ON OPTIMAL HARMONIC SYNTHESIS FROM INACCURATE SPECTRUM

G. G. MAGARIL-IL'YAEV, K. YU. OSIPENKO

Abstract. The problem of optimal recovery of function derivatives on the real line from the Fourier transform of these functions given inaccurately on a finite interval is studied. Optimal recovery methods differing by various methods of input data handling are constructed

The paper is devoted to the construction of a collection of optimal methods for recovery of function derivatives from the Fourier transform of these functions given inaccurately on a finite interval. The precise setting of the problem is the following. Let $n \in \mathbb{N}$ and $W_2^n(\mathbb{R})$ be the Sobolev class of functions $x(\cdot) \in L_2(\mathbb{R})$ for which the $(n-1)$-st derivative is locally absolutely continuous and $\|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})} \le 1$. Further, let $\sigma > 0$, $\Delta_\sigma = [-\sigma, \sigma]$, $1 \le k < n$, and $\delta > 0$. Assume that it is known the Fourier transform $Fx(\cdot)$ of $x(\cdot) \in W_2^n(\mathbb{R})$ given on Δ_σ with an error δ in $L_2(\Delta_\sigma)$-metric, that is, a function $y(\cdot) \in L_2(\Delta_\sigma)$ is known such that $\|Fx(\cdot) - y(\cdot)\|_{L_2(\Delta_\sigma)} \le \delta$. How one can use this information to recover the k-th derivative of $x(\cdot)$ in $L_2(\mathbb{R})$-metric in the best way?

Any map $\varphi\colon L_2(\Delta_\sigma) \to L_2(\mathbb{R})$ we consider as a method of recovery. The error of the method φ is the value

$$e(\varphi) = \sup_{\substack{x(\cdot)\in W_2^n(\mathbb{R}),\ y\in L_2(\Delta_\sigma)\\ \|Fx(\cdot)-y(\cdot)\|_{L_2(\Delta_\sigma)}\le\delta}} \|x^{(k)}(\cdot) - \varphi(y(\cdot))(\cdot)\|_{L_2(\mathbb{R})}.$$

We are interested in the value

$$E = \inf_{m\colon L_2(\Delta_\sigma)\to L_2(\mathbb{R})} e(\varphi),$$

which is called the *error of optimal recovery* and in a method for which the lower bound is delivering which is called an *optimal recovery method.*

Theorem. *Let k, n be integer, $1 \le k < n$, $\sigma > 0$, $\delta > 0$,*

$$\widehat{\sigma} = \left(\frac{n}{k}\right)^{\frac{1}{2(n-k)}} \left(\frac{2\pi}{\delta^2}\right)^{\frac{1}{2n}},$$

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and $\sigma_0 = \min\{\sigma, \widehat{\sigma}\}$. *Then*

$$E = \sigma_0^k \sqrt{\frac{n-k}{2\pi n}\left(\frac{k}{n}\right)^{\frac{k}{n-k}}\delta^2 + \sigma_0^{2(k-n)}}. \tag{1}$$

For all bounded functions $m(\cdot)$ *such that*

$$|m(\xi) - (i\xi)^k\alpha(\xi)| \le |\xi|^k\sqrt{\alpha^2(\xi) + \alpha(\xi)\left(\left(\frac{\xi}{\sigma_0}\right)^{2(n-k)} - 1\right)}, \tag{2}$$

where

$$\alpha(\xi) = \left(1 + \frac{n}{n-k}\left(\frac{n}{k}\right)^{\frac{k}{n-k}}\left(\frac{\xi}{\sigma_0}\right)^{2n}\right)^{-1},$$

methods

$$\varphi(y(\cdot))(t) = \frac{1}{2\pi}\int_{\Delta_\sigma} m(\xi)y(\xi)e^{i\xi t}\,d\xi \tag{3}$$

are optimal.

Corollary. *For all* σ',

$$0 \le \sigma' \le \left(\frac{n-k}{n}\right)^{\frac{1}{2k}}\left(\frac{k}{n}\right)^{\frac{1}{2(n-k)}}\sigma_0,$$

methods

(4)

$$\varphi(y(\cdot))(t) = \frac{1}{2\pi}\int_{|\xi|\le\sigma'}(i\xi)^k y(\xi)e^{i\xi t}\,d\xi + \frac{1}{2\pi}\int_{\sigma'\le|\xi|\le\sigma_0}(i\xi)^k\alpha(\xi)y(\xi)e^{i\xi t}\,d\xi$$

are optimal.

Optimal methods (4) are characterized by the fact that inaccurate input data is "filtering" on the interval $\sigma' \le |\xi| \le \sigma_0$ by the function $\alpha(\cdot)$ and on the interval $|\xi| \le \sigma'$ there is no filtering.

Proof of Theorem. An easy estimate shows that E^2 is not less than the value of the following problem

$$\|x^{(k)}(\cdot)\|^2_{L_2(\mathbb{R})} \to \max, \quad \|Fx(\cdot)\|^2_{L_2(\Delta_\sigma)} \le \delta^2, \quad \|x^{(n)}(\cdot)\|^2_{L_2(\mathbb{R})} \le 1,$$

or passing to Fourier transforms

$$\frac{1}{2\pi}\int_{\mathbb{R}}\xi^{2k}|Fx(\xi)|^2\,d\xi \to \max, \quad \int_{\Delta_\sigma}|Fx(\xi)|^2\,d\xi \le \delta^2,$$
$$\frac{1}{2\pi}\int_{\mathbb{R}}\xi^{2n}|Fx(\xi)|^2\,d\xi \le 1.$$

The value of this problem (that is, the upper bound of the maximizing functional) may be found using duality arguments since it is a linear programming problem with respect to the variable $|Fx(\cdot)|^2$ (see [5]). It is equal to the square of the value in the right hand side of (1) what gives the lower bound for E.

If φ is any method, then by definition the value $e^2(\varphi)$ equals the value of the problem

$$(5)\quad \|x^{(k)}(\cdot)-\varphi(y(\cdot))(\cdot)\|^2_{L_2(\mathbb{R})}\to\max,\quad \|Fx(\cdot)-y(\cdot)\|^2_{L_2(\Delta_\sigma)}\le\delta^2,$$
$$y(\cdot)\in L_2(\Delta_\sigma),\quad \|x^{(n)}(\cdot)\|^2_{L_2(\mathbb{R})}\le 1.$$

We show that if the method φ under the Fourier transform is multiplication by the bounded function $m(\cdot)$ satisfying (2), then the value of problem (5) is not less than E^2 and thus φ is an optimal method.

So let φ be a method of the mentioned type. Then passing to Fourier transforms denoting by $h=(2\pi)^{-1}\int_{|t|>\sigma}\xi^{2n}|Fx(\xi)|^2\,d\xi$, $z(\xi)=Fx(\xi)-y(\xi)$, and $b(\xi)=(i\xi)^k-m(\xi)$ and taking into account that

$$\frac{1}{2\pi}\int_{|t|>\sigma}\xi^{2k}|Fx(\xi)|^2\,d\xi\le\sigma^{2(k-n)}\frac{1}{2\pi}\int_{|t|>\sigma}\xi^{2n}|Fx(\xi)|^2\,d\xi=h\sigma^{2(k-n)},$$

problem (5) may be rewritten in the form

$$(6)\quad \frac{1}{2\pi}\int_{\Delta_\sigma}|m(\xi)z(\xi)+b(\xi)Fx(\xi)|^2\,d\xi+h\sigma^{2(k-n)}\to\max,$$
$$\int_{\Delta_\sigma}|z(\xi)|^2\,d\xi\le\delta^2,\quad \frac{1}{2\pi}\int_{\Delta_\sigma}\xi^{2n}|Fx(\xi)|^2\,d\xi+h\le 1.$$

From the Cauchy-Schwarz-Bunyakovskii inequality $|m(\xi)z(\xi)+b(\xi)Fx(\xi)|^2\le(s^{-1}|a(\xi)|^2+2\pi\xi^{-2n}|b(\xi)|^2)(s|z(\xi)|^2+(2\pi)^{-1}\xi^{2n}|Fx(\xi)|^2)$ which is holding for all $s>0$ it follows that the value of problem (6) is estimated by the value

$$(7)\quad \max_{h\in[0,1]}\left(A_s(s\delta^2+1-h)+h\sigma^{2(k-n)}\right)=A_ss\delta^2+\max(A_s,\sigma^{2(k-n)})$$

where

$$A_s=A_s(m(\cdot))=\frac{1}{2\pi}\sup_{\xi\in\Delta_\sigma}\left(\frac{|m(\xi)|^2}{s}+\frac{2\pi}{\xi^{2n}}|b(\xi)|^2\right)=$$
$$=\frac{1}{2\pi}\sup_{\xi\in\Delta_\sigma}\left(\frac{\xi^{2n}+2\pi s}{s\xi^{2n}}\left|m(\xi)-\frac{2\pi s(i\xi)^k}{\xi^{2n}+2\pi s}\right|^2+\frac{2\pi\xi^{2k}}{\xi^{2n}+2\pi s}\right).$$

If $m(\xi)=\widehat{m}(\xi)=2\pi s(i\xi)^k/(\xi^{2n}+2\pi s)$, then it is easy to obtain that

$$A_s(\widehat{m}(\cdot))=\begin{cases}\dfrac{k}{n}\left(\dfrac{n-k}{2\pi ks}\right)^{1-k/n}, & s\le\dfrac{n-k}{2\pi k}\sigma^{2n},\\[2ex] \dfrac{\sigma^{2k}}{\sigma^{2n}+2\pi s}, & s>\dfrac{n-k}{2\pi k}\sigma^{2n}.\end{cases}$$

Denoting by $\widehat{\varphi}$ the method corresponding to the function $\widehat{m}(\cdot)$, from here and from (7) we obtain that for all $s > 0$

$$e^2(\widehat{\varphi}) \le \begin{cases} A_s(\widehat{m}(\cdot))(s\delta^2 + 1), & s \le \dfrac{n-k}{2\pi k}\left(\dfrac{k}{n}\right)^{\frac{n}{n-k}} \sigma^{2n}, \\ A_s(\widehat{m}(\cdot))s\delta^2 + \sigma^{2(n-k)}, & s > \dfrac{n-k}{2\pi k}\left(\dfrac{k}{n}\right)^{\frac{n}{n-k}} \sigma^{2n}. \end{cases}$$

The minimum of the value in the right hand side over s coincides with the square of the value in the right hand side of (1) and is delivered at the point

$$\widehat{s} = \frac{n-k}{2\pi k}\left(\frac{k}{n}\right)^{\frac{n}{n-k}} \sigma_0^{2n}.$$

Moreover, $A_{\widehat{s}}(\widehat{m}(\cdot)) = \sigma_0^{2(k-n)}$. Thus, if the bounded function $m(\cdot)$ is such that the equality $A_{\widehat{s}}(m(\cdot)) = \sigma_0^{2(k-n)}$ holds, then the corresponding method is optimal. But it is easy to verify that this equality is equivalent to condition (2). $\square$

Corollary immediately follows from Theorem which has been proved.

First results concerning optimal recovery of linear operators from inaccurate information were obtained in [1]. Further development of this subject was given in the papers of authors [2], [3], and [4].

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Moscow State Institute of Radio Engineering, Electronics and Automation (Technology University), Southern Mathematical Institute, Vladikavkaz Research Center, Russian Academy of Sciences

MATI — Russian State Technological University, Southern Mathematical Institute, Vladikavkaz Research Center, Russian Academy of Sciences