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On the Reconstruction of Convolution-Type Operators from Inaccurate Information

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Abstract—We address the problem of optimal reconstruction of the values of a linear operator on $\mathbb{R}^d$ or $\mathbb{Z}^d$ from approximate values of other operators. Each operator acts as the multiplication of the Fourier transform by a certain function. As an application, we present explicit expressions for optimal methods of reconstructing the solution of the heat equation (for continuous and difference models) at a given instant of time from inaccurate measurements of this solution at other time instants.

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1. STATEMENT OF THE PROBLEM AND FORMULATION OF THE MAIN RESULT

Let $T=\mathbb{R}^d$ or $\mathbb{Z}^d$, where $\mathbb{R}$ and $\mathbb{Z}$ are the sets of real and integer numbers, respectively, and d is a positive integer. Let $\widehat{T}=\mathbb{R}^d$ if $T=\mathbb{R}^d$ and $\widehat{T}=\mathbb{T}^d$ ($\mathbb{T}$ is the unit circle) if $T=\mathbb{Z}^d$.

Let $\alpha(\cdot)$ be a continuous (in general, complex-valued) function on $\widehat{T}$, $R>0$, and $F\colon L_2(T)\to L_2(\widehat{T})$ be the Fourier transform. Set

$$X_\alpha^R(T)=\bigl\{x(\cdot)\in L_2(T)\bigm|\alpha^r(\cdot)Fx(\cdot)\in L_2(\widehat{T})\ \forall\, r\in[0,R]\bigr\}.^1$$

For every $r\in[0,R]$, define an operator $A_r\colon X_\alpha^R(T)\to L_2(T)$ by the rule

$$A_rx(\cdot)=F^{-1}\bigl(\alpha^r(\cdot)Fx(\cdot)\bigr)(\cdot),$$

where $F^{-1}\colon L_2(\widehat{T})\to L_2(T)$ is the inverse Fourier transform.

In terms of generalized functions, such an operator can always be expressed as a convolution with some kernel.

Natural problems related to the reconstruction of functions and their derivatives, the solutions of differential equations, etc., can be reduced to the reconstruction of this type of operators. Consider two simple examples. Let $T=\mathbb{R}$ and $\alpha(\xi)=i\xi$. Then $A_rx(\cdot)$ is the rth Weyl (fractional) derivative. If $\alpha(\xi)=e^{-\xi^2}$, then $A_rx(\cdot)$ is the temperature distribution in an infinite rod at time r for the initial distribution $x(\cdot)$.

We address the following problem: reconstruct the values of an operator A_{r_0} from approximate values of operators $A_{r_1},\dots,A_{r_n}$, $r_j\in[0,R]$, $j=0,1,\dots,n$ (in terms of the examples considered, this is the reconstruction of a function and/or its derivatives from approximate values of other derivatives and the reconstruction of the rod temperature at a given instant of time from its approximate measurements at other time instants).

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[1] Here $\alpha^r(\xi)=|\alpha(\xi)|^r\exp(ir\arg\alpha(\xi))$, where arg is the principal value of the argument.

A precise statement is as follows. Suppose given functions $y_j(\cdot) \in L_2(T)$, $j = 1, \dots, n$, such that

$$\|A_{r_j}x(\cdot) - y_j(\cdot)\|_{L_2(T)} \le \delta_j, \qquad j = 1, \dots, n, \quad 0 \le r_1 < \dots < r_n \le R,$$

and $\delta_j > 0$, $j = 1, \dots, n$. By the optimal reconstruction of the values of A_{r_0} from given information we mean the following. Any mapping $m\colon (L_2(T))^n \to L_2(T)$ is declared a reconstruction method. The error of a method m is the quantity

$$e_{r_0}(\overline{r}, \overline{\delta}, m) = \sup_{\substack{x(\cdot), y_1(\cdot), \dots, y_n(\cdot) \in L_2(T) \\ \|A_{r_j}x(\cdot) - y_j(\cdot)\|_{L_2(T)} \le \delta_j,\ j=1,\dots,n}} \|A_{r_0}x(\cdot) - m(\overline{y}(\cdot))(\cdot)\|_{L_2(T)};$$

here $\overline{r} = (r_1, \dots, r_n)$, $\overline{\delta} = (\delta_1, \dots, \delta_n)$, and $\overline{y}(\cdot) = (y_1(\cdot), \dots, y_n(\cdot))$. We are interested in the quantity

$$E_{r_0}(\overline{r}, \overline{\delta}) = \inf_{m\colon (L_2(T))^n \to L_2(T)} e_{r_0}(\overline{r}, \overline{\delta}, m),$$

which is called the *error of optimal reconstruction*, and in the method $\widehat{m}$ for which the lower bound is attained, which is called the *optimal reconstruction method*.

To formulate the main result, we need some definitions. We will say that a continuous function $\alpha(\cdot)$ on $\widehat{T}$ satisfies condition $\mathcal{A}$ if

(i) the infimum $a = \inf_{t \in \widehat{T}} |\alpha(t)|$ is attained on $\widehat{T}$ if $a > 0$, and the supremum $b = \sup_{t \in \widehat{T}} |\alpha(t)|$ is attained on $\widehat{T}$ if $b < \infty$;

(ii) the set of functions $y(\cdot) \in L_2(T)$ such that

$$F^{-1}\biggl(\frac{\alpha^{r_1}(\cdot)}{|\alpha(\cdot)|^{2r_1} + |\alpha(\cdot)|^{2r_2}} Fy(\cdot)\biggr)(\cdot) \in X_\alpha^R(T)$$

for all $r_1, r_2 \in [0, R]$ is dense in $L_2(T)$.

Consider the following set on the plane (t, x):

$$M = \operatorname{co}\bigl\{(r_j, \ln(1/\delta_j)),\ 1 \le j \le n\bigr\} + \bigl\{(t, t\ln(1/a)) \mid t \le 0\bigr\} + \bigl\{(t, t\ln(1/b)) \mid t \ge 0\bigr\},$$

where co denotes the convex hull of the set and the second (third) term should be omitted if $a = 0$ ($b = \infty$). Define a function $\theta(\cdot)$ on $[0, R]$ by the rule $\theta(t) = \max\{x \mid (t, x) \in M\}$ and $\theta(t) = -\infty$ if $(t, x) \notin M$ for any x. It is clear that $\theta(\cdot)$ is a concave polygonal curve on $[r_1, r_n]$. Let $r_{s_1} < \dots < r_{s_k}$ be its salient points (see the figure for $a > 0$ and $b < \infty$).

Theorem 1. *Let $\alpha(\cdot)$ be a continuous function on $\widehat{T}$ that satisfies condition $\mathcal{A}$. Then*

$$E_{r_0}(\overline{r}, \overline{\delta}) = e^{-\theta(r_0)}.$$

If $r_0 \in [r_{s_j}, r_{s_{j+1}}]$, $1 \le j \le k-1$, then the method

$$\widehat{m}(\overline{y}(\cdot))(\cdot) = F^{-1}\Bigl(\alpha^{r_0 - r_{s_j}}(\cdot)\beta_j(\cdot)Fy_{s_j}(\cdot) + \alpha^{r_0 - r_{s_{j+1}}}(\cdot)(1 - \beta_j(\cdot))Fy_{s_{j+1}}(\cdot)\Bigr)(\cdot),$$

where

$$\beta_j(\cdot) = \frac{(r_{s_{j+1}} - r_0)\delta_{s_{j+1}}^2 |\alpha(\cdot)|^{2r_{s_j}}}{(r_{s_{j+1}} - r_0)\delta_{s_{j+1}}^2 |\alpha(\cdot)|^{2r_{s_j}} + (r_0 - r_{s_j})\delta_{s_j}^2 |\alpha(\cdot)|^{2r_{s_{j+1}}}},$$

is optimal.

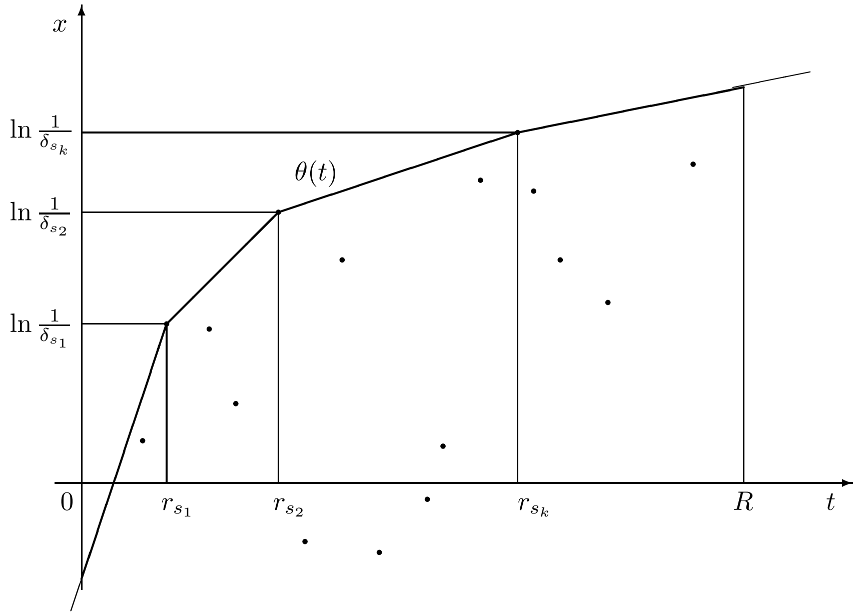

Figure.

If $a > 0$ *and* $0 \le r_0 < r_{s_1}$, *then the method*

$$\widehat{m}(\overline{y}(\cdot))(\cdot) = F^{-1}\bigl(\alpha^{r_0 - r_{s_1}}(\cdot)Fy_{s_1}(\cdot)\bigr)(\cdot)$$

is optimal.

If $b < \infty$ *and* $r_{s_k} < r_0 \le R$, *then the method*

$$\widehat{m}(\overline{y}(\cdot))(\cdot) = F^{-1}\bigl(\alpha^{r_0 - r_{s_k}}(\cdot)Fy_{s_k}(\cdot)\bigr)(\cdot)$$

is optimal.

Note that the optimal method is linear and uses at most two measurements; moreover, these measurements are "smoothed" beforehand.

2. EXAMPLES

1. Optimal reconstruction of the solution of the heat equation. Consider the problem of optimal reconstruction of temperature in $\mathbb{R}^d$ at time instant τ from its approximate measurements at time instants $t_1, \dots, t_n$. The heat propagation in $\mathbb{R}^d$ is described by the equation

$$\frac{\partial u}{\partial t} = \Delta u, \tag{1}$$

where Δ is the Laplace operator, with a given initial distribution of temperature

$$u(0, x) = u_0(x), \qquad x \in \mathbb{R}^d. \tag{2}$$

We assume that $u_0(\cdot) \in L_2(\mathbb{R}^d)$. A unique solution of problem (1), (2) for $t > 0$ is given by the Poisson integral

$$u(t, x) = u(t, x; u_0(\cdot)) = \frac{1}{2\sqrt{\pi t}} \int\limits_{\mathbb{R}^d} e^{-\frac{|x-\xi|^2}{4t}} u_0(\xi)\, d\xi,$$

with $u(t, \cdot) \to u_0(\cdot)$ as $t \to 0$ in the metric of $L_2(\mathbb{R}^d)$.

Suppose that we know approximate temperature distributions $u(t_1, \cdot), \dots, u(t_n, \cdot)$ at instants $0 = t_1 < \ldots < t_n$; i.e., we are given functions $y_j(\cdot) \in L_2(\mathbb{R}^d)$ such that $\|u(t_j, \cdot) - y_j(\cdot)\|_{L_2(\mathbb{R}^d)} \le \delta_j$, where $\delta_j > 0$, $j = 1, \dots, n$. We aim to reconstruct temperature at time $\tau > 0$ from the information about the functions $y_1(\cdot),\dots,y_n(\cdot)$. Here the error of optimal reconstruction has the form

$$E_\tau(\overline{t},\overline{\delta})=\inf_m \sup_{\substack{u_0(\cdot),y_1(\cdot),\dots,y_n(\cdot)\in L_2(\mathbb{R}^d)\\ \|u(t_j,\cdot)-y_j(\cdot)\|_{L_2(\mathbb{R}^d)}\le\delta_j,\ j=1,\dots,n}} \|u(\tau,\cdot)-m(\overline{y}(\cdot))(\cdot)\|_{L_2(\mathbb{R}^d)},$$

where the lower bound is taken over all methods $m\colon (L_2(\mathbb{R}^d))^n\to L_2(\mathbb{R}^d)$ ($\overline{t}=(t_1,\dots,t_n)$, $\overline{y}(\cdot)=(y_1(\cdot),\dots,y_n(\cdot))$, and $\overline{\delta}=(\delta_1,\dots,\delta_n)$).

The Fourier transform of the solution of the heat equation is given by (see, for example, [1])

$$Fu(t,\xi)=e^{-|\xi|^2t}Fu_0(\xi).$$

Thus, our problem is a particular case of the problem considered in Section 1 for $T=\mathbb{R}^d$ and $\alpha(\xi)=e^{-|\xi|^2}$.

We can easily verify that the conditions of Theorem 1 are satisfied. In this case,

$$\inf_{\xi\in\mathbb{R}^d}|\alpha(\xi)|=0,\qquad \sup_{\xi\in\mathbb{R}^d}|\alpha(\xi)|=1;$$

therefore,

$$M=\operatorname{co}\big\{(t_j,\ln 1/\delta_j),\ 1\le j\le n\big\}+\big\{(t,0)\mid t\ge 0\big\}.$$

The function $\theta(\cdot)$ on $[t_1,+\infty)$ is defined by $\theta(t)=\max\{x\mid (t,x)\in M\}$. Let $t_{s_1}<\ldots<t_{s_k}$ be the salient points of $\theta(\cdot)$.

Theorem 1 implies

Theorem 2. *The following equality holds*:

$$E_\tau(\overline{t},\overline{\delta})=e^{-\theta(\tau)}.$$

For $\tau\in[t_{s_j},t_{s_{j+1}}]$, $1\le j\le k-1$, *the method*

$$\widehat{m}(\overline{y}(\cdot))(\cdot)=F^{-1}\Big(e^{-(\tau-t_{s_j})|\xi|^2}\beta_j(\xi)Fy_{s_j}(\xi)+e^{-(\tau-t_{s_{j+1}})|\xi|^2}(1-\beta_j(\xi))Fy_{s_{j+1}}(\xi)\Big)(\cdot),$$

where

$$\beta_j(\xi)=\frac{(t_{s_{j+1}}-\tau)\delta_{s_{j+1}}^2e^{-2t_{s_j}|\xi|^2}}{(t_{s_{j+1}}-\tau)\delta_{s_{j+1}}^2e^{-2t_{s_j}|\xi|^2}+(\tau-t_{s_j})\delta_{s_j}^2e^{-2t_{s_{j+1}}|\xi|^2}},$$

is optimal. For $\tau>t_{s_k}$, *the method*

$$\widehat{m}(\overline{y}(\cdot))(\cdot)=F^{-1}\big(e^{-(\tau-t_{s_k})|\xi|^2}Fy_{s_k}(\xi)\big)(\cdot)$$

is optimal.

2. Reconstruction of the temperature of a rod from inaccurate discrete data. Consider the problem of optimal reconstruction of temperature in $\mathbb{R}^d$ from its approximate values at a discrete set of points at time instants $t_1,\dots,t_n$. We will assume that the temperature distribution is described by the implicit difference scheme

$$\frac{u_{s+1,j}-u_{sj}}{\tau}=\sum_{p=1}^d\frac{u_{s+1,j+e_p}-2u_{s+1,j}+u_{s+1,j-e_p}}{h^2},\tag{3}$$

where $u_{sj}=u(s\tau,jh)$ is the rod temperature at the point jh, $j\in\mathbb{Z}^d$, of the space at the instant $s\tau$, $s\in\mathbb{Z}_+$, and $e_1,\dots,e_d$ is a standard basis in $\mathbb{R}^d$.

Suppose we know approximate values of the rod temperature at the points jh at time instants $t_k = r_k\tau$: $y_k = \{y_{kj}\}_{j\in\mathbb{Z}^d}$, where $0 \le r_1 < \ldots < r_n$, $r_k \in \mathbb{Z}_+$, $k = 1,\ldots,n$. We will assume that $\|u_{r_k} - y_k\|_{l_2} \le \delta_k$, $k = 1,\ldots,n$, where $u_s = \{u_{sj}\}_{j\in\mathbb{Z}^d}$, and

$$\|x\|_{l_2} = \left(\sum_{j\in\mathbb{Z}^d} |x_j|^2\right)^{1/2}$$

for $x = \{x_j\}_{j\in\mathbb{Z}^d}$. It is required to reconstruct the values of temperature at the same points jh at a time instant $r_0\tau$, $r_0 \ge 0$, from the vectors y_k, $k = 1,\ldots,n$. As reconstruction methods, we consider all possible mappings $m\colon (l_2)^n \to l_2$. For a given method m, define its error as

$$e_{r_0}(\overline{r},\overline{\delta},m) = \sup_{\substack{u_0,y_1,\ldots,y_n\in l_2\\ \|u_{r_k}-y_k\|_{l_2}\le\delta_k,\ k=1,\ldots,n}} \|u_r - m(\overline{y})\|_{l_2},$$

where, as above, $\overline{r} = (r_1,\ldots,r_n)$, $\overline{\delta} = (\delta_1,\ldots,\delta_n)$, and $\overline{y}(\cdot) = (y_1(\cdot),\ldots,y_n(\cdot))$. The quantity

$$E_{r_0}(\overline{r},\overline{\delta}) = \inf_{m\colon (l_2)^n\to l_2} e_{r_0}(\overline{r},\overline{\delta},m)$$

is called the error of optimal reconstruction, and the method for which the lower bound is attained is said to be optimal.

The Fourier transform of a vector u_s has the form

$$Fu_s(\xi) = \sum_{j\in\mathbb{Z}^d} u_{sj}e^{-i\langle j,\xi\rangle}.$$

Passing to Fourier transforms in (3), we obtain

$$\frac{Fu_{s+1}(\xi) - Fu_s(\xi)}{\tau} = \sum_{p=1}^{d} \frac{e^{i\xi_p}Fu_{s+1}(\xi) - 2Fu_{s+1}(\xi) + e^{-i\xi_p}Fu_{s+1}(\xi)}{h^2},$$

which yields

$$Fu_{s+1}(\xi) = \left(1 + \frac{2\tau}{h^2}\sum_{p=1}^{d}(1-\cos\xi_p)\right)^{-1} Fu_s(\xi).$$

Thus,

$$Fu_s(\xi) = \alpha^s(\xi)Fu_0(\xi), \qquad \alpha(\xi) = \left(1 + \frac{2\tau}{h^2}\sum_{p=1}^{d}(1-\cos\xi_p)\right)^{-1}.$$

In our case,

$$a = \inf_{\xi\in\mathbb{T}^d}|\alpha(\xi)| = \left(1 + \frac{4d\tau}{h^2}\right)^{-1}, \qquad b = \sup_{\xi\in\mathbb{T}^d}|\alpha(\xi)| = 1.$$

Therefore,

$$M = \operatorname{co}\bigl\{(r_j, \ln(1/\delta_j)),\ 1\le j\le n\bigr\} + \bigl\{(r,0) \mid r\ge 0\bigr\} + \left\{\left(r, -r\ln\left(1+\frac{4d\tau}{h^2}\right)\right) \,\middle|\, r\le 0\right\}.$$

The function $\theta(\cdot)$ on $[0,+\infty)$ is defined by the equality $\theta(t) = \max\{x \mid (t,x)\in M\}$, and $r_{s_1} < \ldots < r_{s_k}$ are the salient points of $\theta(\cdot)$.

Theorem 1 immediately implies an expression for the error of optimal reconstruction and the form of the optimal method for such $\theta(\cdot)$ and $\alpha(\cdot)$.

If the heat propagation in a rod is described by the explicit difference scheme

$$\frac{u_{s+1,j}-u_{sj}}{\tau}=\sum_{p=1}^{d}\frac{u_{s,j+e_p}-2u_{sj}+u_{s,j-e_p}}{h^2},$$

then the Fourier transform of the solution is given by

$$Fu_s(\xi)=\alpha^s(\xi)Fu_0(\xi),\qquad \alpha(\xi)=1-\frac{2\tau}{h^2}\sum_{p=1}^{d}(1-\cos\xi_p).$$

In this case, a similar result can also be easily derived from Theorem 1.

3. PROOF OF THEOREM 1

The proof of the theorem is based on a fact concerning the optimal reconstruction of linear operators. To formulate this fact, we first give a more general statement of the problem of optimal reconstruction.

Let X be a vector space, Z a normed space, $Y_1,\dots,Y_n$ be spaces with inner products $(\cdot,\cdot)_{Y_j}$ and the corresponding norms $\|\cdot\|_{Y_j}$, and $I_j\colon X\to Y_j$, $j=1,\dots,n$, be linear operators. Consider the problem of optimal reconstruction of a linear operator $A\colon X\to Z$ on the class

$$W_k=\bigl\{x\in X\colon\ \|I_jx\|_{Y_j}\le\delta_j,\ 1\le j\le k,\ 0\le k<n\bigr\}$$

(when $k=0$, we assume that $W_0=X$) from inaccurate information about the values of the operators $I_{k+1},\dots,I_n$; i.e., we assume that for every $x\in W_k$ a vector $y=(y_{k+1},\dots,y_n)\in Y_{k+1}\times\dots\times Y_n$ is known such that

$$\|I_jx-y_j\|_{Y_j}\le\delta_j,\qquad j=k+1,\dots,n.$$

As reconstruction methods, we consider all possible mappings $m\colon Y_{k+1}\times\dots\times Y_n\to Z$. The error of a reconstruction method m is the quantity

$$e(A,W_k,I,\delta,m)=\sup_{x\in W_k}\ \sup_{\substack{y=(y_{k+1},\dots,y_n)\in Y_{k+1}\times\dots\times Y_n\\ \|I_jx-y_j\|_{Y_j}\le\delta_j,\ j=k+1,\dots,n}}\|Ax-m(y)\|_Z.$$

We aim to find the error of optimal reconstruction

$$E(A,W_k,I,\delta)=\inf_{m\colon Y_{k+1}\times\dots\times Y_n\to Z}e(A,W_k,I,\delta,m)$$

and the optimal method of reconstruction $\widehat{m}$ for which the lower bound (if it exists) is attained.

Theorem 3. *Suppose that the values of the problems*[2]

$$\|Ax\|_Z^2\to\max,\qquad \|I_jx\|_{Y_j}^2\le\delta_j^2,\qquad j=1,\dots,n,\qquad x\in X,\tag{4}$$

and

$$\|Ax\|_Z^2\to\max,\qquad \sum_{j=1}^{n}\widehat{\lambda}_j\|I_jx\|_{Y_j}^2\le\sum_{j=1}^{n}\widehat{\lambda}_j\delta_j^2,\qquad x\in X,\tag{5}$$

coincide for some $\widehat{\lambda}_j\ge0$, $j=1,\dots,n$.

[2] That is, the values of the functionals to be maximized.

Let also $\widetilde{Y}$ *be a dense subset in* $Y_{k+1}\times\ldots\times Y_n$ *such that, for every* $y=(y_{k+1},\ldots,y_n)\in\widetilde{Y}$, *the problem*

$$\sum_{j=1}^{k}\widehat{\lambda}_j\|I_jx\|^2_{Y_j}+\sum_{j=k+1}^{n}\widehat{\lambda}_j\|I_jx-y_j\|^2_{Y_j}\to\min,\qquad x\in X,\tag{6}$$

has a solution x_y *and, in addition, there exists a linear continuous operator* $\Lambda\colon Y_{k+1}\times\ldots\times Y_n\to Z$,[3] *such that* $\Lambda y=Ax_y$ *for all* $y\in\widetilde{Y}$.

Then $E(A,W_k,I,\delta)=S$, *where* S *is a general solution of problems* (4) *and* (5), *and the method* $\widehat{m}=\Lambda$ *is optimal.*

Proof. First, let us estimate the quantity $E(A,W_k,I,\delta)$ from below. Let $x\in X$, $\|I_jx\|_{Y_j}\le\delta_j$, $j=1,\ldots,k$, and m be an arbitrary method of reconstruction. Then

$$\begin{aligned}2\|Ax\|_Z&=\|Ax-m(0)-(-Ax-m(0))\|_Z\le\|Ax-m(0)\|_Z+\|-Ax-m(0)\|_Z\\&\le 2e(A,W_k,I,\delta,m).\end{aligned}$$

Taking the upper bound over all such x and then the lower bound over all m, we obtain

$$E(A,W_k,I,\delta)\ge\sup_{\|I_jx\|_{Y_j}\le\delta_j,\ j=1,\ldots,n}\|Ax\|_Z=S.$$

Now let us estimate $E(A,W_k,I,\delta)$ from above construct the optimal method. Consider the vector space $E=Y_1\times\ldots\times Y_n$ with the semi-inner product

$$(y^1,y^2)_E=\sum_{j=1}^{n}\widehat{\lambda}_j(y^1_j,y^2_j)_{Y_j},$$

where $y^1=(y^1_1,\ldots,y^1_n)$ and $y^2=(y^2_1,\ldots,y^2_n)$. Then the extremum problem (6) can be rewritten as

$$\|\widetilde{I}x-\widetilde{y}\|^2_E\to\min,\qquad x\in X,\tag{7}$$

where $\widetilde{I}x=(I_1x,\ldots,I_nx)$ and $\widetilde{y}=(0,\ldots,0,y_{k+1},\ldots,y_n)$. One can easily verify that if x_y is a solution of problem (7), then $(\widetilde{I}x_y-\widetilde{y},\widetilde{I}x)_E=0$ for all $x\in X$. Hence we obtain

$$\begin{aligned}\|\widetilde{I}x-\widetilde{y}\|^2_E&=\|\widetilde{I}x-\widetilde{I}x_y+\widetilde{I}x_y-\widetilde{y}\|^2_E\\&=\|\widetilde{I}x-\widetilde{I}x_y\|^2_E-2\operatorname{Re}\big(\widetilde{I}x-\widetilde{I}x_y,\widetilde{I}x_y-\widetilde{y}\big)_E+\|\widetilde{I}x_y-\widetilde{y}\|^2_E\\&=\|\widetilde{I}x-\widetilde{I}x_y\|^2_E+\|\widetilde{I}x_y-\widetilde{y}\|^2_E.\end{aligned}$$

Thus, for all $x\in X$,

$$\|\widetilde{I}x-\widetilde{I}x_y\|^2_E\le\|\widetilde{I}x-\widetilde{y}\|^2_E=\sum_{j=1}^{k}\widehat{\lambda}_j\|I_jx\|^2_{Y_j}+\sum_{j=k+1}^{n}\widehat{\lambda}_j\|I_jx-y_j\|^2_{Y_j}.\tag{8}$$

Let $x\in W_k$ and $y=(y_{k+1},\ldots,y_n)\in Y_{k+1}\times\ldots\times Y_n$ be such that $\|I_jx-y_j\|_{Y_j}\le\delta_j$ for $j=k+1,\ldots,n$. Then, for any $\varepsilon>0$, there exists an element $\widetilde{y}=(\widetilde{y}_{k+1},\ldots,\widetilde{y}_n)\in\widetilde{Y}$ such that $\|y_j-\widetilde{y}_j\|_{Y_j}<\varepsilon$, $j=1,\ldots,n$, and hence

$$\|I_jx-\widetilde{y}_j\|_{Y_j}\le\|I_jx-y_j\|_{Y_j}+\|y_j-\widetilde{y}_j\|_{Y_j}\le\delta_j+\varepsilon,\qquad j=k+1,\ldots,n.$$

[3] The norm of an element $y=(y_{k+1},\ldots,y_n)\in Y_{k+1}\times\ldots\times Y_n$ is defined as $\|y\|=\big(\sum_{j=k+1}^{n}\|y_j\|^2_{Y_j}\big)^{1/2}$.

Set $z = x - x_y$. Then it follows from (8) that

$$\sum_{j=1}^{n} \widehat{\lambda}_j \|I_j z\|_{Y_j}^2 \le \sum_{j=1}^{n} \widehat{\lambda}_j \widetilde{\delta}_j^2, \tag{9}$$

where $\widetilde{\delta}_j = \delta_j$ if $1 \le j \le k$ and $\widetilde{\delta}_j = \delta_j + \varepsilon$ if $k+1 \le j \le n$. One can easily verify that

$$\sup_{\substack{z \in X \\ \sum_{j=1}^{n} \widehat{\lambda}_j \|I_j z\|_{Y_j}^2 \le a^2}} \|Az\|_Z = \frac{c_1}{c_2} \sup_{\substack{x \in X \\ \sum_{j=1}^{n} \widehat{\lambda}_j \|I_j x\|_{Y_j}^2 \le b^2}} \|Ax\|_Z$$

for all $c_1, c_2 > 0$. Therefore, taking into account (9) and the fact that the values of problems (4) and (5) coincide, we obtain

$$\begin{aligned} \|Ax - Ax_{\widetilde{y}}\|_Z = \|Az\|_Z &\le \sup_{\substack{z \in X \\ \sum_{j=1}^{n} \widehat{\lambda}_j \|I_j z\|_{Y_j}^2 \le \sum_{j=1}^{n} \widehat{\lambda}_j \widetilde{\delta}_j^2}} \|Az\|_Z \\ &= \left(\frac{\sum_{j=1}^{n} \widehat{\lambda}_j \widetilde{\delta}_j^2}{\sum_{j=1}^{n} \widehat{\lambda}_j \delta_j^2} \right)^{1/2} \sup_{\substack{z \in X \\ \sum_{j=1}^{m} \widehat{\lambda}_j \|I_j z\|_{Y_j}^2 \le \sum_{j=1}^{n} \widehat{\lambda}_j \delta_j^2}} \|Ax\|_Z \\ &= \left(\frac{\sum_{j=1}^{n} \widehat{\lambda}_j \widetilde{\delta}_j^2}{\sum_{j=1}^{n} \widehat{\lambda}_j \delta_j^2} \right)^{1/2} \sup_{\substack{x \in X \\ \|I_j x\|_{Y_j} \le \delta_j,\ j=1,\dots,n}} \|Ax\|_Z. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\|Ax - \Lambda y\|_Z \le \sup_{\substack{x \in X \\ \|I_j x\|_{Y_j} \le \delta_j,\ j=1,\dots,n}} \|Ax\|_Z = S.$$

In view of the lower estimate proved above, we find that $E(A, W_k, I, \delta) = S$ and $\widehat{m} = \Lambda$ is an optimal method. □

Now, based on this result, we prove Theorem 1.

Proof of Theorem 1. The problem corresponding to problem (4) in Theorem 3 is

$$\|A_{r_0} x(\cdot)\|_{L_2(T)}^2 \to \max, \qquad \|A_{r_j} x(\cdot)\|_{L_2(T)}^2 \le \delta_j^2, \qquad j = 1, \dots, n. \tag{10}$$

Let us find the value of this problem. For the Fourier transforms, by Plancherel's theorem, we obtain

$$\int_{\widehat{T}} |\alpha(\xi)|^{2r_0} |Fx(\xi)|^2 \, d\xi \to \max, \qquad \int_{\widehat{T}} |\alpha(\xi)|^{2r_j} |Fx(\xi)|^2 \, d\xi \le \delta_j^2, \qquad j = 1, \dots, n.$$

It can easily be shown that this problem has no solution; therefore, we consider its extension to the set of all nonnegative measures $d\mu(\cdot)$ on $\widehat{T}$:

$$\int_{\widehat{T}} |\alpha(\xi)|^{2r_0} \, d\mu(\xi) \to \max, \qquad \int_{\widehat{T}} |\alpha(\xi)|^{2r_j} \, d\mu(\xi) \le \delta_j^2, \qquad j = 1, \dots, n. \tag{11}$$

This is a linear (infinite-dimensional) programming problem. Its Lagrange function has the form

$$\mathcal{L}(\mu(\cdot), \lambda) = -\int_{\widehat{T}} |\alpha(\xi)|^{2r_0} \, d\mu(\xi) + \sum_{j=1}^{n} \lambda_j \left(\int_{\widehat{T}} |\alpha(\xi)|^{2r_j} \, d\mu(\xi) - \delta_j^2 \right),$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ is the set of Lagrange multipliers.

If we find a measure $d\widehat{\mu}(\cdot)$ admissible in (11) and Lagrange multipliers $\widehat{\lambda}_j \ge 0$, $j = 1, \dots, n$, such that ($\widehat{\lambda} = (\widehat{\lambda}_1, \dots, \widehat{\lambda}_n)$)

$$\min_{d\mu(\cdot)\ge 0} \mathcal{L}(d\mu(\cdot), \widehat{\lambda}) = \mathcal{L}(d\widehat{\mu}(\cdot), \widehat{\lambda}) \tag{12}$$

and

$$\widehat{\lambda}_j \left(\int_{\widehat{T}} |\alpha(\xi)|^{2r_j}\, d\widehat{\mu}(\xi) - \delta_j^2 \right) = 0, \qquad j = 1, \dots, n, \tag{13}$$

then $d\widehat{\mu}(\cdot)$ is a solution to problem (11). Indeed, let $d\mu(\cdot)$ be an admissible measure in (11). Then, using this fact (and taking into account that $\widehat{\lambda}_j \ge 0$, $j = 1, \dots, n$) and then (12) and (13), we obtain

$$\begin{aligned}
-\int_{\widehat{T}} |\alpha(\xi)|^{2r_0}\, d\mu(\xi) &\ge -\int_{\widehat{T}} |\alpha(\xi)|^{2r_0}\, d\mu(\xi) + \sum_{j=1}^n \widehat{\lambda}_j \left(\int_{\widehat{T}} |\alpha(\xi)|^{2r_j}\, d\mu(\xi) - \delta_j^2 \right) = \mathcal{L}(d\mu(\cdot), \widehat{\lambda}) \\
&\ge \mathcal{L}(d\widehat{\mu}(\cdot), \widehat{\lambda}) = -\int_{\widehat{T}} |\alpha(\xi)|^{2r_0}\, d\widehat{\mu}(\xi) + \sum_{j=1}^n \widehat{\lambda}_j \left(\int_{\widehat{T}} |\alpha(\xi)|^{2r_j}\, d\widehat{\mu}(\xi) - \delta_j^2 \right) \\
&= -\int_{\widehat{T}} |\alpha(\xi)|^{2r_0}\, d\widehat{\mu}(\xi),
\end{aligned}$$

which implies what was required.

Let $r_0 \in [r_{s_j}, r_{s_{j+1}}]$, $1 \le j \le k-1$. We present a measure $d\widehat{\mu}(\cdot)$ admissible in (11) and a set of Lagrange multipliers $\widehat{\lambda}$ that satisfy conditions (12) and (13). Set $d\widehat{\mu}(\xi) = C\delta(\xi - \xi_0)$, where $\delta(\cdot - \xi_0)$ is the Dirac delta function with support shifted to the point ξ_0, and take C and ξ_0 such that the following equalities hold:

$$\int_{\widehat{T}} |\alpha(\xi)|^{2r_p}\, d\widehat{\mu}(\xi) = \delta_p^2, \qquad p = s_j, s_{j+1}. \tag{14}$$

This implies that

$$C = \delta_{s_j}^{\frac{2r_{s_j+1}}{r_{s_{j+1}} - r_{s_j}}} \delta_{s_{j+1}}^{-\frac{2r_{s_j}}{r_{s_{j+1}} - r_{s_j}}}, \qquad \ln \frac{1}{|\alpha(\xi_0)|} = \frac{\ln(1/\delta_{s_{j+1}}) - \ln(1/\delta_{s_j})}{r_{s_{j+1}} - r_{s_j}}. \tag{15}$$

It follows from the form of the set M that, for $a > 0$ and a finite b,

$$\ln \frac{1}{b} < \frac{\ln(1/\delta_{s_{j+1}}) - \ln(1/\delta_{s_j})}{r_{s_{j+1}} - r_{s_j}} < \ln \frac{1}{a}.$$

Hence, by the continuity of $\alpha(\cdot)$, there exists a point $\xi_0 \in \widehat{T}$ for which the second equality in (15) holds. It is also easy to show the existence of such a point in the case when $a = 0$ and/or $b = \infty$.

Set $\widehat{\lambda}_k = 0$, $k \ne s_j, s_{j+1}$, and choose $\widehat{\lambda}_{s_j}$ and $\widehat{\lambda}_{s_{j+1}}$ such that the straight line $y = \widehat{\lambda}_{s_j} + \widehat{\lambda}_{s_{j+1}} x$ is tangent to the curve

$$\begin{cases} y = |\alpha(\xi)|^{2(r_0 - r_{s_j})}, \\ x = |\alpha(\xi)|^{2(r_{s_{j+1}} - r_{s_j})} \end{cases} \tag{16}$$

at the point ξ_0. A straightforward calculation shows that

$$\widehat{\lambda}_{s_j} = \frac{r_{s_{j+1}} - r_0}{r_{s_{j+1}} - r_{s_j}} \left(\frac{\delta_{s_{j+1}}}{\delta_{s_j}}\right)^{\frac{2(r_0 - r_{s_j})}{r_{s_{j+1}} - r_{s_j}}}, \qquad \widehat{\lambda}_{s_{j+1}} = \frac{r_0 - r_{s_j}}{r_{s_{j+1}} - r_{s_j}} \left(\frac{\delta_{s_j}}{\delta_{s_{j+1}}}\right)^{\frac{2(r_{s_{j+1}} - r_0)}{r_{s_{j+1}} - r_{s_j}}}. \tag{17}$$

It is clear that these numbers are positive; therefore, conditions (13) hold for this measure $\widehat{\mu}(\cdot)$ and the set $\widehat{\lambda}$.

Since the curve (16) is concave, we have

$$|\alpha(\xi)|^{2(r_0 - r_{s_j})} \le \widehat{\lambda}_{s_j} + \widehat{\lambda}_{s_{j+1}} |\alpha(\xi)|^{2(r_{s_{j+1}} - r_{s_j})}$$

for any $\xi \in \widehat{T}$, or, equivalently,

$$-|\alpha(\xi)|^{2r_0} + \widehat{\lambda}_{s_j} |\alpha(\xi)|^{2r_{s_j}} + \widehat{\lambda}_{s_{j+1}} |\alpha(\xi)|^{2r_{s_{j+1}}} \ge 0$$

for any $\xi \in \widehat{T}$. Hence we can easily deduce that condition (12) is satisfied.

Finally, since

$$\int\limits_{\widehat{T}} |\alpha(\xi)|^{2r_p}\, d\widehat{\mu}(\xi) = C|\alpha(\xi_0)|^{2r_p} = \delta_{s_j}^{2\frac{r_{s_{j+1}} - r_p}{r_{s_{j+1}} - r_{s_j}}} \delta_{s_{j+1}}^{2\frac{r_p - r_{s_j}}{r_{s_{j+1}} - r_{s_j}}} = e^{-2\theta_j(r_p)} \le e^{-2\theta(r_p)} \le \delta_p^2$$

for any $p = 1, \dots, n$, where $\theta_j(\cdot)$ is a straight line passing through the points $(r_{s_j}, \ln(1/\delta_{s_j}))$ and $(r_{s_{j+1}}, \ln(1/\delta_{s_{j+1}}))$, the measure $d\widehat{\mu}(\cdot)$ is admissible in (11).

Thus, $d\widehat{\mu}(\cdot)$ is a solution to problem (11), and its value is

$$\int\limits_{\widehat{T}} |\alpha(\xi)|^{2r_0}\, d\widehat{\mu}(\xi) = C|\alpha(\xi_0)|^{2r_0} = \delta_{s_j}^{2\frac{r_{s_{j+1}} - r_0}{r_{s_{j+1}} - r_{s_j}}} \delta_{s_{j+1}}^{2\frac{r_0 - r_{s_j}}{r_{s_{j+1}} - r_{s_j}}} = e^{-2\theta(r_0)}.$$

Consider the case when $r_{s_k} < r_0 \le R$. First, let $b < \infty$. Set

$$\widehat{\lambda}_{s_k} = b^{2(r_0 - r_{s_k})}, \qquad \widehat{\lambda}_j = 0, \quad j \ne s_k, \qquad d\widehat{\mu}(\cdot) = \delta_{s_k}^2 b^{-2r_{s_k}} \delta(\cdot - \xi_b),$$

where ξ_b is the point at which the upper bound of $|\alpha(\cdot)|$ is attained (which, recall, is equal to b). One can easily verify that conditions (12) and (13) hold and that the measure is admissible because

$$\int\limits_{\widehat{T}} |\alpha(\xi)|^{2r_p}\, d\widehat{\mu}(\xi) = \delta_{s_k}^2 b^{2(r_p - r_{s_k})} = e^{-2\theta_k(r_p)} \le e^{-2\theta(r_p)} \le \delta_p^2$$

for all $p = 1, \dots, n$, where $\theta_k(\cdot)$ is a straight line that coincides with the polygonal curve $\theta(\cdot)$ for $r > r_{s_k}$. Thus, $d\widehat{\mu}(\cdot)$ is a solution to problem (11), and its value is

$$\int\limits_{\widehat{T}} |\alpha(\xi)|^{2r_0}\, d\widehat{\mu}(\xi) = \delta_{s_k}^2 b^{2(r_0 - r_{s_k})} = e^{-2\theta(r_0)}.$$

Now, let $b = \infty$; then $s_k = n$. There exists a sequence ξ_l such that $b_l = |\alpha(\xi_l)| \to \infty$ as $l \to \infty$. Set $d\widehat{\mu}_l(\cdot) = \delta_n^2 b_l^{-2r_n} \delta(\cdot - \xi_l)$. If the equation of $\theta(\cdot)$ for $r_{s_{k-1}} < r \le r_{s_k} = r_n$ has the form $\theta(r) = \beta(r - r_n) + \ln(1/\delta_n)$, then we have

$$\int\limits_{\widehat{T}} |\alpha(\xi)|^{2r_p}\, d\widehat{\mu}_l(\xi) = \delta_n^2 b_l^{2(r_p - r_n)} \le e^{-2\theta(r_p)} \le \delta_p^2$$

for $b_l > e^{-\beta}$. Thus, the measures $d\widehat{\mu}_l(\cdot)$ are admissible and

$$\int\limits_{\widehat{T}} |\alpha(\xi)|^{2r_0}\, d\widehat{\mu}_l(\xi) = \delta_n^2 b_l^{2(r_0-r_n)} \to \infty$$

as $l \to \infty$. This implies that the value of problem (11) is $+\infty$.

For $r_1 < r_0 < r_{s_1}$ (in this case $a > 0$), set

$$\widehat{\lambda}_{s_1} = a^{2(r_0-r_{s_1})}, \qquad \widehat{\lambda}_j = 0, \quad j \neq s_1, \qquad d\widehat{\mu}(\cdot) = \delta_{s_1}^2 a^{-2r_{s_1}} \delta(\cdot - \xi_a),$$

where ξ_a is the point at which the lower bound of $|\alpha(\cdot)|$ is attained. As in the previous cases, one can show that the measure $d\widehat{\mu}(\cdot)$ is a solution to problem (11) and its value is $e^{-2\theta(r_0)}$.

When $a = 0$, the value of problem (11) is $+\infty$, and the arguments here are the same as in the case of $b = \infty$.

Thus, we have found the value of problem (11) in all possible cases. Applying the standard approximation of the delta function by δ-like sequences, we find that the value of this problem coincides with the value of problem (10). Now, let us show that the value of problem (10) coincides, in turn, with the value of the following problem:

$$\|A_{r_0}x(\cdot)\|_{L_2(T)}^2 \to \max, \qquad \sum_{j=1}^n \widehat{\lambda}_j \|A_{r_j}x(\cdot)\|_{L_2(T)}^2 \le \sum_{j=1}^n \widehat{\lambda}_j \delta_j^2, \tag{18}$$

where $(\widehat{\lambda}_1, \dots, \widehat{\lambda}_n)$ is the set of Lagrange multipliers found above. This problem corresponds to problem (5) from Theorem 3.

Again, passing to the Fourier transforms and then to nonnegative measures, we reduce (18) to the problem

$$\int\limits_{\widehat{T}} |\alpha(\xi)|^{2r_0}\, d\mu(\xi) \to \max, \qquad \int\limits_{\widehat{T}} \sum_{j=1}^n \widehat{\lambda}_j |\alpha(\xi)|^{2r_j}\, d\mu(\xi) \le \sum_{j=1}^n \widehat{\lambda}_j \delta_j^2. \tag{19}$$

Its Lagrange function is given by

$$\mathcal{L}_1(d\mu(\cdot), \nu) = -\int\limits_{\widehat{T}} |\alpha(\xi)|^{2r_0}\, d\mu(\xi) + \nu \left(\int\limits_{\widehat{T}} \sum_{j=1}^n \widehat{\lambda}_j |\alpha(\xi)|^{2r_j}\, d\mu(\xi) - \sum_{j=1}^n \widehat{\lambda}_j \delta_j^2 \right).$$

If $d\widehat{\mu}(\cdot)$ is a solution to problem (11) (it is obvious that the measure $d\widehat{\mu}(\cdot)$ is admissible in (19)) and $\widehat{\nu} = 1$, then analogs of conditions (12) and (13) clearly hold in this case and, hence, $d\widehat{\mu}(\cdot)$ is a solution to problem (19). Next, by the same arguments as above, the values of problems (19) and (18) coincide. However, since the values of problems (11) and (19) coincide, the values of problems (18) and (10) also coincide.

Now, let us construct an optimal method in the case when $r_0 \in [r_{s_j}, r_{s_{j+1}}]$. According to the general Theorem 3, consider the problem

$$\widehat{\lambda}_{s_j} \|A_{r_{s_j}}x(\cdot) - y_{s_j}(\cdot)\|_{L_2(T)}^2 + \widehat{\lambda}_{s_{j+1}} \|A_{r_{s_{j+1}}}x(\cdot) - y_{s_{j+1}}(\cdot)\|_{L_2(T)}^2 \to \min, \qquad x(\cdot) \in X_\alpha^R(T). \tag{20}$$

For all $y_{s_j}(\cdot)$ and $y_{s_{j+1}}(\cdot)$ in a dense subset of $(L_2(T))^n$ (see condition $\mathcal{A}$, which holds for $\alpha(\cdot)$), the Fourier transform of the solution $\widehat{x}(\cdot)$ to problem (20) has the form

$$F\widehat{x}(\cdot) = \frac{\widehat{\lambda}_{s_j} \overline{\alpha^{r_{s_j}}(\cdot)} F y_{s_j}(\cdot) + \widehat{\lambda}_{s_{j+1}} \overline{\alpha^{r_{s_{j+1}}}(\cdot)} F y_{s_{j+1}}(\cdot)}{\widehat{\lambda}_{s_j} |\alpha(\cdot)|^{2r_{s_j}} + \widehat{\lambda}_{s_{j+1}} |\alpha(\cdot)|^{2r_{s_{j+1}}}}.$$

This defines a continuous linear operator from a dense subset of $(L_2(T))^n$ to $L_2(T)$. Taking the superposition of this operator with the operator A_{r_0} and extending it by continuity to the whole space $(L_2(T))^n$ (and replacing the Lagrange multipliers with their expressions from (17)), we obtain an optimal method for the situation under consideration.

The cases when $a > 0$, $0 \le r_0 < r_{s_1}$ and $b < \infty$, $r_{s_k} < r_0 \le R$ are treated in a similar way. □

Let us make a few concluding remarks. While the problems of reconstructing linear functionals have been studied in a rather general form (see, for example, [2–4]), some progress in analogous problems with linear operators has been achieved only in Euclidean spaces with the use of specific features of the latter. The first results of this type were obtained in [5]. This direction was further developed in our papers [6–8], where we applied an approach based on the general principles of extremum theory.

The application of the theory of optimal reconstruction of linear operators to problems of mathematical physics is described in [9–13]. The result presented in Theorem 2 as an example of application of the general Theorem 3 was proved in [14].

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