Suppose that ξ and η are any two continuous curves on $f(K)$. Let us construct a metric space Ξ of all continuous curves on $f(K)$ by defining the distance as

$$\rho(\xi,\eta) = \inf \rho(f_1,f_2).$$

Here, the lower bound is taken by all admissible pairs of parametric representations for ξ and η which are continuous functions $f_1(t)$ and $f_2(t)$ $(0 \le t \le 1)$, and the distance between functions f_1 and f_2 is defined as

$$\rho(f_1,f_2) = \sup_{0\le t\le 1} \rho(f_1(t),f_2(t)).$$

Lemma 1. *Suppose that $\xi = \bigcup\limits_{i=1}^{k} \xi^i$, where each ξ^k is a closed and continuous curve on $f(K)$, and Π_ξ is a side view of the surface piece defined by ξ. Then the function*

$$S(\xi) = \int\limits_{\Pi_\xi} \sqrt{1 + f_x^2 + f_y^2} d\mu$$

is upper semi-continuous in Ξ^k.

Proof. Consider a sequence $\xi_n = \bigcup\limits_{i=1}^{k} \xi_n^i$, where each sequence ξ_n^i converges to ξ^i with respect to the above metric in Ξ. If we denote

$$S(\eta_n) = \sup_{i \geq n} S(\xi_i) \quad \text{with} \quad \inf_{\xi_j \in \bigcup\limits_{i \geq n} \xi_i} \rho_{max}(\eta_n, \xi_j) = 0,$$

$$\rho_{max}(\xi_1, \xi_2) = \max_{1 \leq i \leq k} \rho(\xi_1^i, \xi_2^i),$$

then we have

$$S(\eta_n) = S(\xi) + \int\limits_{\Pi_{\eta_n} \setminus (\Pi_{\eta_n} \cap \Pi_\xi)} \sqrt{1 + f_x^2 + f_y^2} d\mu - \int\limits_{\Pi_\xi \setminus (\Pi_{\xi_0} \cap \Pi_\xi)} \sqrt{1 + f_x^2 + f_y^2} d\mu\,.$$

The first integral tends to zero, but the second integral tends to -$\mu(\xi_0)$, where ξ_0 is a piece of the curve ξ. Therefore, $S(\xi) \geq \overline{\lim\limits_{n \to \infty}} S(\xi_n)$ and the lemma is proved.

For any $O \in f(K)$, we introduce a notation $O^p = f^{-1}(O)$.

Theorem 3. *Function* $G(O)$ $(G(O^p))$ *given in* (1.1) *is upper semi-continuous on* $f(K)$ (K).

Proof. Let a sequence $O_n^p \to O^p$ in K. Then the sequence $O_n = f(O_n^p)$ also tends to O in $f(K)$. Suppose that ξ_n is a boundary of $A_r(O_n)$ consisting of k closed continuous curves. It is clear that $O_n \to O$ $(O_n^p \to O^p)$ implies $\xi_n \to \xi$. By previous lemma, the function $G(O)$ $(G(O^p))$ is also upper semi-continuous.

Corollary 2. *If* $Q(f^{-1}(Q))$ *is compact, then problem* (1.1) *has a solution on* $Q(f^{-1}(Q))$.

In the next two parts of the paper, we assume that f is a linear function.

2 On the forms of exploiting pasture territories in simple cases

At first, we assume that the pasture territory is $\overline{\Re^2 \setminus B(O_1, R)}$, where $B(O_1, R)$ is a disk generated by circle $C(O_1, R)$ presenting the watering-place

$\overline{W}$. Our goal is to define the exploiting area $A_r(O,\bar{W})$ in cases of: $R=0$ (a well), $R=\infty$ (a straight bank of a river or a straight brook), $0<R<\infty$ (a bank of a deep lake). In all cases we assume that the nomadic residence is located at distance $k<r$ from the watering-place [5].

We study each case, separately.

1. Suppose $R=0$. In this case, $\overline{W}$ consists of unique point. Let this point be $O_1(0,-\frac{k}{2})$ and the nomadic residence is located at the point $O(0,\frac{k}{2})$ (Fig.1a). The pasture territory is the whole plane. It is clear that the maximal exploiting area $A_r(O,\bar{W})$ is an ellipse given by the inequality

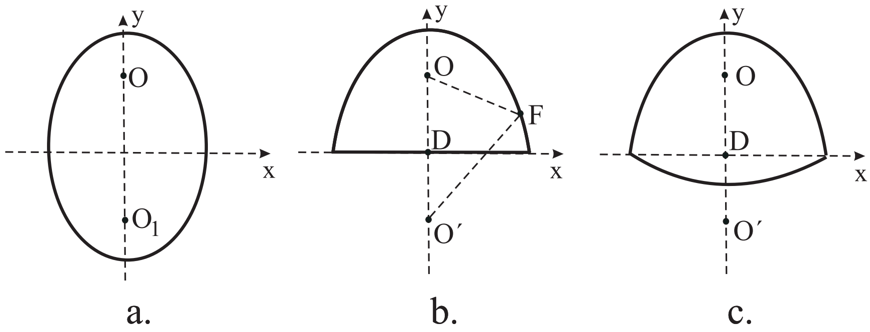

Fig. 1.

$$\sqrt{x^2+\left(\frac{k}{2}-y\right)^2}+\sqrt{x^2+\left(\frac{k}{2}+y\right)^2}\le 2r-k.$$

2. Suppose $R=\infty$. Let us consider two cases.
a. A bank of a river. In this case, the pasture territory is a half-plane, where the livestock can not cross the river and the maximal exploiting area $A_r(O,\bar{W})$ is a semi-ellipse bounded by a straight bank of a river. In fact, if we assume that the nomadic residence is located at the point $O(0,k)$, $D(0,0)$ is the origin of coordinates (Fig.1b), and denote $O'(0,-k)$, then for any point $F(x,y)$ of the curve $\partial A_r(O,\bar{W})$ the following equality holds

$$\rho(O,F)+\rho(O',F)=2r$$

holds. Hence, we have the following inequalities for the exploiting area $A_r(O,\bar{W})$:

$$\sqrt{x^2+(k-y)^2}+\sqrt{x^2+(k+y)^2}\le 2r,\ y\ge 0\ .$$

b. A brook. In this case, the livestock can cross the brook and the pasture territory is the whole plane. The part of the maximal exploiting area $A_r(O,\bar{W})$

on the other side of the brook is a segment of a disk with a center $O(0,k)$ and a radius r (Fig.1c). Therefore, $A_r(O,\bar{W})$ is defined as follows

$$\begin{cases} \sqrt{x^2+(k-y)^2}+\sqrt{x^2+(k+y)^2}\le 2r,\ y\ge 0,\\ x^2+(k-y)^2\le r^2,\ y<0\ . \end{cases}$$

3. Suppose $0<R<\infty$. In this case, the pasture territory is the closure $\overline{R^2\setminus B(O_1,R)}$ of the complement of the disk $B(O_1,R)$ on the plane. We suppose that $B(O_1,R)$ is a deep like. Without losing generality we assume that $R=1$ and O_1 is the origin of coordinates. Then the nomadic residence $O(0,m)$ is located at distance $m=k+1$ from the origin of coordinates (Fig.2). Clearly, the boundary curve $\xi(O)$ of the maximal exploiting area $A_r(O,\bar{W})$ is closed and symmetric with respect to the ordinate. Depending on values of m and r, the boundary curve $\xi(O)$ has different forms.

Theorem 4. i. *If $r\le\sqrt{m^2-1}$, then the lower part of $\xi(O)$ is an arc F_1F_2 and its upper part is an envelope Γ of the family of ellipses with focuses O and E:*

$$\rho(O,E)+\rho(O,M)+\rho(E,M)=2r, \tag{2.1}$$

where E is a point on the arc F_1F_2 $(OF_1=OF_2=r)$ and M is a point on the envelope (Fig.2a).
ii. *If $\sqrt{m^2-1}+(\frac{\pi}{2}+\arcsin\frac{1}{m})\ge r>\sqrt{m^2-1}$, then the upper part of $\xi(O)$ is the same as the previous envelope Γ generated by (2.1). The lower part of $\xi(O)$ is an arc C_1B_1BC, where B and B_1 are the contact points of tangents from O to $C(O_1,1)$. But the middle two parts of $\xi(O)$ are generated by the endpoints of minimal curves of length r starting from O and without passing the interior of disk $B(O_1,1)$ (Fig.2b).*
iii. *If $r>\sqrt{m^2-1}+\frac{\pi}{2}+\arcsin\frac{1}{m}$, then the upper part of $\xi(O)$ is the same envelope Γ generated by (2.1). But the lower part of $\xi(O)$ is generated by the endpoints of minimal curves of length r as in the previous case (Fig.2c).*

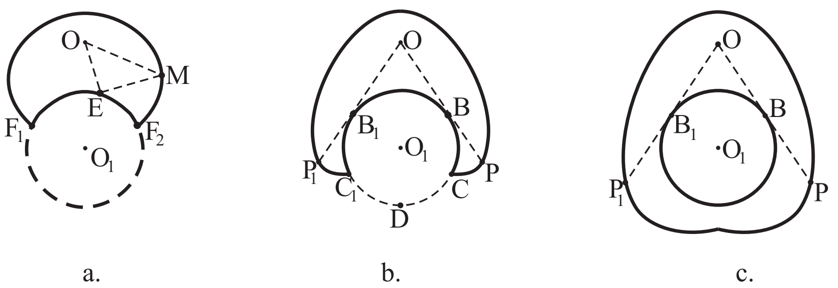

Fig. 2.

Proof. The length of tangents OB and OB_1 is equal to $\sqrt{m^2-1}$. When $r \le \sqrt{m^2-1}$, the boundary curve $\xi(O)$ of the exploiting area $A_r(O,\bar{W})$ must contain any point M satisfying (2.1). Therefore, the upper part of $\xi(O)$ must be an envelope Γ and for any point $M(x,y)$ of Γ, the segments OE and ME have the same reflection angle to the circle $C(O_1,1)$ at the point E.
Denote $\angle OO_1E = \alpha$. Then after some simple calculations we have the following parametric system for the envelope Γ (Fig.2a):

$$\begin{cases} (x\cos\alpha - y\sin\alpha)(m\cos\alpha-1)-(x\sin\alpha+y\cos\alpha-1)m\sin\alpha=0, \\ (1+\frac{x}{m}ctg\alpha-\frac{y}{m})\sqrt{1+m^2-2m\cos\alpha}-2r- \\ -\sqrt{x^2+y^2+1-2y\cos\alpha-2x\sin\alpha} \qquad\qquad =0. \end{cases} \tag{2.2}$$

The endpoints of envelope Γ are the points F_1 and F_2 of arc BB_1 such that $OF_1 = OF_2 = r$. The first part of the theorem is proved.
The curve consisting of tangent OB (OB_1) and arc BD (B_1D) has a length of

$$\sqrt{m^2-1}+\frac{\pi}{2}+\arcsin\frac{1}{m}.$$

Therefore, when

$$\sqrt{m^2-1}+\frac{\pi}{2}+\arcsin\frac{1}{m} \ge r > \sqrt{m^2-1}\ ,$$

the system (2.2) expresses only the top part of the boundary curve $\xi(O)$. The endpoints of this part coincide with the ends of tangent OP and OP_1(Fig.2b). But the lower ends of the boundary curve $\xi(O)$ are located at points C and C_1 of the circle $C(O_1,O)$, where the sum of lengths of arc BC (B_1C_1) and tangent OB (OB_1) is equal to r. Thus, any point of the boundary curve $\xi(O)$ locating between C (C_1) and P (P_1) is defined by the endpoints of the minimal curve of length r. For constructing such curves, we use Cruggs's theorem on the shortest curves with barriers [4]. This theorem claims that the shortest path consists from tangents and geodesics on barrier sets. By this theorem the minimal curve consists of two tangents and an arc. A simple calculation shows that the coordinates of the endpoint satisfy

$$\begin{aligned} &\sqrt{m^2-1}+\sqrt{x^2+y^2-1}+\pi-\arccos\tfrac{1}{m}-\arcsin\sqrt{\tfrac{x^2+y^2-1}{x^2+y^2}} \\ &-\arcsin\sqrt{\tfrac{x^2}{x^2+y^2}}=r \end{aligned}\ . \tag{2.3}$$

The second part of the theorem is proved.

When

$$r > \sqrt{m^2-1}+\frac{\pi}{2}+\arcsin\frac{1}{m},$$

the boundary curve $\xi(O)$ consists of an inner part which is the circle $C(O_1,1)$ (fig.2c) and an outer part of which any point satisfies either the equation (2.2) or the equation (2.3).

Now let us consider the case, where the pasture territory is the upper half-plane with a half-disk of radius R ($R \leq r$). Assume that the nomadic residence is located at the origin of coordinates O (Fig.3). We also assume that the boundary curve of the pasture territory is the watering-place.

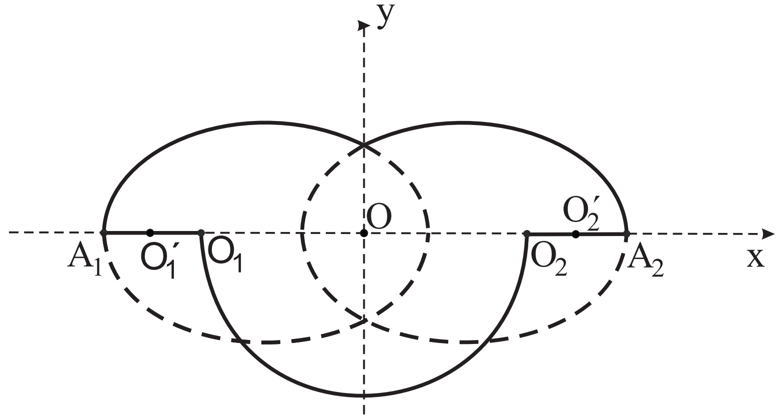

Fig. 3.

When $R = r$, the maximal exploiting area $A_r(O, \bar{W})$ is a half-disk:

$$x^2 + y^2 \leq R^2, \ y \leq 0.$$

Our goal is to find the useful pasture territory, namely, its boundary curve, when $r > R$.

Proposition 1. *Let O_1 and O_2 be the two points of circle $C(O, R)$ on the abscissa. Then for $r > R$ the upper bound of the maximal exploiting area $A_r(O, \bar{W})$ is defined as a union of the upper bounds of ellipses $E_r(O, O_i)$:*

$$\mid OO_i \mid + \mid OM \mid + \mid MO_i \mid \leq 2r, \ M \in \Re^2_+, \ i = 1, 2.$$

But the lower bound consists of two sections O_1A_1, O_2A_2 and the semicircle O_1DO_2, where $\mid OA_1 \mid = \mid OA_2 \mid = r$.

Proof. It is clear that the ellipse $E_r(O, O_i)$ contains the ellipse $E_r(O, O'_i)$, where O'_i is an arbitrary point on A_iO_i satisfying $\mid A_iO'_i \mid < \mid A_iO_i \mid$, $i = 1, 2$. Also, the set $(E_r(O, O_1) \cup E_r(O, O_2)) \cap A_r(O, \bar{W})$ contains $E_r(O, N) \cap A_r(O, \bar{W})$ for any point N of the arc O_1DO_2 of circle $C(O, R)$. Hence, the upper part of $\partial(E_r(O, O_1) \cup E_r(O, O_2))$ is also the upper part of $\xi(O)$. The proposition is proved.

3 Some solution properties of the main maximization problem on the plane

We suppose that $f(K) = Q = \bar{W} = \Re^2$ and the herbage density $\mu(\Re^2)$ has a positive Lebesgue measure only for some closed and connected set $M \subseteq \Re^2$ (or $\mu(\Re^2 \setminus M) = 0$) satisfying $\overline{int\, M} = M$. In this case, evidently, $A_r(O, \bar{W}) = B(O, r)$. Let Q_r^* be the set of solutions of the problem (1.1). It is required to define the set $Q_r^* \subseteq Q$ as the area $S(M \cap B(O, r))$ is maximal for any $O \in Q_r^*$. Denote r_M by the maximal radius of the inscribed circles contained in M, and R_r by the minimal radius of the described circles containing M.

Lemma 2. *Suppose that M is a closed and convex set and O_1O_2 is a closed interval. Then for any $r > 0$ and for any $O \in O_1O_2$, the following inequality holds [2].*

$$S(B(O, r) \cap M) \geq min(S(B(O_1, r) \cap M), S(B(O_2, r) \cap M)),$$

where we denote S as the area of a domain.

Proof. Since $O \in O_1O_2$, there exists a number $\alpha \in [0, 1]$ such that

$$O = \alpha O_1 + (1 - \alpha)O_2 \text{ and } B(O, r) = \alpha B(O_1, r) + (1 - \alpha)B(O_2, r).$$

By convexity of M, we have

$$(\alpha B(O_1, r) \cap M) + ((1 - \alpha)B(O_2, r) \cap M) \subseteq B(O, r),$$

$$(\alpha B(O_1, r) \cap M) + ((1 - \alpha)B(O_2, r) \cap M) \subseteq M,$$

and this implies

$$(\alpha B(O_1, r) \cap M) + ((1 - \alpha)B(O_2, r) \cap M) \subseteq (B(O, r) \cap M).$$

Using the Brunn-Minkowski inequality [6], we have

$$\begin{aligned}\sqrt{S(B(O, r) \cap M)} &\geq \sqrt{S(\alpha B(O_1, r) \cap M + (1 - \alpha)B(O_2, r) \cap M)} \\ &\geq \alpha\sqrt{S(B(O_1, r) \cap M)} + (1 - \alpha)\sqrt{S(B(O_2, r) \cap M)} \\ &\geq \min(\sqrt{S(B(O_1, r) \cap M)}, \sqrt{S(B(O_2, r) \cap M)}).\end{aligned}$$

and the lemma is proved.

Theorem 5. *Following statements are hold.*
1a. *If $r < r_M$ or $r > R_M$, then int $Q_r^* \neq \emptyset$.*
1b. *If $r = r_M$ or $r = R_M$, then int $Q_r^* = \emptyset$ and Q_r^* consists of unique point.*
2. *If $r > R_M$ or M is convex, then Q_r^* is convex and compact.*
3. *If M is a simply connected set, then int $Q_r^* = \emptyset$ for $r_M < r < R_M$.*

Proof. Statements 1a, 1b and statement 2 in case $r > R_M$ are evident. Statement 2 follows from the lemma 2 when M is convex.
Now we consider the statement 3. We have

$$\pi r_M^2 < S(B(O, r) \cap M) < \pi R_M^2$$

for any $O \in Q_r^*$. On the contrary, we assume that $int\ Q_r^* \neq \emptyset$. Then there exists a small scalar $\varepsilon > 0$ for every point $O \in int\ Q_r^*$ such that $S(B(O', r) \cap M)$ is constant for any $O' \in B(O, \varepsilon)$.
Hence, it follows that either the ring $(B(O, r + \varepsilon) \setminus B(O, r - \varepsilon))$ consists of points of M or $M \cap (B(O, r + \varepsilon) \setminus B(O, r - \varepsilon)) = \emptyset$. In first case, from the simply connectedness of M it follows that $r < r_M$ which contradicts to $r > r_M$. In second case, from the connectedness of M it follows that $r > R_M$ which contradicts to $r < R_M$. The theorem is proved [1].

Note that for the statement 3, the simply connectedness of M is necessary. In fact, if M is a ring $B(O, R) \setminus (int\ B(O, R_1))$, where $R > R_1$ and $\frac{R+R_1}{2} < r - R$, then Q_r^* contains $B(O, R - r)$ so that $int\ Q_r^* \neq \emptyset$.

Now we consider some primary propositions which may be useful. It is clear that the function

$$g(r) = \max_{O \in R^2} S(B(O, r) \cap M)$$

is strongly increasing on the interval $[r_M, R_M]$.

Proposition 2. *If M is convex, then $Q_r^* \subseteq M$ for any $r \in [0, R_M]$.*

Proof. The statement of proposition in case of $r \in [0, r_M]$ is evident. We consider the case when $r \in (r_M, R_M]$. Suppose $O \in Q_r^*$ and $O \notin M$. Then there exists $O_1 \in Q_r^*$ such that $\rho(O, O_1) = \min_{A \in M} \rho(O, A)$. Passing through O_1, we can construct a line separating M and O, and perpendicular to the straight line OO_1. It is clear that $B(O, r) \cap M$ is included in $int\ B(O_1, r)$. Then, there exists $\varepsilon > 0$ such that $(B(O, r) \cap M) \subseteq B(O_1, r - \varepsilon)$. Therefore,

$$g(r - \varepsilon) \geq S(B(O_1, r - \varepsilon) \cap M) \geq S(B(O, r) \cap M) = g(r)$$

which contradicts to the strongly monotonicity of $g(r)$.

When $r \geq R_M$, then the nomadic residence must be located at the point O which is the center of the minimal circle describing M.

Proposition 3. *O is either the center of the describing circle of an acute triangle $\triangle ABC$, where $A, B, C \in C(O, R_M) \cap M$, or the middle point of the diameter of M.*

Proof. If $C(O, R_M) \cap M$ contains some acute triangle, then O indeed coincides with the center of the describing circle of this triangle. Otherwise, there exists a half-disk including $C(O, R_M) \cap M$. If the both ends of the diameter of this half-disk do not belong to M, then by moving O slightly we can obtain another circle $B(O_1, R_1)$ $(R_1 < R_M)$ containing M. This contradicts to the fact that R_M is the radius of the minimal circle describing M.

Now we assume that the possible location Q for the nomadic residence is a line l and $R_M \leq r$. Let O be the center of the describing circle $C(O, R_{M,l})$ $(R_{M,l} \geq R_M)$ of M. We consider a line η which is perpendicular to l and passes through O. This line η separates $C(O, R_{M,l})$ into two parts: $C^+(O, R_{M,l})$ and $C^-(O, R_{M,l})$ none of which contain an end of the separating diameter.

Proposition 4. *Either there exist two points $A \in C^+(O, R_{M,l}) \cap M$ and $B \in C^-(O, R_{M,l}) \cap M$ or there exists a point $C \in M \cap \eta \cap C(O, R_{M,l})$.*

Proof. If neither A and B nor C exists, then all points of the set $C(O, R_{M,l}) \cap M$ are located on either $C^+(O, R_{M,l})$ or $C^-(O, R_{M,l})$. Therefore, by moving O to $O_1 \in l$ slightly, we can construct a disk $B(O_1, R_1)$ satisfying $M \subseteq B(O_1, R_1), R_1 < R_{M,l}$. This contradicts to the fact that $R_{M,l}$ is the radius of the minimal describing circle of M with a center belonging to l.

Now, again we assume that $Q = \Re^2,\ r_M \leq r \leq R_M$.

Theorem 6. *Let $O \in Q_r^*$ and M is a triangle or a diagonally symmetric convex quadrangle or any regular convex polygon. Then there exists a number $r_{max} \leq R_M$ such that for any r, $r_M < r < r_{max}$ the ratio of the chord generated by $C(O, r) \cap M$ and the lenght of the corresponding side is constant.*

Proof. The statement of the theorem for regular convex polygon is evident because $O \in Q_r^*$ is the center of polygon, where $r_{max} = R_M$.

Let M is a triangle. Assume that a triangle $\triangle ABC$ with edges a, b and c is given, and its largest angle is $\angle ABC$. Let a circle with radius r is given.

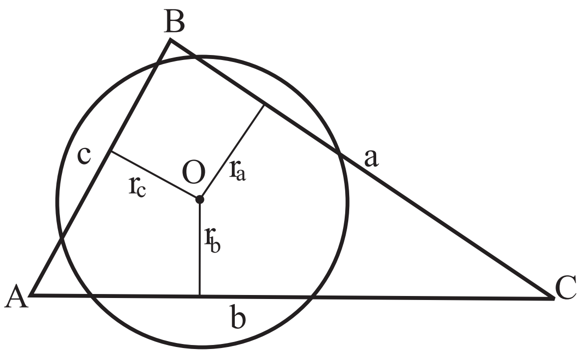

Fig. 4.

We denote by r_a, r_b and r_c distances measured from the center O of the circle to edges a, b and c of the triangle, respectively, where $r \leq \min\{OB, OA, OC\}$ (Fig.4). We construct the following Lagrange function

$$L(r_a, r_b, r_c, \lambda) = r^2 \arccos\frac{r_a}{r} + r^2 \arccos\frac{r_b}{r} + r^2\frac{r_c}{r} - \sqrt{r^2 - r_a^2}r_a \\ - \sqrt{r^2 - r_b^2}r_b - \sqrt{r^2 - r_c^2}r_c + \lambda(r_a a + r_b b + r_c c - a - b - c)$$

and consider the maximization problem

$$L(r_a, r_b, r_c, \lambda) \rightarrow \max;$$

$$0 < r_a,\ 0 < r_b,\ 0 < r_c.$$

By Lagrange rule, the partial derivatives of the Lagrange function are equal to zero, we obtain

$$\frac{2\sqrt{r^2 - r_a^2}}{a} = \frac{2\sqrt{r^2 - r_b^2}}{b} = \frac{2\sqrt{r^2 - r_c^2}}{c} = \lambda.$$

If $\angle ABC \leq \frac{\pi}{2}$, then $r_{\max} = R_M$, otherwise $r_{max} < R_M$ and $r_{max} =$ OB.

Now, let us consider a diagonally symmetric quadrangle $ABCD$ with edges $AB = AD = a$, $BC = DC = b$. Clearly, the center O of the maximal circle with radius r always lies on the axis of symmetry AC, and the Lagrange function for this circle has the following form

$$L(r_a, r_b, \lambda) = 2r\big(\arccos\frac{r_a}{r} + \arccos\frac{r_b}{r}\big) \\ - 2\sqrt{r^2 - r_a^2} \cdot r_a - 2\sqrt{r^2 - r_b^2} \cdot r_b + 2\lambda(ar_a + br_b - a - b).$$

Corresponding Lagrange problem is

$$L(r_a, r_b, \lambda) \ \rightarrow \ \max,$$

$$r_a > 0,\ r_b > 0.$$

By Lagrange rule, the partial derivatives of the Lagrange function are equal to zero, we obtain

$$\frac{2\sqrt{r^2 - r_a^2}}{a} = \frac{2\sqrt{r^2 - r_b^2}}{b} = \lambda.$$

If BD < AC, then $r_{\max} = OB$. But, if BD $\geq$ AC, then

$$r_{\max} = \begin{cases} \text{OC, if } \angle BAD \leq \angle BCD, \\ \text{OA, if } \angle BAD > \angle BCD \end{cases}.$$

The proof is completed.

4 Conclusion

Nowadays, the world civilization is divided into two forms: settled and nomadic. The nomadic civilization is closely connected with the nature, and ecological and economical problems of nomads are regulated simultaneously. Therefore, research activities in this field are increasing more and more, and many international conferences are being organized every year.

Mongolia is one of the few countries where the nomadic civilization still exists in classical form. Fifty percent of the population is involved somehow in stock nomadic breeding. Since Mongolian has extreme climate, it is very important for nomads to determine optimal choices for roaming places, i.e., the location for the nomadic residence depending on the seasons. While the settled civilization is well studied and modeled mathematically, the study of the nomadic civilization is practically ignored and less. Therefore, our work may be regarded as new in mathematical modeling.

In this work, we consider extremal problems on pasture surface, define its basic elements, present and solve the problem of determining optimal locations for the nomadic residence, and prove some related and existence theorems. This research is realized within the Russia-Mongolian joint grant "Economic and geometry extremal problems on equipped surfaces".

We have used mathematical apparatus such as geometry, functional analysis and theory of extremal problems in our study.

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