ON OPTIMAL HARMONIC SYNTHESIS FROM INACCURATE SPECTRUM

G. G. MAGARIL-IL’YAEV, K. YU. OSIPENKO

Abstract. The problem of optimal recovery of function derivatives on the real line from the Fourier transform of these functions given inaccurately on a finite interval is studied. Optimal recovery methods differing by various methods of input data handling are constructed.

The paper is devoted to the construction of a collection of optimal methods for recovery of function derivatives from the Fourier transform of these functions given inaccurately on a finite interval. The precise setting of the problem is the following. Let \( n \in \mathbb{N} \) and \( W^n_2(\mathbb{R}) \) be the Sobolev class of functions \( x(\cdot) \in L^2(\mathbb{R}) \) for which the \((n-1)\)-st derivative is locally absolutely continuous and \( \|x^{(n)}(\cdot)\|_{L^2(\mathbb{R})} \leq 1 \). Further, let \( \sigma > 0 \), \( \Delta_\sigma = [\sigma, \sigma] \), \( 1 \leq k < n \), and \( \delta > 0 \). Assume that it is known the Fourier transform \( Fx(\cdot) \) of \( x(\cdot) \in W^n_2(\mathbb{R}) \) given on \( \Delta_\sigma \) with an error \( \delta \) in \( L^2(\Delta_\sigma) \)-metric, that is, a function \( y(\cdot) \in L^2(\Delta_\sigma) \) is known such that \( \|Fx(\cdot) - y(\cdot)\|_{L^2(\Delta_\sigma)} \leq \delta \). How one can use this information to recover the \( k \)-th derivative of \( x(\cdot) \) in \( L^2(\mathbb{R}) \)-metric in the best way?

Any map \( \varphi : L^2(\Delta_\sigma) \to L^2(\mathbb{R}) \) we consider as a method of recovery. The error of the method \( \varphi \) is the value

\[
e(\varphi) = \sup_{x(\cdot) \in W^n_2(\mathbb{R}), \; y \in L^2(\Delta_\sigma), \|Fx(\cdot) - y(\cdot)\|_{L^2(\Delta_\sigma)} \leq \delta} \|x^{(k)}(\cdot) - \varphi(y(\cdot))(\cdot)\|_{L^2(\mathbb{R})}.
\]

We are interested in the value

\[
E = \inf_{m : L^2(\Delta_\sigma) \to L^2(\mathbb{R})} e(\varphi),
\]

which is called the error of optimal recovery and in a method for which the lower bound is delivering which is called an optimal recovery method.

Theorem. Let \( k, n \) be integer, \( 1 \leq k < n \), \( \sigma > 0 \), \( \delta > 0 \),

\[
\hat{\sigma} = \left( \frac{n}{k} \right)^{\frac{1}{2(n-k)}} \frac{2\pi}{(\delta^2)^{\frac{1}{2n}}},
\]

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and $\sigma_0 = \min\{\sigma, \sigma_0\}$. Then
\begin{equation}
E = \sigma_0^k \left( \frac{n - k}{2\pi n} \right) \frac{k}{n} \frac{n - k}{\sigma_0^2} \delta^2 + \sigma_0^{2(k-n)}.
\end{equation}

For all bounded functions $m(\cdot)$ such that
\begin{equation}
|m(\xi) - (i\xi)^k \alpha(\xi)| \leq |\xi|\kappa \sqrt{\alpha^2(\xi) + \alpha(\xi) \left( \frac{\xi}{\sigma_0^2} \right)^{2(n-k)}} - 1,
\end{equation}
where
\begin{equation}
\alpha(\xi) = \left( 1 + \frac{n}{n - k} \left( \frac{k}{n} \right) \frac{n - k}{\sigma_0} \left( \frac{\xi}{\sigma_0^2} \right) \right)^{\frac{1}{2(n-k)}},
\end{equation}
methods
\begin{equation}
\varphi(y(\cdot))(t) = \frac{1}{2\pi} \int_{\Delta_\sigma} m(\xi) y(\xi) e^{i\xi t} d\xi + \frac{1}{2\pi} \int_{\sigma' \leq |\xi| \leq \sigma_0} (i\xi)^k \alpha(\xi) y(\xi) e^{i\xi t} d\xi
\end{equation}
are optimal.

**Corollary.** For all $\sigma'$,
\begin{equation}
0 \leq \sigma' \leq \left( \frac{n - k}{\sigma_0} \right)^{\frac{1}{2(n-k)}} \frac{k}{n} \sigma_0,
\end{equation}
methods
\begin{equation}
\varphi(y(\cdot))(t) = \frac{1}{2\pi} \int_{\xi \leq \sigma'} (i\xi)^k y(\xi) e^{i\xi t} d\xi + \frac{1}{2\pi} \int_{\sigma' \leq |\xi| \leq \sigma_0} (i\xi)^k \alpha(\xi) y(\xi) e^{i\xi t} d\xi
\end{equation}
are optimal.

Optimal methods (4) are characterized by the fact that inaccurate input data is “filtering” on the interval $\sigma' \leq |\xi| \leq \sigma_0$ by the function $\alpha(\cdot)$ and on the interval $|\xi| \leq \sigma'$ there is no filtering.

**Proof of Theorem.** An easy estimate shows that $E^2$ is not less than the value of the following problem
\begin{equation}
\|x^{(k)}(\cdot)\|_{L^2(\mathbb{R})}^2 \rightarrow \max, \quad \|F x(\cdot)\|_{L^2(\Delta_\sigma)}^2 \leq \delta^2, \quad \|x^{(n)}(\cdot)\|_{L^2(\mathbb{R})}^2 \leq 1,
\end{equation}
or passing to Fourier transforms
\begin{equation}
\frac{1}{2\pi} \int_{\mathbb{R}} \xi^{2k} |F x(\xi)|^2 d\xi \rightarrow \max, \quad \int_{\Delta_\sigma} |F x(\xi)|^2 d\xi \leq \delta^2,
\end{equation}
\begin{equation}
\frac{1}{2\pi} \int_{\mathbb{R}} \xi^{2n} |F x(\xi)|^2 d\xi \leq 1.
\end{equation}
The value of this problem (that is, the upper bound of the maximizing functional) may be found using duality arguments since it is a linear programming problem with respect to the variable $|F x(\cdot)|^2$ (see [5]). It
is equal to the square of the value in the right hand side of (1) what gives the lower bound for \( E \).

If \( \varphi \) is any method, then by definition the value \( e^2(\varphi) \) equals the value of the problem

\[
\begin{align*}
(5) \quad & \|x^{(k)}(\cdot) - \varphi(y(\cdot))(\cdot)\|^2_{L_2(\mathbb{R})} \to \max, \quad \|Fx(\cdot) - y(\cdot)\|^2_{L_2(\Delta_\sigma)} \leq \delta^2, \\
& y(\cdot) \in L_2(\Delta_\sigma), \quad \|x^{(n)}(\cdot)\|^2_{L_2(\mathbb{R})} \leq 1.
\end{align*}
\]

We show that if the method \( \varphi \) under the Fourier transform is multiplication by the bounded function \( m(\cdot) \) satisfying (2), then the value of problem (5) is not less than \( E^2 \) and thus \( \varphi \) is an optimal method.

So let \( \varphi \) be a method of the mentioned type. Then passing to Fourier transforms denoting by \( h = (2\pi)^{-1} \int_{|t| > \sigma}^{2n}|Fx(\xi)|^2 \, d\xi \), \( z(\xi) = Fx(\xi) - y(\xi) \), and \( b(\xi) = (i\xi)^k - m(\xi) \) and taking into account that

\[
\frac{1}{2\pi} \int_{|t| > \sigma}^{2n}|Fx(\xi)|^2 \, d\xi \leq \sigma^{2(k-n)} \frac{1}{2\pi} \int_{|t| > \sigma}^{2n}|Fx(\xi)|^2 \, d\xi = h\sigma^{2(k-n)},
\]

problem (5) may be rewritten in the form

\[
\begin{align*}
(6) \quad & \frac{1}{2\pi} \int_{\Delta_\sigma} \left| m(\xi)z(\xi) + b(\xi)Fx(\xi) \right|^2 \, d\xi + h\sigma^{2(k-n)} \to \max, \\
& \int_{\Delta_\sigma} |z(\xi)|^2 \, d\xi \leq \delta^2, \quad \frac{1}{2\pi} \int_{\Delta_\sigma} \xi^{2n}|Fx(\xi)|^2 \, d\xi + h \leq 1.
\end{align*}
\]

From the Cauchy-Schwarz-Bunyakovskii inequality \( |m(\xi)z(\xi) + b(\xi)Fx(\xi)|^2 \leq (s^{-1}|a(\xi)|^2 + 2\pi\xi^{-2n}|b(\xi)|^2)|z(\xi)|^2 + (2\pi)^{-1}\xi^{2n}|Fx(\xi)|^2) \) which is holding for all \( s > 0 \) it follows that the value of problem (6) is estimated by the value

\[
\begin{align*}
(7) \quad & \max_{h \in [0,1]} \left( A_s(s\delta^2 + 1 - h) + h\sigma^{2(k-n)} \right) = A_s s\delta^2 + \max(A_s, \sigma^{2(k-n)})
\end{align*}
\]

where

\[
A_s = A_s(m(\cdot)) = \frac{1}{2\pi} \sup_{\xi \in \Delta_\sigma} \left( \frac{|m(\xi)|^2}{s} + \frac{2\pi}{\xi^{2n}}|b(\xi)|^2 \right) = \frac{1}{2\pi} \sup_{\xi \in \Delta_\sigma} \left( \frac{\xi^{2n} + 2\pi s}{s^{2n}} \left| m(\xi) - \frac{2\pi s(i\xi)^k}{\xi^{2n} + 2\pi s} \right|^2 + \frac{2\pi \xi^{2k}}{\xi^{2n} + 2\pi s} \right).
\]

If \( m(\xi) = \hat{m}(\xi) = 2\pi s(i\xi)^k/(\xi^{2n} + 2\pi s) \), then it is easy to obtain that

\[
A_s(\hat{m}(\cdot)) = \begin{cases} 
\frac{k}{n} \left( \frac{n - k}{2\pi k s} \right)^{1-k/n}, & s \leq \frac{n - k}{2\pi k} \sigma^{2n}, \\
\frac{\sigma^{2k}}{\sigma^{2n} + 2\pi s}, & s > \frac{n - k}{2\pi k} \sigma^{2n}.
\end{cases}
\]
Denoting by $\hat{\phi}$ the method corresponding to the function $\hat{m}(\cdot)$, from here and from (7) we obtain that for all $s > 0$

$$e^2(\hat{\phi}) \leq \begin{cases} A_s(\hat{m}(\cdot))(s\delta^2 + 1), & s \leq \frac{n - k}{2\pi k} \left(\frac{k}{n}\right)^{\frac{n}{2\pi}} \sigma^{2n}, \\ A_s(\hat{m}(\cdot))s\delta^2 + \sigma^2(n-k), & s > \frac{n - k}{2\pi k} \left(\frac{k}{n}\right)^{\frac{n}{2\pi}} \sigma^{2n}. \end{cases}$$

The minimum of the value in the right hand side over $s$ coincides with the square of the value in the right hand side of (1) and is delivered at the point

$$\hat{s} = \frac{n - k}{2\pi k} \left(\frac{k}{n}\right)^{\frac{n}{2\pi}} \sigma^{2n}.$$

Moreover, $A_{\hat{s}}(\hat{m}(\cdot)) = \sigma_0^{2(k-n)}$. Thus, if the bounded function $m(\cdot)$ is such that the equality $A_{\hat{s}}(m(\cdot)) = \sigma_0^{2(k-n)}$ holds, then the corresponding method is optimal. But it is easy to verify that this equality is equivalent to condition (2). 

Corollary immediately follows from Theorem which has been proved.

First results concerning optimal recovery of linear operators from inaccurate information were obtained in [1]. Further development of this subject was given in the papers of authors [2], [3], and [4].

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REFERENCES


