ON THE CONSTRUCTION OF FAMILIES OF OPTIMAL RECOVERY METHODS FOR LINEAR OPERATORS

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ABSTRACT. The paper proposes some approach to the construction of families of optimal methods for the recovery of linear operators from inaccurately given information. The proposed construction method is used to recover derivatives from inaccurately specified other derivatives in the multidimensional case and to recover solutions of the heat equation from inaccurately specified temperature distributions at some instants of time.

1. INTRODUCTION

Let X be a linear space, Y, Z be normed linear spaces. The problem of optimal recovery of the linear operator $\Lambda\colon X \to Z$ by inaccurately given values of the linear operator $I\colon X \to Y$ on the set $W \subset X$ is posed as a problem of finding the value

$$E(\Lambda, W, I, \delta) = \inf_{\varphi\colon Y \to Z} \sup_{\substack{x \in W,\ y \in Y \\ \|Ix - y\|_Y \le \delta}} \|\Lambda x - \varphi(y)\|_Z,$$

called the *error of optimal recovery*, and the mapping φ on which the lower bound is attained, called the *optimal recovery method* (here $\delta \ge 0$ is a parameter that characterizes the error of setting the values of the operator I). Initially, this problem was posed for the case when Λ is a linear functional, Y is a finite-dimensional space and the information is known exactly ($\delta = 0$), by S. A. Smolyak [1]. In fact, this statement was a generalization of A. N. Kolmogorov's problem about the best quadrature formula on the class of functions [2], in which the integral and the values of the functions are replaced by arbitrary linear functionals and there is no condition for the linearity of the recovery method. Subsequently, much research has been devoted to extensions of this problem (see [3]–[10], and the references given therein).

One of the first papers in which the problem of constructing an optimal recovery method for a linear operator was considered was the paper [4]. This topic was further developed in the papers [11]–[19]. It turned out that in some cases it is possible to construct a whole family of optimal recovery methods for a linear operator. The study of such families began in [20] and continued in [21], [22], [14], and [19].

The aim of this paper is to propose some approach to the construction of families of optimal recovery methods for linear operators and demonstrate its application to a number of particular problems.

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2. General setting and construction of families of optimal recovery methods

We will consider the case when in the optimal recovery problem the set W (a priori information about elements from X) is given in the form of constraints associated with a certain set of linear operators. Let $Y_0, Y_1, \dots, Y_n$ be normed linear spaces and $I_j\colon X \to Y_j$, $j = 0, 1, \dots, n$, be linear operators. Let, in addition, the numbers $\delta_1, \dots, \delta_n \ge 0$ are given and the set of natural numbers $J \subset \{1, \dots, n\}$ is given. Put $\overline{J} = \{1, \dots, n\} \setminus J$.

The problem is to optimally recover the operator I_0 on the set

$$W_J = \{\, x \in X : I_j x\|_{Y_j} \le \delta_j,\ j \in J \,\}$$

by the values of the operators I_j, given with errors δ_j, $j \in \overline{J}$ (when $J = \emptyset$ we assume $W = X$). More precisely, we will assume that, for each $x \in W$, we know the vector

$$y = \{y_j\}_{j\in\overline{J}} \in Y_{\overline{J}} = \prod_{j\in\overline{J}} Y_j$$

such that $\|I_j x - y_j\|_{Y_j} \le \delta_j$, $j \in \overline{J}$. As recovery methods we will consider arbitrary mappings $\varphi\colon Y_{\overline{J}} \to Y_0$.

The *error of a method* $\varphi(\cdot)$ is defined as

$$e_J(I, \delta, \varphi) = \sup_{\substack{x\in W_J,\ y\in Y_{\overline{J}} \\ \|I_j x - y_j\|_{Y_j} \le \delta_j,\ j\in\overline{J}}} \|I_0 x - \varphi(y)\|_{Y_0},$$

and the quantity

$$E_J(I, \delta) = \inf_{\varphi\colon Y_{\overline{J}} \to Y_0} e_J(I, \delta, \varphi) \tag{1}$$

is known as the *optimal recovery error* (here $I = (I_0, I_1 \dots, I_n)$, $\delta = (\delta_1, \dots, \delta_n)$). The methods on which the lower bound in (1) is attained (if any exist) are called *optimal*.

Theorem 1. *Let $1 \le p < +\infty$. Assume that there exist $\widehat{\lambda}_j \ge 0$, $j = 1, \dots, n$, such that*

$$\sup_{\substack{x\in X \\ \|I_j x\|_{Y_j} \le \delta_j,\ j=1,\dots,n}} \|I_0 x\|_{Y_0}^p \ge \sum_{j=1}^n \widehat{\lambda}_j \delta_j^p.$$

Moreover, let the set of linear operators $S_j\colon Y_j \to Y_0$, $j = 1, \dots, n$, be such that

$$I_0 = \sum_{j=1}^n S_j I_j \tag{2}$$

and

$$\Bigg\| \sum_{j=1}^n S_j z_j \Bigg\|_{Y_0}^p \le \sum_{j=1}^n \widehat{\lambda}_j \|z_j\|_{Y_j}^p \tag{3}$$

for all $z_j \in Y_j$, $j = 1, \dots, n$. Then for any $J \in \{1, \dots, n\}$ methods

$$\widehat{\varphi}(y) = \sum_{j\in\overline{J}} S_j y_j \tag{4}$$

are optimal for the corresponding optimal recovery problem, and for the error of optimal recovery the equality

$$E_J(I,\delta)=\left(\sum_{j=1}^n\widehat{\lambda}_j\delta_j^p\right)^{1/p} \tag{5}$$

holds.

Proof. Let $\varphi\colon Y_{\overline{J}}\to Y_0$ be an arbitrary method of recovery and $x\in X$ such that $\|I_jx\|_{Y_j}\le\delta_j$, $j=1,\dots,n$. Then

$$\begin{aligned}2\|I_0x\|_{Y_0}&=\|I_0x-\varphi(0)-(I_0(-x)-\varphi(0))\|_{Y_0}\\&\le\|I_0x-\varphi(0)\|_{Y_0}+\|I_0(-x)-\varphi(0)\|_{Y_0}\le 2e_J(I,\delta,\varphi).\end{aligned}$$

Hence

$$e_J^p(I,\delta,\varphi)\ge\sup_{\substack{x\in X\\ \|I_jx\|_{Y_j}\le\delta_j,\ j=1,\dots,n}}\|I_0x\|_{Y_0}^p\ge\sum_{j=1}^n\widehat{\lambda}_j\delta_j^p.$$

Since $\varphi(\cdot)$ is arbitrary, we obtain

$$E_J^p(I,\delta)\ge\sum_{j=1}^n\widehat{\lambda}_j\delta_j^p. \tag{6}$$

To estimate the p-th power of error of method $\widehat{\varphi}(\cdot)$, it is necessary to estimate the value of the following extremal problem

$$\Big\|I_0x-\sum_{j\in\overline{J}}S_jy_j\Big\|_{Y_0}^p\to\max,\quad \|I_jx\|_{Y_j}\le\delta_j,\ j\in J,$$
$$\|I_jx-y_j\|_{Y_j}\le\delta_j,\ j\in\overline{J},\quad x\in X.$$

Put $z_j=I_jx-y_j$, $j\in\overline{J}$. Then this problem is rewritten as follows

$$\Big\|\Big(I_0-\sum_{j\in\overline{J}}S_jI_j\Big)x+\sum_{j\in\overline{J}}S_jz_j\Big\|_{Y_0}^p\to\max,\quad \|I_jx\|_{Y_j}\le\delta_j,\ j\in J, \tag{7}$$
$$\|z_j\|_{Y_j}\le\delta_j,\ j\in\overline{J},\quad x\in X.$$

In view of (2) and condition (3) we obtain

$$\begin{aligned}\Big\|\Big(I_0-\sum_{j\in\overline{J}}S_jI_j\Big)x+\sum_{j\in\overline{J}}S_jz_j\Big\|_{Y_0}^p&=\Big\|\sum_{j\in J}S_jI_jx+\sum_{j\in\overline{J}}S_jz_j\Big\|_{Y_0}^p\\&\le\sum_{j\in J}\widehat{\lambda}_j\|I_jx\|_{Y_j}^p+\sum_{j\in\overline{J}}\widehat{\lambda}_j\|z_j\|_{Y_j}^p\le\sum_{j=1}^n\widehat{\lambda}_j\delta_j^p.\end{aligned}$$

Thus,

$$E_J^p(I,\delta)\le e_J^p(I,\delta,\widehat{\varphi})\le\sum_{j=1}^n\widehat{\lambda}_j\delta_j^p,$$

which together with (6) proves the theorem. □

Note that the dual extremal problem

$$\|I_0x\|_{Y_0} \to \max, \quad \|I_jx\|_{Y_j} \le \delta_j, \quad j = 1, \dots, n,$$

"does not distinguish" which of the operators I_j are informational, and which of them define the class on which the recovery problem is being considered. In other words, the dual extremal problem does not distinguish a priori information from a posteriori information. Because of this, it follows from Theorem 1 that if the operators $S_j\colon Y_j \to Y_0$, $j = 1, \dots, n$, are found such that they satisfy the conditions (2) and (3), then 2^n recovery problems are immediately solved. Moreover, to obtain the appropriate optimal method, it is sufficient to put $y_j = 0$, $j \in J$, in the method

$$\widehat{\varphi}(y) = S_1y_1 + \dots + S_ny_n.$$

3. Recovery in $L_p(\mathbb{R}^d)$

Denote by $L_p(\mathbb{R}^d)$, $1 \le p < \infty$, the set of all measurable functions $x(\cdot)$ for which

$$\|x(\cdot)\|_{L_p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |x(\xi)|^p \, d\xi\right)^{1/p} < \infty.$$

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d_+$. For $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ we set $(i\xi)^\alpha = (i\xi_1)^{\alpha_1} \dots (i\xi_d)^{\alpha_d}$, $|\xi|^\alpha = |\xi_1|^{\alpha_1} \dots |\xi_d|^{\alpha_d}$. For $\alpha^0, \alpha^1, \dots, \alpha^n \in \mathbb{R}^d_+$ put

$$I_jx(\xi) = (i\xi)^{\alpha^j} x(\xi), \quad j = 0, 1, \dots, n.$$

We denote by X the set of all measurable functions $x(\cdot)$ for which $\|I_jx(\cdot)\|_{L_p(\mathbb{R}^d)} < \infty$, $j = 1, \dots, n$. Consider problem (1) for $Y_0 = Y_1 = \dots = Y_n = L_p(\mathbb{R}^d)$.

Put

$$Q = \operatorname{co}\{(\alpha^1, \ln 1/\delta_1), \dots, (\alpha^n, \ln 1/\delta_n)\},$$

where $\operatorname{co} M$ denotes the convex hull of the set M, and define the function $S(\cdot)$ on $\mathbb{R}^d$ by the formula

$$S(\alpha) = \max\{\, z \in \mathbb{R} : (\alpha, z) \in Q \,\}, \tag{8}$$

assuming that $S(\alpha) = -\infty$ if the set in curly brackets is empty.

Let $\alpha^0 \in \operatorname{co}\{\alpha^1, \dots, \alpha^n\}$. Then the point $(\alpha^0, S(\alpha^0))$ belongs to the boundary of the convex polyhedron Q. We draw a hyperplane of support to the convex polyhedron Q at the point $(\alpha^0, S(\alpha^0))$. It can be written as $z = \langle \alpha, \widehat{\eta} \rangle + \widehat{a}$ for some $\widehat{\eta} = (\widehat{\eta}_1, \dots, \widehat{\eta}_d) \in \mathbb{R}^d$ and $\widehat{a} \in \mathbb{R}$ ($\langle \alpha, \widehat{\eta} \rangle$ denotes the scalar product of the vectors α and $\widehat{\eta}$). According to the Caratheodory theorem, there exist points $(\alpha^{j_k}, \ln 1/\delta_{j_k})$, $k = 1, \dots, s$, $s \le d + 1$, from this hyperplane such that

$$\alpha^0 = \sum_{k=1}^{s} \theta_{j_k} \alpha^{j_k}, \quad \theta_{j_k} > 0, \ k = 1, \dots, s, \quad \sum_{k=1}^{s} \theta_{j_k} = 1. \tag{9}$$

Put $J_0 = \{j_1, \dots, j_s\}$ and

$$\widehat{\lambda}_j = \frac{\theta_j}{\delta_j^p} e^{-pS(\alpha^0)}, \quad j \in J_0.$$

Theorem 2. *Let $\alpha^0 \in \operatorname{co}\{\alpha^1, \dots, \alpha^n\}$. Then for any $J \in \{1, \dots, n\}$*

$$E_J(I, \delta) = e^{-S(\alpha^0)}.$$

Moreover, all methods

$$\widehat{\varphi}(y) = \sum_{j\in\overline{J}\cap J_0} a_j(\xi)y_j,$$

where measurable functions $a_j(\cdot)$, $j\in J_0$, *satisfy the conditions*

$$\sum_{j\in J_0}(i\xi)^{\alpha^j}a_j(\xi) = (i\xi)^{\alpha^0}, \tag{10}$$

$$\sum_{j\in J_0}\frac{|a_j(\xi)|^{p'}}{\widehat{\lambda}_j^{p'/p}} \le 1, \quad 1/p+1/p'=1, \quad \text{if } 1<p<\infty, \tag{11}$$

$$\max_{j\in J_0}\frac{|a_j(\xi)|}{\widehat{\lambda}_j} \le 1, \quad \text{if } p=1, \tag{12}$$

for almost all $\xi\in\mathbb{R}^d$, *are optimal for the corresponding optimal recovery problem.*

Proof. We estimate the value of the extremal problem

$$\int_{\mathbb{R}^d}|(i\xi)^{\alpha^0}x(\xi)|^p\,d\xi\to\max, \quad \int_{\mathbb{R}^d}|(i\xi)^{\alpha^j}x(\xi)|^p\,d\xi\le\delta_j^p, \quad j=1,\dots,n. \tag{13}$$

Put $\widehat{A}=e^{-p\widehat{a}}$, $\widehat{\xi}_j=e^{-\widehat{\eta}_j}$, $j=1,\dots,d$, $\widehat{\xi}=(\widehat{\xi}_1,\dots,\widehat{\xi}_d)$. For sufficiently small $\varepsilon>0$ consider the cube

$$B_\varepsilon=\{\,\xi=(\xi_1,\dots,\xi_d)\in\mathbb{R}^d:\widehat{\xi}_j-\varepsilon\le\xi_j\le\widehat{\xi}_j,\ j=1,\dots,d\,\}$$

and the function

$$x_\varepsilon(\xi)=\begin{cases}\left(\widehat{A}/|B_\varepsilon|\right)^{1/p}, & \xi\in B_\varepsilon,\\ 0, & \xi\notin B_\varepsilon\end{cases}$$

($|B_\varepsilon|$ denotes the volume of the cube B_ε). Then

$$\int_{\mathbb{R}^d}|(i\xi)^{\alpha^j}x_\varepsilon(\xi)|^p\,d\xi\le\widehat{A}|\widehat{\xi}|^{p\alpha^j}=e^{-p(\langle\alpha^j,\widehat{\eta}\rangle+\widehat{a})}.$$

In view of the fact that $z=\langle\alpha,\widehat{\eta}\rangle+\widehat{a}$ is the hyperplane of support to Q, we have

$$\langle\alpha^j,\widehat{\eta}\rangle+\widehat{a}\ge\ln 1/\delta_j.$$

It follows that

$$\int_{\mathbb{R}^d}|(i\xi)^{\alpha^j}x_\varepsilon(\xi)|^p\,d\xi\le\delta_j^p, \quad j=1,\dots,n.$$

Thus, $x_\varepsilon(\cdot)$ is an admissible function for problem (13). Consequently,

$$\sup_{\substack{x\in X\\ \|I_jx\|_{Y_j}\le\delta_j,\ j=1,\dots,n}}\|I_0x\|_{Y_0}^p\ge\int_{\mathbb{R}^d}|(i\xi)^{\alpha^0}x_\varepsilon(\xi)|^p\,d\xi\ge\widehat{A}|\widehat{\xi}_\varepsilon|^{p\alpha^0},$$

where

$$\widehat{\xi}_\varepsilon=(\widehat{\xi}_1-\varepsilon,\dots,\widehat{\xi}_d-\varepsilon).$$

Making ε tends to zero, we have

$$\sup_{\substack{x\in X\\ \|I_jx\|_{Y_j}\le\delta_j,\ j=1,\dots,n}}\|I_0x\|_{Y_0}^p\ge e^{-pa}|\widehat{\xi}|^{p\alpha^0}=e^{-p(\langle\alpha^0,\widehat{\eta}\rangle+\widehat{a})}=e^{-pS(\alpha^0)}.$$

Thus,

$$\sup_{\substack{x\in X\\ \|I_jx\|_{Y_j}\le\delta_j,\ j=1,\dots,n}}\|I_0x\|_{Y_0}^p\ge\sum_{j\in J_0}\widehat{\lambda}_j\delta_j^p.$$

We define operators $S_j\colon L_p(\mathbb{R}^d)\to L_p(\mathbb{R}^d)$, $j=1,\dots,n$, as follows

$$S_jz(\xi)=\begin{cases}a_j(\xi)z(\xi), & j\in J_0,\\ 0, & j\notin J_0,\end{cases}$$

where $a_j(\cdot)$, $j\in J_0$, satisfy conditions (10)–(12). We have

$$\left\|\sum_{j=1}^n S_jz_j(\cdot)\right\|^p_{L_p(\mathbb{R}^d)}=\int_{\mathbb{R}^d}\left|\sum_{j\in J_0}a_j(\xi)z_j(\xi)\right|^p d\xi. \tag{14}$$

By Hölder's inequality for $1<p<\infty$

$$\left|\sum_{j\in J_0}a_j(\xi)z_j(\xi)\right|=\left|\sum_{j\in J_0}\frac{a_j(\xi)}{\widehat{\lambda}_j^{1/p}}\widehat{\lambda}_j^{1/p}z_j(\xi)\right|\le\Omega(\xi)\biggl(\sum_{j\in J_0}\widehat{\lambda}_j|z_j(\xi)|^p\biggr)^{1/p},$$

where

$$\Omega(\xi)=\biggl(\sum_{j\in J_0}\frac{|a_j(\xi)|^{p'}}{\widehat{\lambda}_j^{p'/p}}\biggr)^{1/p'},\quad 1/p+1/p'=1.$$

For $p=1$ we obtain the inequality

$$\left|\sum_{j\in J_0}a_j(\xi)z_j(\xi)\right|\le\Omega(\xi)\biggl(\sum_{j\in J_0}\widehat{\lambda}_j|z_j(\xi)|\biggr),$$

in which

$$\Omega(\xi)=\max_{j\in J_0}\frac{|a_j(\xi)|}{\widehat{\lambda}_j}.$$

Using the obtained inequalities, it follows from (14) that

$$\left\|\sum_{j=1}^n S_jz_j(\cdot)\right\|^p_{L_p(\mathbb{R}^d)}\le\int_{\mathbb{R}^d}\Omega^p(\xi)\biggl(\sum_{j\in J_0}\widehat{\lambda}_j|z_j(\xi)|^p\biggr)d\xi.$$

In view of conditions (11)–(12) we get

$$\left\|\sum_{j=1}^n S_jz_j(\cdot)\right\|^p_{L_p(\mathbb{R}^d)}\le\sum_{j\in J_0}\widehat{\lambda}_j\|z_j(\cdot)\|^p_{L_p(\mathbb{R}^d)}.$$

It remains to show that the set of functions $a_j(\cdot)$, $j\in J_0$, satisfying conditions (10)–(12) is nonempty. Consider the function

$$f(\eta)=-1+\sum_{j\in J_0}\widehat{\lambda}_je^{-p\langle\alpha^j-\alpha^0,\eta\rangle}$$

on $\mathbb{R}^d$. This is obviously a convex function, and it is easy to verify that $f(\widehat{\eta})=0$ and the derivative of this function at the point $\widehat{\eta}$ is also zero. It follows that $f(\eta)\ge0$ for all $\eta\in\mathbb{R}^d$. Consequently,

$$-e^{-p\langle\alpha^0,\eta\rangle}+\sum_{j\in J_0}\widehat{\lambda}_je^{-p\langle\alpha^j,\eta\rangle}\ge0.$$

Putting $e^{-\eta_j}=|\xi_j|$, $j=1,\dots,d$, we obtain that

$$-|\xi|^{p\alpha^0}+\sum_{j\in J_0}\widehat{\lambda}_j|\xi|^{p\alpha^j}\ge0 \tag{15}$$

for all $\xi \in \mathbb{R}^d$. Set

$$a_j(\xi) = (i\xi)^{\alpha^0} \frac{\widehat{\lambda}_j(-i\xi)^{\alpha^j} |\xi|^{(p-2)\alpha^j}}{\sum\limits_{j \in J_0} \widehat{\lambda}_j |\xi|^{p\alpha^j}}, \quad j \in J_0.$$

It is easy to check that condition (10) is valid. For $p = 1$, taking into account (15), we obtain

$$\frac{|a_j(\xi)|}{\widehat{\lambda}_j} = \frac{|\xi|^{\alpha^0}}{\sum\limits_{j \in J_0} \widehat{\lambda}_j |\xi|^{\alpha^j}} \le 1.$$

If $p > 1$, then

$$\sum_{j \in J_0} \frac{|a_j(\xi)|^{p'}}{\widehat{\lambda}_j^{p'/p}} = \sum_{j \in J_0} \frac{|\xi|^{p'\alpha^0} \widehat{\lambda}_j^{p'} |\xi|^{(p-1)p'\alpha^j}}{\widehat{\lambda}_j^{p'/p} \left(\sum\limits_{j \in J_0} \widehat{\lambda}_j |\xi|^{p\alpha^j} \right)^{p'}} = \frac{|\xi|^{p'\alpha^0} \sum\limits_{j \in J_0} \widehat{\lambda}_j |\xi|^{p\alpha^j}}{\left(\sum\limits_{j \in J_0} \widehat{\lambda}_j |\xi|^{p\alpha^j} \right)^{p'}}$$

$$= \left(\frac{|\xi|^{p\alpha^0}}{\sum\limits_{j \in J_0} \widehat{\lambda}_j |\xi|^{p\alpha^j}} \right)^{p'-1}.$$

Now it follows from (15) that

$$\sum_{j \in J_0} \frac{|a_j(\xi)|^{p'}}{\widehat{\lambda}_j^{p'/p}} \le 1.$$

□

Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}_+^d$. For $x(\cdot) \in L_2(\mathbb{R}^d)$ denote by $D^\alpha x(\cdot)$ the Weyl derivative of order α, which is defined as follows

$$D^\alpha x(t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (i\xi)^\alpha Fx(\xi) e^{i\langle \xi, t \rangle} \, d\xi,$$

where $Fx(\cdot)$ is the Fourier transform of $x(\cdot)$.

Let $\alpha^0, \alpha^1, \ldots, \alpha^n \in \mathbb{R}_+^d$. Put

$$I_j = D^{\alpha^j} \quad j = 0, 1, \ldots, n.$$

Denote by X the set of measurable functions $x(\cdot)$, for which $\|D^{\alpha^j} x(\cdot)\|_{L_2(\mathbb{R}^d)} < \infty$, $j = 1, \ldots, n$. Consider problem (1) for $Y_0 = Y_1 = \ldots = Y_n = L_2(\mathbb{R}^d)$. Using the previously introduced notation for $p = 2$, we get

Theorem 3. *Let $\alpha^0 \in \operatorname{co}\{\alpha^1, \ldots, \alpha^n\}$. Then for any $J \in \{1, \ldots, n\}$*

$$E_J(I, \delta) = e^{-S(\alpha^0)}.$$

Moreover, all methods

$$\widehat{\varphi}(y) = \sum_{j \in \overline{J} \cap J_0} \Lambda_j y_j,$$

where $\Lambda_j\colon L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)$, $j\in J_0$, are linear continuous operators whose actions in Fourier images have the form: $F\Lambda_j y_j(\cdot)=a_j(\cdot)Fy_j(\cdot)$, and measurable functions $a_j(\cdot)$, $j\in J_0$, satisfy conditions

$$\sum_{j\in J_0}(i\xi)^{\alpha^j}a_j(\xi)=(i\xi)^{\alpha^0},$$
$$\sum_{j\in J_0}\frac{|a_j(\xi)|^2}{\widehat{\lambda}_j}\le 1,$$

for almost all $\xi\in\mathbb{R}^d$, are optimal for the corresponding optimal recovery problem.

Proof. Passing to the Fourier images and using the Parseval equality, conditions

$$\|D^{\alpha^j}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}\le\delta_j^2,$$
$$\|D^{\alpha^j}x(\cdot)-y_j(\cdot)\|^2_{L_2(\mathbb{R}^d)}\le\delta_j^2,$$

may be rewritten in the form

$$\int_{\mathbb{R}^d}|\xi|^{2\alpha^j}|f(\xi)|^2\,d\xi\le\delta_j^2,$$
$$\int_{\mathbb{R}^d}|(i\xi)^{\alpha^j}f(\xi)-Y_j(\xi)|^2\le\delta_j^2,$$

where

$$f(\cdot)=\frac{1}{(2\pi)^{d/2}}Fx(\cdot),\quad Y_j(\cdot)=\frac{1}{(2\pi)^{d/2}}Fy_j(\cdot).$$

For any recovery method $\varphi\colon(L_2(\mathbb{R}^d))^m\to L_2(\mathbb{R}^d)$, $m=card\overline{J}$,

$$\|D^{\alpha^0}x(\cdot)-\varphi(y)(\cdot)\|^2_{L_2(\mathbb{R}^d)}=\int_{\mathbb{R}^d}|(i\xi)^{\alpha^0}f(\xi)-\Phi(y)(\xi)|^2\,d\xi,$$

where

$$\Phi(y)(\cdot)=\frac{1}{(2\pi)^{d/2}}F\varphi(y)(\cdot).$$

Thus the problem under consideration is equivalent to the problem, the solution of which is given in Theorem 2 (for $p=2$). □

Note that Theorems 1 and 2 imply the equality

$$\sup_{\|D^{\alpha^j}x(\cdot)\|_{L_2(\mathbb{R}^d)}\le\delta_j,\ j=1,\dots,n}\|D^{\alpha^0}x(\cdot)\|_{L_2(\mathbb{R}^d)}=e^{-S(\alpha^0)}=\prod_{j\in J_0}\delta_j^{\theta_j}. \tag{16}$$

The extremal problem in the left-hand side of (16) is closely related to finding the exact constant in the generalized Hardy–Littlewood–Polya inequality, which in the case under consideration has the form

$$\|D^{\alpha^0}x(\cdot)\|_{L_2(\mathbb{R}^d)}\le\prod_{j\in J_0}\|D^{\alpha^j}x(\cdot)\|^{\theta_j}_{L_2(\mathbb{R}^d)}$$

(for more information, see [23]).

4. Generalized heat equation on a sphere

Set

$$\mathbb{S}^{d-1} = \{\, x \in \mathbb{R}^d : |x| = 1 \,\}, \quad d \ge 2,$$

where $|x| = \sqrt{x_1^2 + \ldots + x_d^2}$. The Laplace–Beltrami operator Δ_S is defined for functions defined on the unit sphere $\mathbb{S}^{d-1}$ as follows

$$\Delta_S Y(x') = \Delta Y\left(\frac{x}{|x|}\right)_{\big| x=x'},$$

where Δ is the Laplace operator. Denote by $\mathcal{H}_k$ the set of spherical harmonics of order k. It is known (see [24]) that $L_2(\mathbb{S}^{d-1}) = \sum_{k=0}^{\infty} \mathcal{H}_k$, while $\dim \mathcal{H}_0 = a_0 = 1$,

$$\dim \mathcal{H}_k = a_k = (d+2k-2)\frac{(d+k-3)!}{(d-2)!k!}, \quad k = 1, 2, \ldots.$$

Choose in $\mathcal{H}_k$ the orthonormal basis $Y_j^{(k)}(\cdot)$, $j = 1, \ldots, a_k$. For $\alpha > 0$ the operator $(-\Delta_S)^{\alpha/2}$ is defined by the equality

$$(-\Delta_S)^{\alpha/2} Y(\cdot) = \sum_{k=1}^{\infty} \Lambda_k^{\alpha/2} \sum_{j=1}^{a_k} c_{kj} Y_j^{(k)}(\cdot),$$

where

$$Y(\cdot) = \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} c_{kj} Y_j^{(k)}(\cdot),$$

and $\Lambda_k = k(k+d-2)$ are the eigenvalues of the operator $-\Delta_S$.

Consider the problem of finding a solution of the equation

$$u_t + (-\Delta_S)^{\alpha/2} u = 0, \tag{17}$$

with initial condition

$$u(\cdot, 0) = f(\cdot),$$

where $f(\cdot) \in L_2(\mathbb{S}^{d-1})$. If

$$f(\cdot) = \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} c_{kj} Y_j^{(k)}(\cdot), \tag{18}$$

then, using the Fourier method, it is not difficult to obtain a solution of this problem

$$u(x', t) = \sum_{k=0}^{\infty} e^{-\Lambda_k^{\alpha/2} t} \sum_{j=1}^{a_k} c_{kj} Y_j^{(k)}(x').$$

Assume that the solutions of the problem under consideration are approximately known at $t = 0, T$. It is required to recover the solution at the instant of time τ, $0 < \tau < T$. For functions $f(\cdot) \in L_2(\mathbb{S}^{d-1})$, having expansion (18), we put $I_1 f(\cdot) = f(\cdot)$,

$$I_0 f(\cdot) = \sum_{k=0}^{\infty} e^{-\Lambda_k^{\alpha/2} \tau} \sum_{j=1}^{a_k} c_{kj} Y_j^{(k)}(\cdot),$$

$$I_2 f(\cdot) = \sum_{k=0}^{\infty} e^{-\Lambda_k^{\alpha/2} T} \sum_{j=1}^{a_k} c_{kj} Y_j^{(k)}(\cdot).$$

Thus we come to the problem (1) for $X = Y_0 = Y_1 = Y_2 = L_2(\mathbb{S}^{d-1})$, $p = 2$ and $J = \emptyset$.

Theorem 4. *If* $\delta_1/\delta_2 \in \left[e^{\Lambda_m^{\alpha/2}T}, e^{\Lambda_{m+1}^{\alpha/2}T}\right]$ *for some* $m \in \mathbb{Z}_+$, *then for all* α_{kj}, $k = 0, 1, \ldots$, $j = 1, \ldots, a_k$, *satisfying the condition*

$$\frac{\left(e^{\Lambda_k^{\alpha/2}(T-\tau)} - \alpha_{kj}\right)^2}{\lambda_1 e^{2\Lambda_k^{\alpha/2}T}} + \frac{\alpha_{kj}^2}{\lambda_2} \le 1, \tag{19}$$

where

$$\lambda_1 = \frac{e^{2\Lambda_{m+1}^{\alpha/2}(T-\tau)} - e^{2\Lambda_m^{\alpha/2}(T-\tau)}}{e^{2\Lambda_{m+1}^{\alpha/2}T} - e^{2\Lambda_m^{\alpha/2}T}},$$

$$\lambda_2 = \frac{e^{-2\Lambda_m^{\alpha/2}\tau} - e^{-2\Lambda_{m+1}^{\alpha/2}\tau}}{e^{-2\Lambda_m^{\alpha/2}T} - e^{-2\Lambda_{m+1}^{\alpha/2}T}},$$

methods

$$\widehat{\varphi}(y_1, y_2)(\cdot) = \sum_{k=0}^{\infty}\sum_{j=1}^{a_k}\left(e^{-\Lambda_k^{\alpha/2}T}\left(e^{\Lambda_k^{\alpha/2}(T-\tau)} - \alpha_{kj}\right)y_{kj}^{(1)} + \alpha_{kj}y_{kj}^{(2)}\right)Y_j^{(k)}(\cdot),$$

where

$$y_s(\cdot) = \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} y_{kj}^{(s)}Y_j^{(k)}(\cdot), \quad s = 1, 2,$$

are optimal and

$$E_\emptyset(I, \delta) = \sqrt{\lambda_1\delta_1^2 + \lambda_2\delta_2^2}.$$

If $\delta_1/\delta_2 \in (0, 1]$, *then the method*

$$\widehat{\varphi}(y_1, y_2)(\cdot) = \sum_{k=0}^{\infty} e^{-\Lambda_k^{\alpha/2}\tau}\sum_{j=1}^{a_k} y_{kj}^{(1)}Y_j^{(k)}(\cdot)$$

is optimal and $E_\emptyset(I, \delta) = \delta_1$.

Proof. Consider the extremal problem

$$\|I_0f(\cdot)\|_{L_2(\mathbb{S}^{d-1})}^2 \to \max, \quad \|I_jf(\cdot)\|_{L_2(\mathbb{S}^{d-1})}^2 \le \delta_j^2, \quad j = 1, 2.$$

This problem may be written in the form

$$\sum_{k=0}^{\infty} e^{-2\Lambda_k^{\alpha/2}\tau}f_k^2 \to \max, \quad \sum_{k=0}^{\infty} f_k^2 \le \delta_1^2, \quad \sum_{k=0}^{\infty} e^{-2\Lambda_k^{\alpha/2}T}f_k^2 \le \delta_2^2, \tag{20}$$

where

$$f_k^2 = \sum_{j=1}^{a_k} c_{jk}^2, \quad k = 0, 1, \ldots.$$

Let $\delta_1/\delta_2 \in \left[e^{\Lambda_m^{\alpha/2}T}, e^{\Lambda_{m+1}^{\alpha/2}T}\right]$. Define f_m and f_{m+1} from the conditions

$$f_m^2 + f_{m+1}^2 = \delta_1^2,$$
$$e^{-2\Lambda_m^{\alpha/2}T}f_m^2 + e^{-2\Lambda_{m+1}^{\alpha/2}T}f_{m+1}^2 = \delta_2^2.$$

We have

$$f_m^2=\frac{\delta_2^2-\delta_1^2e^{-2\Lambda_{m+1}^{\alpha/2}T}}{e^{-2\Lambda_m^{\alpha/2}T}-e^{-2\Lambda_{m+1}^{\alpha/2}T}},$$
$$f_{m+1}^2=\frac{\delta_1^2e^{-2\Lambda_m^{\alpha/2}T}-\delta_2^2}{e^{-2\Lambda_m^{\alpha/2}T}-e^{-2\Lambda_{m+1}^{\alpha/2}T}}.$$

The sequence $\{f_k\}$, in which $f_k=0$ for $k\neq m,m+1$, is admissible in the extremal problem (20). Therefore,

$$\sup_{\substack{f(\cdot)\in L_2(\mathbb{S}^{d-1})\\ \|I_jf(\cdot)\|_{L_2(\mathbb{S}^{d-1})}\le\delta_j,\ j=1,2}}\|I_0f(\cdot)\|^2_{L_2(\mathbb{S}^{d-1})}\ge e^{-2\Lambda_m^{\alpha/2}\tau}f_m^2+e^{-2\Lambda_{m+1}^{\alpha/2}\tau}f_{m+1}^2$$
$$=\lambda_1\delta_1^2+\lambda_2\delta_2^2.$$

If $\delta_1/\delta_2\in(0,1]$, then the sequence $\{f_k\}$, in which $f_0=\delta_1^2$, and $f_k=0$ for $k\ge1$, is admissible in the extremal problem (20). Therefore, in this case

$$\sup_{\substack{f(\cdot)\in L_2(\mathbb{S}^{d-1})\\ \|I_jf(\cdot)\|_{L_2(\mathbb{S}^{d-1})}\le\delta_j,\ j=1,2}}\|I_0f(\cdot)\|^2_{L_2(\mathbb{S}^{d-1})}\ge f_0^2=\delta_1^2.$$

Let again $\delta_1/\delta_2\in\left[e^{\Lambda_m^{\alpha/2}T},e^{\Lambda_{m+1}^{\alpha/2}T}\right]$. For functions $f(\cdot)\in L_2(\mathbb{S}^{d-1})$, having expansion (18), define operators $S_j\colon L_2(\mathbb{S}^{d-1})\to L_2(\mathbb{S}^{d-1})$, $j=1,2$, by equalities

$$S_1f(\cdot)=\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}e^{-\Lambda_k^{\alpha/2}T}\left(e^{\Lambda_k^{\alpha/2}(T-\tau)}-\alpha_{kj}\right)c_{kj}Y_j^{(k)}(\cdot),$$
$$S_2f(\cdot)=\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}\alpha_{kj}c_{kj}Y_j^{(k)}(\cdot),$$

where α_{kj} satisfy condition (19). It is easy to see that $I_0=S_1I_1+S_2I_2$. For $f_1(\cdot),f_2(\cdot)\in L_2(\mathbb{S}^{d-1})$ we have

$$\|S_1f_1(\cdot)+S_2f_2(\cdot)\|^2_{L_2(\mathbb{S}^{d-1})}$$
$$=\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}\left(e^{-\Lambda_k^{\alpha/2}T}\left(e^{\Lambda_k^{\alpha/2}(T-\tau)}-\alpha_{kj}\right)f_{kj}^{(1)}+\alpha_{kj}f_{kj}^{(2)}\right)^2,$$

where $f_{kj}^{(1)},f_{kj}^{(2)}$ are the Fourier coefficients of $f_1(\cdot),f_2(\cdot)$. From the Cauchy–Schwartz–Bunyakovskii inequality, taking into account (19), we get

$$\left(e^{-\Lambda_k^{\alpha/2}T}\left(e^{\Lambda_k^{\alpha/2}(T-\tau)}-\alpha_{kj}\right)f_{kj}^{(1)}+\alpha_{kj}f_{kj}^{(2)}\right)^2$$
$$\le\left(\frac{\left(e^{\Lambda_k^{\alpha/2}(T-\tau)}-\alpha_{kj}\right)^2}{\lambda_1e^{2\Lambda_k^{\alpha/2}T}}+\frac{\alpha_{kj}^2}{\lambda_2}\right)\left(\lambda_1(f_{kj}^{(1)})^2+\lambda_2(f_{kj}^{(2)})^2\right)$$
$$\le\lambda_1(f_{kj}^{(1)})^2+\lambda_2(f_{kj}^{(2)})^2.$$

Thus,

$$\|S_1f_1(\cdot)+S_2f_2(\cdot)\|^2_{L_2(\mathbb{S}^{d-1})}\le\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}\Big(\lambda_1(f_{kj}^{(1)})^2+\lambda_2(f_{kj}^{(2)})^2\Big)$$
$$=\lambda_1\|f_1(\cdot)\|^2_{L_2(\mathbb{S}^{d-1})}+\lambda_2\|f_2(\cdot)\|^2_{L_2(\mathbb{S}^{d-1})}.$$

We show that there are α_{kj}, $k=0,1,\ldots$, $j=1,\ldots,a_k$, satisfying condition (19). Consider on the plane (x,y) a set of points with coordinates

$$x_k=e^{-2\Lambda_k^{\alpha/2}T},$$
$$y_k=e^{-2\Lambda_k^{\alpha/2}\tau},\quad k=0,1,\ldots.$$

This set of points lies on a concave curve $y=x^{\tau/T}$. Draw a straight line through the points (x_{m+1},y_{m+1}) and (x_m,y_m). It is easy to verify that the equation of this line is written as $y=\lambda_1+\lambda_2x$. Due to the concavity of the curve on which the points under consideration lie, we have

$$y_k\le\lambda_1+\lambda_2x_k,\quad k=0,1,\ldots.$$

Consequently, for all $k=0,1,\ldots$

$$\frac{e^{-2\Lambda_k^{\alpha/2}\tau}}{\lambda_1+\lambda_2e^{-2\Lambda_k^{\alpha/2}T}}\le 1.$$

Put

$$\widehat{\alpha}_{kj}=\frac{\lambda_2e^{\Lambda_k^{\alpha/2}(T-\tau)}}{\lambda_1e^{2\Lambda_k^{\alpha/2}T}+\lambda_2}.$$

Then

$$\frac{\Big(e^{\Lambda_k^{\alpha/2}(T-\tau)}-\widehat{\alpha}_{kj}\Big)^2}{\lambda_1e^{2\Lambda_k^{\alpha/2}T}}+\frac{\widehat{\alpha}_{kj}^2}{\lambda_2}=\frac{e^{2\Lambda_k^{\alpha/2}(T-\tau)}}{\lambda_1e^{2\Lambda_k^{\alpha/2}T}+\lambda_2}=\frac{e^{-2\Lambda_k^{\alpha/2}\tau}}{\lambda_1+\lambda_2e^{-2\Lambda_k^{\alpha/2}T}}\le 1.$$

If $\delta_1/\delta_2\in(0,1]$, then we set $S_1=I_0$ and $S_2=0$. Then

$$\|S_1f_1(\cdot)+S_2f_2(\cdot)\|^2_{L_2(\mathbb{S}^{d-1})}=\|I_0f_1(\cdot)\|^2_{L_2(\mathbb{S}^{d-1})}=\sum_{k=0}^{\infty}e^{-2\Lambda_k^{\alpha/2}\tau}\sum_{j=1}^{a_k}(f_{kj}^{(1)})^2$$
$$\le\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}(f_{kj}^{(1)})^2=\|f_1(\cdot)\|^2_{L_2(\mathbb{S}^{d-1})}.$$

Now the statement of the theorem being proved follows from Theorem 1. □

Condition (19) can be written in the equivalent form

$$(\alpha_{kj}-\widehat{\alpha}_{kj})^2\le\lambda_1\lambda_2e^{4\Lambda_k^{\alpha/2}T}\frac{-e^{-2\Lambda_k^{\alpha/2}\tau}+\lambda_1+\lambda_2e^{-2\Lambda_k^{\alpha/2}T}}{\Big(\lambda_1+\lambda_2e^{-2\Lambda_k^{\alpha/2}T}\Big)^2}.$$

Thus, all α_{kj} satisfying condition (19) have the form

$$\alpha_{kj}=\widehat{\alpha}_{kj}+\theta_{kj}e^{2\Lambda_k^{\alpha/2}T}\sqrt{\lambda_1\lambda_2}\frac{\sqrt{-e^{-2\Lambda_k^{\alpha/2}\tau}+\lambda_1+\lambda_2e^{-2\Lambda_k^{\alpha/2}T}}}{\lambda_1+\lambda_2e^{-2\Lambda_k^{\alpha/2}T}},$$

where $|\theta_{kj}|\le 1$.

If we consider the problem of optimal recovery of the solution at the instant of time τ by an inaccurately given solution at the instant of time $T > \tau$ on the class

$$W = \{\, f(\cdot) \in L_2(\mathbb{S}^{d-1}) : \|f(\cdot)\|_{L_2(\mathbb{S}^{d-1})} \le \delta_1 \,\},$$

then from the same Theorem 1 (for $J = \{1\}$) will follow that the methods $\widehat{\varphi}(0, y_2)(\cdot)$ will be optimal. It turns out that among this family of optimal methods there is a subfamily of optimal methods that have some advantage over the rest.

In order to specify this subfamily, we first formulate an extended version of the problem under consideration. Let some class of functions $\mathcal{F} \subset L_2(\mathbb{S}^{d-1})$ be given. Set

$$e(\mathcal{F}, \delta, \varphi) = \sup_{\substack{f(\cdot)\in\Omega,\ y(\cdot)\in L_2(\mathbb{S}^{d-1}) \\ \|u(\cdot,T)-y(\cdot)\|_{L_2(\mathbb{S}^{d-1})}\le\delta}} \|u(\cdot,\tau) - \varphi(y)(\cdot)\|_{L_2(\mathbb{S}^{d-1})},$$

$$E(\mathcal{F}, \delta) = \inf_{\varphi\colon\, L_2(\mathbb{S}^{d-1})\to L_2(\mathbb{S}^{d-1})} e(\mathcal{F}, \delta, \varphi).$$

The problem of finding the error of optimal recovery $E(\mathcal{F}, \delta)$ and the corresponding optimal method differs from the one considered earlier only by an arbitrary class $\mathcal{F}$.

We will say that the method $\varphi(y)(\cdot)$ is exact on the set $L \subset L_2(\mathbb{S}^{d-1})$ if $\varphi(u(\cdot, T))(\cdot) = u(\cdot, \tau)$ for all $f(\cdot) \in L$.

Proposition 1. *If $\widehat{\varphi}(y)(\cdot)$ is an optimal method for the class $\mathcal{F}$, which is linear and exact on the set $L \subset L_2(\mathbb{S}^{d-1})$ containing zero, then it is optimal on the class $\mathcal{F} + L$. Moreover,*

$$E(\mathcal{F}, \delta) = E(\mathcal{F} + L, \delta). \tag{21}$$

Proof. Let $f(\cdot) \in \mathcal{F} + L$, $f(\cdot) = f_1(\cdot) + f_2(\cdot)$, where $f_1(\cdot) \in \mathcal{F}$, $f_2(\cdot) \in L$. Denote by $u_j(\cdot, \cdot)$ the solution of equation (17) with the initial function $f_j(\cdot)$, $j = 1, 2$. Let $y(\cdot) \in L_2(\mathbb{S}^{d-1})$ such that $\|u(\cdot, T) - y(\cdot)\|_{L_2(\mathbb{S}^{d-1})} \le \delta$. Put $y_1(\cdot) = y(\cdot) - u_2(\cdot, T)$. It is clear that $y_1(\cdot) \in L_2(\mathbb{S}^{d-1})$. Since $u_1(\cdot, T) - y_1(\cdot) = u(\cdot, T) - y(\cdot)$ we have

$$\|u_1(\cdot, T) - y_1(\cdot)\|_{L_2(\mathbb{S}^{d-1})} \le \delta. \tag{22}$$

From linearity and exactness $\widehat{\varphi}(y)(\cdot)$ on L follows the equality

$$\|u(\cdot, \tau) - \widehat{\varphi}(y)(\cdot)\|_{L_2(\mathbb{S}^{d-1})} = \|u_1(\cdot, \tau) - \widehat{\varphi}(y_1)(\cdot)\|_{L_2(\mathbb{S}^{d-1})}. \tag{23}$$

The expression in the the right-hand side in (23) by virtue of (22) does not exceed the value $e(\mathcal{F}, \delta, \widehat{\varphi})$, which is equal to $E(\mathcal{F}, \delta)$, since the method $\widehat{\varphi}(y)(\cdot)$ is optimal. Taking into account this fact and going to the upper bound by $f(\cdot) \in \mathcal{F} + L$ and the corresponding $y(\cdot)$ we get that

$$e(\mathcal{F} + L, \delta, \widehat{\varphi}) \le E(\mathcal{F}, \delta).$$

Hence and from the fact that $\mathcal{F} \subset \mathfrak{F} + L$, we have

$$E(\mathcal{F}, \delta) \le E(\mathcal{F} + L, \delta) \le e(\mathcal{F} + L, \delta, \widehat{\varphi}) \le E(\mathcal{F}, \delta).$$

Consequently, $\widehat{\varphi}(y)(\cdot)$ is an optimal method for the class $\mathcal{F}+L$ and (21) is valid. $\square$

Assume that $\delta_1/\delta_2 \in \left[e^{\Lambda_m^{\alpha/2}T}, e^{\Lambda_{m+1}^{\alpha/2}T}\right]$. It is easy to show that for sufficiently large m the inequality $\lambda_2 \ge 1$ holds. Thus, if δ_1 is fixed, then for sufficiently small

δ_2 the inequality $\lambda_2 \ge 1$ is satisfied. In this case we put

$$\widehat{k} = \max\left\{ k \in \mathbb{Z}_+ : \Lambda_k \le \left(\frac{\ln \lambda_2}{2(T-\tau)}\right)^{2/\alpha} \right\}.$$

It is easy to check that

$$\widehat{k} = \left[\sqrt{\frac{(d-2)^2}{4} + \left(\frac{\ln \lambda_2}{2(T-\tau)}\right)^{2/\alpha}} - \frac{d-2}{2}\right]$$

($[a]$ is the integer part of a).

Consider the methods

$$\widehat{\varphi}_0(y)(\cdot) = \sum_{k=0}^{\widehat{k}} \sum_{j=1}^{a_k} e^{\Lambda_k^{\alpha/2}(T-\tau)} Y_j^{(k)}(\cdot) + \sum_{k=\widehat{k}+1}^{\infty} \sum_{j=1}^{a_k} \alpha_{kj} y_{kj} Y_j^{(k)}(\cdot),$$

where α_{kj}, $k = \widehat{k}+1, \widehat{k}+2, \ldots$, $j = 1, \ldots, a_k$, are satisfied condition (19). In view of the fact that for

$$\alpha_{kj} = e^{\Lambda_k^{\alpha/2}(T-\tau)}, \quad k = 0, 1, \ldots \widehat{k}, \quad j = 1, \ldots, a_k,$$

condition (19) is valid, methods $\widehat{\varphi}_0(y)(\cdot)$ are optimal on the class W.

Moreover, methods $\widehat{\varphi}_0(y)(\cdot)$ are exact on the subspace

$$L_{\widehat{k}} = \sum_{k=0}^{\widehat{k}} \mathcal{H}_k.$$

Indeed, let $f(\cdot) \in L_{\widehat{k}}$. Then

$$f(\cdot) = \sum_{k=0}^{\widehat{k}} \sum_{j=1}^{a_k} c_{kj} Y_j^{(k)}(\cdot).$$

Therefore,

$$u(x', T) = \sum_{k=0}^{\widehat{k}} e^{-\Lambda_k^{\alpha/2} T} \sum_{j=1}^{a_k} c_{kj} Y_j^{(k)}(x').$$

Consequently,

$$\widehat{\varphi}_0(u(\cdot, T))(\cdot) = \sum_{k=0}^{\widehat{k}} e^{-\Lambda_k^{\alpha/2} \tau} \sum_{j=1}^{a_k} c_{kj} Y_j^{(k)}(\cdot) = u(\cdot, \tau).$$

Thus, it follows from Proposition 1 that methods $\widehat{\varphi}_0(y)(\cdot)$ are not only optimal on the class W, but they remain optimal on the wider class $W + L_{\widehat{k}}$.

5. Optimal recovery of solutions of difference equations

Let us consider the process of heat propagation in an infinite rod described by a discrete model, namely, by an implicit difference scheme

$$\frac{u_{s+1,j} - u_{sj}}{\tau} = \frac{u_{s+1,j+1} - 2u_{s+1,j} + u_{s+1,j-1}}{h^2}. \tag{24}$$

Here τ and h are positive numbers, $(s, j) \in \mathbb{Z}_+ \times \mathbb{Z}$, $u_{s,j}$ is the temperature of the rod at the instant of time $s\tau$ at the point jh.

Denote by $l_{2,h}$ the set of vectors $x=\{x_j\}_{j\in\mathbb{Z}}$ for which

$$\|x\|_{l_{2,h}}=\left(h\sum_{j\in\mathbb{Z}}|x_j|^2\right)^{1/2}<\infty,\quad h>0.$$

Suppose that the temperature of the rod is approximately measured at the instant of time zero and at the instant of time $n\tau$, i.e. the vectors $u_0=\{u_{0,j}\}$ and $u_n=\{u_{n,j}\}$ are approximately known, or, more precisely, we know the vectors $y_1,y_2\in l_{2,h}$ such that

$$\|u_0-y_1\|_{l_{2,h}}\le\delta_1,\quad \|u_n-y_2\|_{l_{2,h}}\le\delta_2,$$

where $\delta_j>0$, $j=1,2$. According to this information, it is required to recover the vector $u_m=\{u_{m,j}\}$, where $0<m<n$, i.e. recover the value of the rod temperature at the instant of time $m\tau$.

Thus we come again to problem (1), in which $X=Y_0=Y_1=Y_2=l_2$, $p=2$, $J=\emptyset$, and the operators $I_j\colon l_{2,h}\to l_{2,h}$, $j=0,1,2$, are defined by the equalities

$$I_0u_0=u_m,\quad I_1u_0=u_0,\quad I_2u_0=u_n.$$

By the Fourier transform of the sequence $x=\{x_j\}_{j\in\mathbb{Z}}\in l_{2,h}$ we mean the function

$$Fx(\xi)=h\sum_{j\in\mathbb{Z}}x_je^{-ijh\xi}.$$

It is easy to verify that $Fx(\cdot)\in L_2([-\pi/h,\pi/h])$ and

$$\|Fx(\cdot)\|^2_{L_2([-\pi/h,\pi/h])}=2\pi\|x\|^2_{l_{2,h}}. \tag{25}$$

Let us apply the Fourier transform to the both parts of the equality (24)

$$h\sum_{j\in\mathbb{Z}}\frac{u_{s+1,j}-u_{sj}}{\tau}e^{-ijh\xi}=h\sum_{j\in\mathbb{Z}}\frac{u_{s+1,j+1}-2u_{s+1,j}+u_{s+1,j-1}}{h^2}e^{-ijh\xi}.$$

Hence

$$\frac{U_{s+1}(\xi)-U_s(\xi)}{\tau}=\frac{e^{ih\xi}-2+e^{-ih\xi}}{h^2}U_{s+1}(\xi),$$

where

$$U_s(\xi)=h\sum_{j\in\mathbb{Z}}u_{s,j}e^{-ijh\xi}.$$

Thus,

$$U_{s+1}(\xi)=\left(1+\frac{4\tau}{h^2}\sin^2\frac{h\xi}{2}\right)^{-1}U_s(\xi).$$

Consequently,

$$U_s(\xi)=\Lambda^s(\xi)U_0(\xi),\quad \Lambda(\xi)=\left(1+\frac{4\tau}{h^2}\sin^2\frac{h\xi}{2}\right)^{-1}.$$

Put $a=(1+4\tau/h^2)^{-1}$,

$$\lambda_1=\begin{cases}0, & \delta_2/\delta_1\in(0,a^n],\\ \left(1-\dfrac{m}{n}\right)\left(\dfrac{\delta_2}{\delta_1}\right)^{2m/n}, & \delta_2/\delta_1\in(a^n,1),\\ 1, & \delta_2/\delta_1\in[1,+\infty),\end{cases}$$

$$\lambda_2=\begin{cases}a^{2(m-n)}, & \delta_2/\delta_1\in(0,a^n],\\ \dfrac{m}{n}\left(\dfrac{\delta_2}{\delta_1}\right)^{2(m/n-1)}, & \delta_2/\delta_1\in(a^n,1),\\ 0, & \delta_2/\delta_1\in[1,+\infty).\end{cases}.$$

Theorem 5. *The following equality holds:*

$$E_\emptyset(I,\delta)=\sqrt{\lambda_1\delta_1^2+\lambda_2\delta_2^2}.$$

For all $\alpha(\cdot)$ *satisfying for* $\delta_2/\delta_1\in(a^n,1)$ *the condition*

$$\Lambda^{2m}(\xi)\left(\frac{|1-\alpha(\xi)|^2}{\lambda_1}+\Lambda^{-2n}(\xi)\frac{|\alpha(\xi)|^2}{\lambda_2}\right)\le 1, \tag{26}$$

and in other cases, equality

$$\alpha(\xi)=\begin{cases}1, & \delta_2/\delta_1\in(0,a^n],\\ 0, & \delta_2/\delta_1\in[1,+\infty),\end{cases}$$

methods

$$\widehat{\varphi}(y_1,y_2)=F^{-1}(\Lambda^m(\cdot)(1-\alpha(\cdot))Fy_1(\cdot)+\Lambda^{m-n}(\cdot)\alpha(\cdot)Fy_2(\cdot))$$

are optimal.

Proof. Consider the extremal problem

$$\|u_m\|^2_{l_{2,h}}\to\max,\quad \|u_0\|^2_{l_{2,h}}\le\delta_1^2,\quad \|u_n\|^2_{l_{2,h}}\le\delta_2^2.$$

Passing to the Fourier images, we obtain the following problem

$$\frac{1}{2\pi}\|\Lambda^m(\cdot)U_0(\cdot)\|^2_{L_2([-\pi/h,\pi/h])}\to\max,\quad \frac{1}{2\pi}\|U_0(\cdot)\|^2_{L_2([-\pi/h,\pi/h])}\le\delta_1^2,\quad \frac{1}{2\pi}\|\Lambda^n(\cdot)U_0(\cdot)\|^2_{L_2([-\pi/h,\pi/h])}\le\delta_2^2. \tag{27}$$

Assume that $\delta_2/\delta_1\in(a^n,1)$. For $\xi\in[0,\pi/h]$ the function $\Lambda(\xi)$ monotonically decreases from 1 to a. Therefore, there will be $\widehat{\xi}\in(0,\pi/h)$ such that $\Lambda^n(\xi)=\delta_2/\delta_1$. For sufficiently small $\varepsilon>0$ put

$$\widehat{U}_0(\xi)=\begin{cases}\sqrt{\dfrac{2\pi}{\varepsilon}}\delta_1, & \xi\in(\widehat{\xi},\widehat{\xi}+\varepsilon),\\ 0, & \xi\notin(\widehat{\xi},\widehat{\xi}+\varepsilon).\end{cases}$$

We have

$$\frac{1}{2\pi}\|\widehat{U}_0(\cdot)\|^2_{L_2([-\pi/h,\pi/h])}=\delta_1^2$$

and

$$\frac{1}{2\pi}\|\Lambda^n(\cdot)\widehat{U}_0(\cdot)\|^2_{L_2([-\pi/h,\pi/h])}=\frac{\delta_1^2}{\varepsilon}\int_{\widehat{\xi}}^{\widehat{\xi}+\varepsilon}\Lambda^{2n}(\xi)\,d\xi\le\delta_1^2\Lambda^{2n}(\widehat{\xi})=\delta_2^2.$$

Thus $\widehat{U}_0(\cdot)$ is admissible in problem (27). Consequently,

$$\sup_{\substack{u_0\in l_{2,h}\\ \|u_0\|^2_{l_{2,h}}\le\delta_1^2\\ \|u_n\|^2_{l_{2,h}}\le\delta_2^2}}\|u_m\|^2_{l_{2,h}}\ge\frac{1}{2\pi}\|\Lambda^m(\cdot)\widehat{U}_0(\cdot)\|^2_{L_2([-\pi/h,\pi/h])}=\frac{\delta_1^2}{\varepsilon}\int_{\widehat{\xi}}^{\widehat{\xi}+\varepsilon}\Lambda^{2m}(\xi)\,d\xi$$
$$=\delta_1^2\Lambda^{2m}(c),$$

where $c\in[\widehat{\xi},\widehat{\xi}+\varepsilon]$. Passing to the limit as $\varepsilon\to 0$, we obtain

$$\sup_{\substack{u_0\in l_{2,h}\\ \|u_0\|^2_{l_{2,h}}\le\delta_1^2\\ \|u_n\|^2_{l_{2,h}}\le\delta_2^2}}\|u_m\|^2_{l_{2,h}}\ge\delta_1^2\Lambda^{2m}(\widehat{\xi})=\delta_1^{2(1-m/n)}\delta_2^{2m/n}=\lambda_1\delta_1^2+\lambda_2\delta_2^2.$$

Assume that $\delta_2/\delta_1\in(0,a^n]$. For sufficiently small $\varepsilon>0$ put

$$\widehat{U}_0(\xi)=\begin{cases}\sqrt{\dfrac{2\pi}{\varepsilon}}\dfrac{\delta_2}{\Lambda^n(\xi)}, & \xi\in(\pi/h-\varepsilon,\pi/h],\\ 0, & \xi\notin(\pi/h-\varepsilon,\pi/h].\end{cases}$$

Then

$$\frac{1}{2\pi}\|\Lambda^n(\cdot)\widehat{U}_0(\cdot)\|^2_{L_2([-\pi/h,\pi/h])}=\delta_2^2$$

and

$$\frac{1}{2\pi}\|\widehat{U}_0(\cdot)\|^2_{L_2([-\pi/h,\pi/h])}=\frac{\delta_2^2}{\varepsilon}\int_{\pi/h-\varepsilon}^{\pi/h}\Lambda^{-2n}(\xi)\,d\xi\le\delta_2^2a^{-2n}\le\delta_1^2.$$

Thus $\widehat{U}_0(\cdot)$ is admissible in problem (27). Consequently,

$$\sup_{\substack{u_0\in l_{2,h}\\ \|u_0\|^2_{l_{2,h}}\le\delta_1^2\\ \|u_n\|^2_{l_{2,h}}\le\delta_2^2}}\|u_m\|^2_{l_{2,h}}\ge\frac{1}{2\pi}\|\Lambda^m(\cdot)\widehat{U}_0(\cdot)\|^2_{L_2([-\pi/h,\pi/h])}=\frac{\delta_2^2}{\varepsilon}\int_{\pi/h-\varepsilon}^{\pi/h}\Lambda^{2(m-n)}(\xi)\,d\xi$$
$$=\delta_2^2\Lambda^{2(m-n)}(c),$$

where $c\in[\pi/h-\varepsilon,\pi/h]$. Passing to the limit as $\varepsilon\to 0$, we obtain

$$\sup_{\substack{u_0\in l_{2,h}\\ \|u_0\|^2_{l_{2,h}}\le\delta_1^2\\ \|u_n\|^2_{l_{2,h}}\le\delta_2^2}}\|u_m\|^2_{l_{2,h}}\ge\delta_2^2a^{2(m-n)}=\lambda_2\delta_2^2.$$

If, finally, $\delta_2/\delta_1\in[1,+\infty)$ for sufficiently small $\varepsilon>0$ we put

$$\widehat{U}_0(\xi)=\begin{cases}\sqrt{\dfrac{2\pi}{\varepsilon}}\delta_1, & \xi\in(0,\varepsilon),\\ 0, & \xi\notin(0,\varepsilon).\end{cases}$$

Then

$$\frac{1}{2\pi}\|\widehat{U}_0(\cdot)\|^2_{L_2([-\pi/h,\pi/h])}=\delta_1^2$$

and

$$\frac{1}{2\pi}\|\Lambda^n(\cdot)\widehat{U}_0(\cdot)\|^2_{L_2([-\pi/h,\pi/h])}=\frac{\delta_1^2}{\varepsilon}\int_0^{\varepsilon}\Lambda^{2n}(\xi)\,d\xi\le\delta_1^2\le\delta_2^2.$$

Thus $\widehat{U}_0(\cdot)$ is admissible in problem (27). Consequently,

$$\sup_{\substack{u_0\in l_{2,h}\\ \|u_0\|^2_{l_{2,h}}\le\delta_1^2\\ \|u_n\|^2_{l_{2,h}}\le\delta_2^2}} \|u_m\|^2_{l_{2,h}} \ge \frac{1}{2\pi}\|\Lambda^m(\cdot)\widehat{U}_0(\cdot)\|^2_{L_2([-\pi/h,\pi/h])} = \frac{\delta_1^2}{\varepsilon}\int_0^\varepsilon \Lambda^{2m}(\xi)\,d\xi$$

$$= \delta_1^2\Lambda^{2m}(c),$$

where $c\in[0,\varepsilon]$. Passing to the limit as $\varepsilon\to 0$, we obtain

$$\sup_{\substack{u_0\in l_{2,h}\\ \|u_0\|^2_{l_{2,h}}\le\delta_1^2\\ \|u_n\|^2_{l_{2,h}}\le\delta_2^2}} \|u_m\|^2_{l_{2,h}} \ge \delta_1^2.$$

Now we concerned with estimate (3). Let $\delta_2/\delta_1\in(a^n,1)$. Define the operators $S_j\colon l_{2,h}\to l_{2,h}$, $j=1,2$, so that

$$F(S_1u)(\cdot)=\Lambda^m(\cdot)(1-\alpha(\cdot))Fu(\cdot),$$
$$F(S_2u)(\cdot)=\Lambda^{m-n}(\cdot)\alpha(\cdot)Fu(\cdot).$$

It is easy to verify that for all $u_0\in l_{2,h}$

$$F((I_0-S_1I_1-S_2I_2)u)(\cdot)\equiv 0.$$

Therefore, $I_0=S_1I_1+S_2I_2$. In view of (25) we get

$$\|S_1z_1+S_2z_2\|^2_{l_{2,h}} = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h}\Lambda^{2m}(\xi)\left|(1-\alpha(\xi)Fz_1(\xi)+\Lambda^{-n}(\xi)\alpha(\xi)Fz_2(\xi)\right|^2\,d\xi.$$

It follows from the Cauchy–Schwartz–Bunyakovskii inequality that

$$\Lambda^{2m}(\xi)\left|(1-\alpha(\xi)Fz_1(\xi)+\Lambda^{-n}\alpha(\xi)Fz_2(\xi)\right|^2\le\Omega(\xi)(\lambda_1|Fz_1(\xi)|^2+\lambda_2|Fz_2(\xi)|^2),$$

where

$$\Omega(\xi)=\Lambda^{2m}(\xi)\left(\frac{|1-\alpha(\xi)|^2}{\lambda_1}+\Lambda^{-2n}(\xi)\frac{|\alpha(\xi)|^2}{\lambda_2}\right).$$

In view of (26) we obtain

$$\|S_1z_1+S_2z_2\|^2_{l_{2,h}}\le\frac{1}{2\pi}\int_{-\pi/h}^{\pi/h}(\lambda_1|Fz_1(\xi)|^2+\lambda_2|Fz_2(\xi)|^2)\,d\xi$$

$$=\lambda_1\|z_1\|^2_{l_{2,h}}+\lambda_2\|z_2\|^2_{l_{2,h}}.$$

It follows from Theorem 1 that in the case under consideration, the methods

$$\widehat{\varphi}(y_1,y_2)=S_1y_1+S_2y_2$$

are optimal and

$$E_\emptyset(I,\delta)=\sqrt{\lambda_1\delta_1^2+\lambda_2\delta_2^2}.$$

Now consider the case when $\delta_2/\delta_1\in(0,a^n]$. Define the operator $S_2\colon l_{2,h}\to l_{2,h}$ so that

$$F(S_2u)(\cdot)=\Lambda^{m-n}(\cdot)Fu(\cdot).$$

Since

$$F((I_0-S_2I_2)u_0)(\xi)\equiv 0,$$

then $I_0 = S_2 I_2$. Moreover,

$$\|S_2 z_2\|^2_{l_{2,h}} = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} \Lambda^{2(m-n)}(\xi)|Fz_2(\xi)|^2\, d\xi \le a^{2(m-n)}\|z_2\|^2_{l_{2,h}}.$$

It follows from Theorem 1 that in the case under consideration, the method

$$\widehat{\varphi}(y_1, y_2) = S_2 y_2$$

is optimal and

$$E_\emptyset(I, \delta) = a^{m-n}\delta_2.$$

Finally, if $\delta_2 \ge \delta_1$ we define the operator $S_1 \colon l_{2,h} \to l_{2,h}$ so that

$$F(S_1 u)(\cdot) = \Lambda^m(\cdot) F u(\cdot).$$

Then $I_0 = S_1 I_1$ and

$$\|S_1 z_1\|^2_{l_{2,h}} = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} \Lambda^{2m}(\xi)|Fz_1(\xi)|^2\, d\xi \le \|z_1\|^2_{l_{2,h}}.$$

It follows from Theorem 1 that the method

$$\widehat{\varphi}(y_1, y_2) = S_1 y_1$$

is optimal and

$$E_\emptyset(I, \delta) = \delta_1.$$

We prove that for $\delta_2/\delta_1 \in (a^n, 1)$ the set of functions $\alpha(\cdot)$, satisfying condition (26) is nonempty. Consider the concave function

$$y = x^{m/n}, \quad x \ge 0. \tag{28}$$

Draw a tangent to the graph of this function at the point $x_0 > 0$. It is easy to verify that the tangent will have the form $y = \widehat{\lambda}_1 + \widehat{\lambda}_2 x$, where

$$\widehat{\lambda}_1 = \left(1 - \frac{m}{n}\right) x_0^{m/n}, \quad \widehat{\lambda}_2 = \frac{m}{n} x_0^{m/n-1}.$$

Due to the concavity of the curve (28), for all $x \ge 0$ the inequality

$$x^{m/n} \le \widehat{\lambda}_1 + \widehat{\lambda}_2 x$$

will hold.

Put

$$x = \Lambda^{2n}(\xi), \quad x_0 = \left(\frac{\delta_2}{\delta_1}\right)^2.$$

Then $\widehat{\lambda}_j = \lambda_j$, $j = 1, 2$, and for all $\xi \in [-\pi/h, \pi/h]$ the inequality

$$\Lambda^{2m}(\xi) \le \lambda_1 + \lambda_2 \Lambda^{2n}(\xi)$$

holds. It follows that

$$\frac{\Lambda^{2m}(\xi)}{\lambda_1 + \lambda_2 \Lambda^{2n}(\xi)} \le 1.$$

Putting

$$\alpha(\xi) = \frac{\lambda_2 \Lambda^{2n}(\xi)}{\lambda_1 + \lambda_2 \Lambda^{2n}(\xi)},$$

we obtain

$$\Lambda^{2m}(\xi)\left(\frac{|1-\alpha(\xi)|^2}{\lambda_1} + \Lambda^{-2n}(\xi)\frac{|\alpha(\xi)|^2}{\lambda_2}\right) = \frac{\Lambda^{2m}(\xi)}{\lambda_1 + \lambda_2 \Lambda^{2n}(\xi)} \le 1.$$

□

If we consider the problem of optimal recovery of the solution at the instant of time $m\tau$ by an inaccurately given solution at the instant of time $n\tau$ on the class

$$W = \{\, u_0 \in l_{2,h} : \|u_0\|_{l_{2,h}} \le \delta_1 \,\},$$

then from the same Theorem 1 it will follow that the methods $\widehat{\varphi}(0, y_2)(\cdot)$ will be optimal.

Note that for a continuous model of heat propagation, the result obtained in [25] for $t_1 = 0$, $t_2 = T$ $(n = 2)$ and the intermediate point τ_0, in which it is required to recover the temperature distribution, in the one-dimensional case will coincide with the limiting error of recovery and one of the methods constructed in Theorem 5 for $h \to 0$ and $\tau \to 0$ (in this case, we must put $a = 0$).

We also note that a problem similar to the one considered when the process of heat propagation occurs on a circle was considered in [22].

References

[1] S. A. Smolyak, On optimal recovery of functions and functionals of them, Diss.... Cand. Phys.-Math. Sci., Moscow State Univ., Moscow, 1965. (Russian)

[2] S. M. Nikol'skii, "On the question of estimates for approximation by quadrature formulae", *Uspekhi Mat. Nauk* **5**:2(36) (1950), 165–177. (Russian)

[3] C. A. Micchelli and T. J. Rivlin, "A survey of optimal recovery", *Optimal estimation in approximation theory*, Proc. Internat. Sympos. (Freudenstadt 1976), Plenum, New York 1977, pp. 1–54.

[4] A. A. Melkman and C. A. Micchelli, "Optimal estimation of linear operators in Hilbert spaces from inaccurate data" *SIAM J. Numer. Anal.*, **16** (1979), 87–105.

[5] J. F. Traub and H. Woźniakowski, *A general theory of optimal algorithms*, ACM MMonograph Series, Academic Press, New York–London 1980; Russian transl., Mir, Moscow 1983.

[6] V. V. Arestov, "Optimal recovery of operators and related problems", *Proc. All-Union School in Function Theory* (Dushanbe, August 1986), Trudy Mat. Inst. Steklov., vol. 189, Nauka, Moscow 1989, pp. 3–20; English transl., *Proc. Steklov Inst. Math.* **189**:4 (1990), 1–20.

[7] G. G. Magaril-Il'yaev and K. Yu. Osipenko, "Optimal recovery of functionals based on inaccurate data", *Mat. Zametki* **50**:6 (1991), 85–93; English transl., *Math. Notes* **50**:6 (1991), 1274–1279.

[8] L. Plaskota, *Noisy information and computational complexity*, Cambridge University Press, Cambridge 1996.

[9] K. Yu. Osipenko, *Optimal recovery of analytic functions*, Nova Science Publ., Inc., Huntington, New York 2000.

[10] G. G. Magaril-Il'yaev and V. M. Tikhomirov, *Convex analysis: theory and applications*, Editorial URSS, Moscow 2011; English transl. of 1st ed., Transl. Math. Monogr., vol. 222, Amer. Math. Soc., Providence, RI 2003.

[11] G. G. Magaril-Il'yaev and K. Yu. Osipenko, "Optimal recovery of functions and their derivatives from Fourier coefficients prescribed with an error", *Mat. Sb.* **193**:3 (2002), 79–100; English transl., *Sb. Math.* **193**:3 (2002), 387–407.

[12] G. G. Magaril-Il'yaev and K. Yu. Osipenko, "Optimal recovery of functions and their derivatives from inaccurate information about a spectrum and inequalities for derivatives", *Funktsional. Anal. l Prilozhen.* **37**:3 (2003), 51–64; English transl., *Funct. Anal. Appl.* **37**:3 (2003), 203–214.

[13] K. Yu. Osipenko, "The Hardy–Littlewood–Pólya inequality for analytic functions from Hardy–Sobolev spaces", , *Mat. Sb.* **197**:3 (2006), 15–34; English transl., *Sb. Math.* **197**:3 (2006), 315–334.

[14] K. Yu. Osipenko, "Optimal recovery of linear operators in non-Euclidean metrics", *Mat. Sb.* **205**:10 (2014), 77–106; English transl., *Sb. Math.* **205**:10 (2014), 1442–1472.

[15] K. Yu. Osipenko, "Optimal recovery of operators and multidimensional Carlson type inequalities", *J. Complexity* **32**:1 (2016), 53–73.

[16] V. V. Arestov, “Best uniform approximation of the differentiation operator by operators bounded in the space L_2”, *Tr. Inst. Mat. Mekh.* **24**:4 (2018), 34–56; English transl., *Proc. Steklov Inst. Math.* **308**:1 (2030), 9–30.
[17] V. V. Arestov, “Best approximation of a differentiation operator on the set of smooth functions with exactly or approximately given Fourier transform”, *Mathematical Optimization Theory and Operation Research. MOTOR 2019*. Lecture Notes in Comput. Sci., vol. 11548, Springer, Cham 2019, pp. 434–448.
[18] V. V. Arestov, “Uniform approximation of differentiation operators by bounded linear operators in the space L_r”, *Anal. Math.*, **46**:3 (2020), 425–445.
[19] K. Yu. Osipenko, “Optimal recovery in weighted spaces with homogeneous weights”, *Mat. Sb.* **213**:3 (2022), 111–138; English transl., *Sb. Math.* **213**:3 (2022), 385–411.
[20] G. G. Magaril-Il’yaev and K. Yu. Osipenko, “On optimal harmonic synthesis from inaccurate spectral data”, *Funktsional. Anal. l Prilozhen.* **44**:3 (2010), 76–79; English transl., *Funct. Anal. Appl.* **44**:3 (2010), 223–225.
[21] G. G. Magaril-Il’yaev and K. Yu. Osipenko, “The Hardy–Littlewood–Pólya inequality and the reconstruction of derivatives from inaccurate data”, *Dokl. Ross. Akad. Nauk* **438**:3 (2011), 300–302; English transl., *Dokl. Math.* **83**:3 (2011), 337–339.
[22] G. G. Magaril-Il’yaev and K. Yu. Osipenko, “On optimal recovery of solutions to difference equations from inaccurate data”, *Probl. Mat. Anal.* **69** (2013), 47–54; English transl., *J. Math. Sci., New York* **189**:4 (2013), 596–603.
[23] G. G. Magaril-Il’yaev and V. M. Tikhomirov, “Kolmogorov-type inequalities for derivatives”, *Mat. Sb.* **188**:12 (1997), 73–106; English transl., *Sb. Math.* **188**:12 (1997), 1799–1832.
[24] E. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, NJ 1971; Russian transl., Mir, Moscow 1974.
[25] G. G. Magaril-Il’yaev and K. Yu. Osipenko, “Optimal recovery of the solution of the heat equation from inaccurate data”, *Mat. Sb.* **200**:5 (2009), 37–54; English transl., *Sb. Math.* **200**:5 (2009), 665–682.

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