ON OPTIMAL RECOVERY METHODS IN HARDY–SOBOLEV SPACES

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Abstract. In the paper a general approach to the construction of optimal recovery methods for linear functionals by a known solution of a dual extremal problem is proposed based on some parametrization of the solution of this dual problem. Using the proposed approach we succeeded in solving of series optimal recovery problems in Hardy–Sobolev classes such as optimal recovery of functions by the information of the Fourier coefficients or function values at some system of nodes in periodic and non-periodic cases.

1. Setting of problem and method of parametrization

Let W be some set of a linear space X. Consider the problem of optimal recovery of a linear functional L on this set by the values of linear functionals $l_1, \dots, l_n$. For $x \in W$ we set

$$Ix := (l_1 x, \dots, l_n x).$$

The operator $I\colon W \to K^n$, where $K = \mathbb{R}$ or $\mathbb{C}$ depending on whether X is a real or complex space, is called an *information* operator. The value

$$e(L, W, I) := \inf_{S\colon K^n \to K} \sup_{x \in W} |Lx - S(Ix)|$$

is called the *error* of optimal recovery of functional L on the set W. Any method S_0 for which

$$\sup_{x \in W} |Lx - S_0(Ix)| = e(L, W, I),$$

is said to be an *optimal* method of recovery.

S. A. Smolyak [1] proved that in the real case for a convex and centrally symmetric set W among optimal methods of recovery there exists a linear method and

(1) $$e(L, W, I) = \sup_{\substack{x \in W \\ Ix = 0}} |Lx|.$$

The analogous result in the complex case for a convex and balanced set W was proved in [2] (more general settings of optimal recovery problems and appropriate results may be found in [3] and in the literature cited there).

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Any element $x_0 \in W$ for which $Ix_0 = 0$ and

$$|Lx_0| = \sup_{\substack{x \in W \\ Ix=0}} |Lx|$$

we call *extremal*. The problem of finding an extremal element often turns out more simple than the problem of finding an optimal recovery method. The purpose of this paper is to offer a way allowing to obtain an optimal method of recovery by extremal element (in the presence of some its parametrization) and to construct a number of optimal recovery methods, using the offered scheme.

Theorem 1. *Let X be a real linear space, W a convex centrally symmetric set from X, and x_0 an extremal element in the problem of optimal recovery of a linear functional L on the set W by the values of linear functionals $l_1x, \dots, l_nx$. Assume that for all $M = (t_1, \dots, t_{s+n}) \in \mathbb{R}^{n+s}$ from some neighborhood of $M_0 \in \mathbb{R}^{n+s}$ there exist $x(M) \in X$ such that $x(M_0) = x_0$ and for the given functions $\psi_1(M), \dots, \psi_s(M)$ such that $\psi_j(M_0) = 0$, $j = 1, \dots, s$, for all M from a neighborhood of M_0, satisfying the condition $\psi_j(M) = 0$, $j = 1, \dots, s$, $x(M) \in W$ (in the case when $s = 0$ we assume that for all M from a neighborhood of M_0, $x(M) \in W$). Then if the functions $\varphi(M) := Lx(M)$, $\varphi_j(M) := l_jx(M)$, $j = 1, \dots, n$, and $\psi_j(M)$, $j = 1, \dots, s$, have continuous partial derivatives with respect to all variables in a neighborhood of M_0 and the determinant of the matrix*

$$J(M) = \begin{pmatrix} \dfrac{\partial\varphi_1}{\partial t_1} & \dots & \dfrac{\partial\varphi_n}{\partial t_1} & \dfrac{\partial\psi_1}{\partial t_1} & \dots & \dfrac{\partial\psi_s}{\partial t_1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dfrac{\partial\varphi_1}{\partial t_{n+s}} & \dots & \dfrac{\partial\varphi_n}{\partial t_{n+s}} & \dfrac{\partial\psi_1}{\partial t_{n+s}} & \dots & \dfrac{\partial\psi_s}{\partial t_{n+s}} \end{pmatrix}$$

does not vanish at M_0, then the method

$$Lx \approx \sum_{j=1}^{n} C_j l_j x,$$

where $C_1, \dots, C_n$ are solutions of the system

$$J(M_0)\mathbf{C} = \operatorname{grad} \varphi\big|_{M_0},$$

in which $\mathbf{C} = (C_1, \dots, C_{n+s})$, is the unique linear optimal method of recovery.

Proof. Let

$$Lx \approx \sum_{j=1}^{n} C_j l_j x$$

be an optimal method of recovery. Then in view of the fact that x_0 is an extremal element we have for all $x \in W$

$$|Lx - \sum_{j=1}^{n} C_j l_j x| \leq |Lx_0|.$$

Consequently for all M from some neighborhood of M_0 such that $\psi_j(M_0) = 0$, $j = 1, \dots, s$, the inequality

$$|\varphi(M) - \sum_{j=1}^{n} C_j \varphi_j(M)| \leq |\varphi(M_0)|$$

holds. Since $\varphi_j(M_0) = 0$, $j = 1, \dots, n$, from here it easily follows that the function

$$\varphi(M) - \sum_{j=1}^{n} C_j \varphi_j(M)$$

has an relative extremum at the point M_0. The method of Lagrange leads to necessary conditions

$$\frac{\partial \varphi}{\partial t_m} - \sum_{j=1}^{n} C_j \frac{\partial \varphi_j}{\partial t_m} - \sum_{j=1}^{s} C_{n+j} \frac{\partial \psi_j}{\partial t_m} = 0, \quad m = 1, \dots, n+s,$$

from which $C_1, \dots, C_n$ are uniquely determined. □

We cite one simple example. Let $\mathcal{H}_\infty^{\mathbb{R}}$ be the space of functions analytic in the unit disk

$$D := \{ z \in \mathbb{C} : |z| < 1 \},$$

bounded, and real in the interval $(-1, 1)$. As the set W we consider $H_\infty^{\mathbb{R}}$ which is the set of functions from $\mathcal{H}_\infty^{\mathbb{R}}$ satisfying the condition

$$\sup_{z \in D} |f(z)| \leq 1.$$

For the problem of optimal recovery of functions from $H_\infty^{\mathbb{R}}$ at the point $\tau \in (-1, 1)$ by their values at zero the dual problem (1) may be solved immediately using the Schwartz lemma:

$$\sup_{\substack{f \in H_\infty^{\mathbb{R}} \\ f(0)=0}} |f(\tau)| = |\tau|.$$

Thus the function $f_0(z) = z$ is extremal for the considered problem. Set

$$f_1(z, t) = \frac{z + t}{1 + tz}.$$

It is easy to see that $f_1(z, t) \in H_\infty^{\mathbb{R}}$ for all $t \in (-1, 1)$. Moreover, $f_1(z, 0) = f_0(z)$ and $f_1(0, t) = t$. From Theorem 1 we get that the method

$$f(\tau) \approx \left(\frac{\partial f_1}{\partial t}(0, 0) \right)^{-1} \frac{\partial f_1}{\partial t}(\tau, 0) f(0) = (1 - \tau^2) f(0)$$

is the unique linear optimal method of recovery. More general results concerning the considered problem may be found in [2] and [4] (they also can be obtained by the proposed method).

2. Optimal recovery by Fourier coefficients

We shall call the Hardy–Sobolev class $H^r_{\infty,\beta}$ the set of 2π-periodic functions analytic in the strip $S_\beta := \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$ and satisfying the condition $|f^{(r)}(z)| \le 1$, $z \in S_\beta$. By $H^{r,\mathbb{R}}_{\infty,\beta}$ we denote the class of functions from $H^r_{\infty,\beta}$ that are real on the real axis. In the case $r = 0$ we denote these classes by $H_{\infty,\beta}$ and $H^{\mathbb{R}}_{\infty,\beta}$, respectively.

Put

$$\begin{aligned} a_j(f) &:= \frac{1}{\pi}\int_{\mathbb{T}} f(x)\cos jx\,dx, \quad j = 0, 1, \dots\ , \\ b_j(f) &:= \frac{1}{\pi}\int_{\mathbb{T}} f(x)\sin jx\,dx, \quad j = 1, 2, \dots\ , \end{aligned}$$

where $\mathbb{T} := [0, 2\pi)$. Consider the problem of optimal recovery of $f(\xi)$, $f \in H^r_{\infty,\beta}$, $\xi \in \mathbb{T}$, by the values of the information operator

$$If = (a_0(f), a_1(f), \dots, a_{n-1}(f), b_1(f), \dots, b_{n-1}(f)).$$

In view of a translate invariance of the considered class the optimal recovery error does not depend on ξ. We denote it by $e(H^r_{\infty,\beta}, I)$.

The solution of the investigated problem gives in terms of the elliptic function theory. We recall some definitions from this theory. The complete elliptic integrals of the first kind with moduli k and $k' := \sqrt{1-k^2}$ are defined by

$$K := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad K' := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}.$$

In what follows we assume that k is chosen from the condition

$$\frac{\pi K'}{2K} = \beta.$$

In this case k is defined by the equation (see, for example, [5])

$$k = 4e^{-\beta}\left(\frac{\sum_{m=0}^{\infty} e^{-2\beta m(m+1)}}{1 + 2\sum_{m=1}^{\infty} e^{-2\beta m^2}}\right)^2.$$

We shall use the standard notation $\operatorname{sn}(z,k)$, $\operatorname{cn}(z,k)$, and $\operatorname{dn}(z,k)$ for the Jacobi elliptic functions omitting the dependence on the modulus when it equals k.

Set

$$\Phi^\beta_{n,0}(z) := \sqrt{\lambda}\,\operatorname{sn}\left(\frac{2n\Lambda}{\pi}z, \lambda\right), \quad \Phi^\beta_{n,r} := D_r * \Phi_{n,0}, \quad r \ge 1,$$

where Λ is the complete elliptic integral of the first kind for the modulus λ defined by

$$\frac{\Lambda'}{\Lambda} = 2n\frac{K'}{K} = \frac{4\beta n}{\pi},$$

$$D_r(t) = 2\sum_{m=1}^{\infty}\frac{\cos(mt-\pi r/2)}{m^r}, \quad r=1,2,\dots\ ,$$

is the Bernoulli kernel, and

$$(f*g)(z) := \frac{1}{2\pi}\int_{\mathbb{T}} f(z-t)g(t)\,dt.$$

The functions $\Phi_{n,r}^{\beta}$ introduced in [6] have the properties similar to the ones of the Euler perfect splines (see, for example, [7, p. 72]). The Euler perfect splines are solutions of number of classical extremal problems for the Sobolev classes (about exact values of n-widths, about the Kolmogorov type inequality for derivatives, and others), and the functions $\Phi_{n,r}^{\beta}$ turns out the solutions of analogous problems for analytic functions from the Hardy–Sobolev classes. It follows from [6] that

$$\left.\begin{aligned}\Phi_{n,r}^{\beta}(z) &= \frac{\pi}{\sqrt{\lambda}\Lambda n^r}\sum_{m=0}^{\infty}\frac{\sin((2m+1)nz-\pi r/2)}{(2m+1)^r\sinh((2m+1)2n\beta)},\\ \|\Phi_{n,r}^{\beta}\|_\infty &= \frac{\pi}{\sqrt{\lambda}\Lambda n^r}\sum_{m=0}^{\infty}\frac{(-1)^{m(r+1)}}{(2m+1)^r\sinh((2m+1)2n\beta)},\end{aligned}\right\} \quad r=0,1,\dots$$

(we denote by $\|\cdot\|_\infty$ the standard norm in $L_\infty(\mathbb{T})$).

It was proved in [8] that

$$e(H_{\infty,\beta}^r, I) = \|\Phi_{n,r}^{\beta}\|_\infty$$

and the extremal function for the problem of optimal recovery of $f(0)$ on the class $H_{\infty,\beta}^r$ by the information operator I is the function

$$\varphi_{n,r}^{\beta}(z) := \begin{cases}\Phi_{n,r}^{\beta}\left(z+\dfrac{\pi}{2n}\right), & r=2l,\\ \Phi_{n,r}^{\beta}(z), & r=2l+1.\end{cases}$$

Nevertheless the question about optimal recovery method remained open. Using Theorem 1, we shall construct a linear optimal method of recovery for this problem.

First we prove one auxiliary result. Set

$$\operatorname{ctn} z := \frac{\operatorname{cn} z\,\operatorname{dn} z}{\operatorname{sn} z}.$$

Lemma 1. *For all $0 \le t_1 < \ldots < t_{2n} < 2K$ the system of functions*

$$1, \operatorname{ctn}\left(\frac{K}{\pi}z - t_1\right), \dots, \operatorname{ctn}\left(\frac{K}{\pi}z - t_{2n}\right)$$

is a Chebyshev system on the set $\mathbb{T}\setminus\{\pi t_1/K, \dots, \pi t_{2n}/K\}$.

Proof. Assume that there exist real $C_0, C_1, \dots, C_{2n}$ not all equal zero for which the function

$$C_0 + \sum_{j=1}^{2n} C_j \operatorname{ctn}\left(\frac{K}{\pi}z - t_j\right)$$

has $2n+1$ zeros on the set $\mathbb{T} \setminus \{\pi t_1/K, \dots, \pi t_{2n}/K\}$. Then the function

$$\begin{aligned} g(z) &= C_0 \prod_{j=1}^{2n} \operatorname{sn}\left(\frac{K}{\pi}z - t_j\right) \\ &\quad + \sum_{m=1}^{2n} C_m \operatorname{cn}\left(\frac{K}{\pi}z - t_m\right) \operatorname{dn}\left(\frac{K}{\pi}z - t_m\right) \prod_{\substack{j=1 \\ j \neq m}}^{2n} \operatorname{sn}\left(\frac{K}{\pi}z - t_j\right) \end{aligned}$$

must have at least $2n+1$ zeros on $\mathbb{T}$. The function $g(z)$ is an elliptic functions with periods 2π, $2\pi K'/K$. By the Liouville theorem (see [5, p. 14]) it follows that the number of zeros of $g(z)$, counting multiplicities, in the parallelogram of periods coincides with the number of poles. The number of poles of $g(z)$ in the parallelogram of periods does not exceed $2n+1$. Since the number of zeros counting multiplicities of 2π-periodic function should be even it does not exceed $2n$ what contradicts with made assumption. □

Put

$$\sigma(z) := \operatorname{sn}\left(\frac{2n\Lambda}{\pi}z, \lambda\right) \operatorname{ctn} \frac{K}{\pi}z.$$

Theorem 2. *For all $\xi \in \mathbb{T}$ the method*

$$f(\xi) \approx d_0 \frac{a_0(f)}{2} + \sum_{j=1}^{n-1} d_j (a_j(f) \cos j\xi + b_j(f) \sin j\xi),$$

where

$$d_j = \frac{2}{n a_j(\sigma)} \sum_{m=1}^{n} (-1)^{m+1} \operatorname{ctn} \frac{2m-1}{2n} K \cos j \frac{2m-1}{2n}\pi,$$
$$j = 0, \dots, n-1,$$

is optimal for the class $H_{\infty,\beta}$. For $r \geq 1$ and all $\xi \in \mathbb{T}$ the method

$$f(\xi) \approx \frac{a_0(f)}{2} + \frac{2}{n} \sum_{j=1}^{n-1} \frac{j^r d_{jr}}{a_j(\sigma)} (a_j(f) \cos j\xi + b_j(f) \sin j\xi),$$

where

$$d_{jr} = \begin{cases} (-1)^{r/2} \displaystyle\sum_{m=1}^{n} (D_r * \sigma)\left(\frac{2m-1}{2n}\pi\right) \cos j \frac{2m-1}{2n}\pi, & r = 2l, \\ (-1)^{(r-1)/2} \displaystyle\sum_{m=1}^{n-1} (D_r * \sigma)\left(\frac{m}{n}\pi\right) \sin j \frac{m}{n}\pi, & r = 2l+1, \end{cases}$$

is optimal for the class $H^r_{\infty,\beta}$.

Proof. First we consider the problem of optimal recovery of $f(0)$ for the class $H^{r,\mathbb{R}}_{\infty,\beta}$ by the information operator

$$I_0 f := (a_0(f), a_1(f), \dots, a_{n-1}(f)).$$

We have

$$e(H^{r,\mathbb{R}}_{\infty,\beta}, I_0) = \sup_{\substack{f\in H^{r,\mathbb{R}}_{\infty,\beta}\\ I_0 f=0}} |f(0)| \geq |\varphi^\beta_{n,r}(0)|.$$

On the other hand, if $f \in H^{r,\mathbb{R}}_{\infty,\beta}$ and $I_0 f = 0$, then, putting

$$f_0(z) := \frac{f(z) + f(-z)}{2},$$

we have $f_0 \in H^{r,\mathbb{R}}_{\infty,\beta}$, $I_0 f_0 = 0$, and furthermore, since f_0 is an even function, $b_m(f) = 0$, $m \in \mathbb{N}$. Consequently,

$$f(0) = f_0(0) \leq \sup_{\substack{f\in H^{r}_{\infty,\beta}\\ If=0}} |f(0)| = |\varphi^\beta_{n,r}(0)|.$$

Thus,

$$e(H^{r,\mathbb{R}}_{\infty,\beta}, I_0) = |\varphi^\beta_{n,r}(0)|.$$

Using the first principal transform of elliptic functions of degree $2n$, it can be shown (see [9]) that

$$\varphi^\beta_{n,0}(z) = \sqrt{\lambda}\,\mathrm{sn}\left(\frac{2n\Lambda}{\pi}z + \Lambda, \lambda\right) = k^n \prod_{m=1}^{2n} \mathrm{sn}\left(\frac{K}{\pi}z - \frac{2m-1}{2n}K\right).$$

For $M = (t_1, \dots, t_n)$ put

$$h_M(z) := k^n \prod_{m=1}^{n} \mathrm{sn}\left(\frac{K}{\pi}z - t_m\right) \prod_{m=n+1}^{2n} \mathrm{sn}\left(\frac{K}{\pi}z - \frac{2m-1}{2n}K\right).$$

Then for all $M \in [0, 2K)^n$, $h_M \in H^{\mathbb{R}}_{\infty,\beta}$, and for

$$t^0_m := \frac{2m-1}{2n}K$$

and $M_0 := (t^0_1, \dots, t^0_n)$, $h_{M_0} = \varphi^\beta_{n,0}$.

Let $r = 0$. Let us show that the determinant of the matrix consisting of the elements

$$\frac{\partial}{\partial t_m} a_j(h_M)\Big|_{M_0} = a_j\left(\frac{\partial h_M}{\partial t_m}\Big|_{M_0}\right), \quad m = 1, \dots, n,\ j = 0, \dots, n-1,$$

does not vanish. If we assume the converse, then there exist real $C_1, \dots, C_n$ not all equal zero such that for the function

$$g = \sum_{m=1}^{n} C_m \frac{\partial h_M}{\partial t_m}\Big|_{M_0}$$

the equalities $a_0(g) = \ldots = a_{n-1}(g) = 0$ hold. Consequently, for the even function $g_0(z) := g(z) + g(-z)$ the equalities

$$a_0(g_0) = a_1(g_0) = b_1(g_0) = \ldots = a_{n-1}(g_0) = b_{n-1}(g_0) = 0$$

hold. Since

$$\frac{\partial h_M}{\partial t_m}(z) = -h_M(z)\operatorname{ctn}\left(\frac{K}{\pi}z - t_m\right),$$

in view of the evenness of h_{M_0} we have

$$\begin{aligned} g_0(z) &= -h_{M_0}(z)\sum_{m=1}^{n} C_m\left(\operatorname{ctn}\left(\frac{K}{\pi}z - t_m^0\right) - \operatorname{ctn}\left(\frac{K}{\pi}z + t_m^0\right)\right) \\ &= -h_{M_0}(z)\sum_{m=1}^{n} C_m\left(\operatorname{ctn}\left(\frac{K}{\pi}z - t_m^0\right) - \operatorname{ctn}\left(\frac{K}{\pi}z - \tau_m^0\right)\right), \end{aligned}$$

where $\tau_m^0 = 2K - t_m^0$. Put

$$F := g_0 - C_0 h_{M_0},$$

where $C_0 = g_0(0)/h_{M_0}(0)$. For the function F the equalities

$$a_0(F) = a_1(F) = b_1(F) = \ldots = a_{n-1}(F) = b_{n-1}(F) = 0$$

hold. In view of the fact that the trigonometric system

$$1, \cos x, \sin x, \ldots, \cos(n-1)x, \sin(n-1)x$$

is a Chebyshev system the function F must have at least $2n$ sign changes on $\mathbb{T}$ (see [10, p. 41]). In addition, in view of the evenness $F(z)$ has a zero at $z = 0$ without sign changes. Thus F has at least $2n + 1$ distinct zeros on $\mathbb{T}$, which contradicts Lemma 1.

By Theorem 1 (for $s = 0$) it follows that for finding an optimal recovery method it remains to solve the system

$$\text{(2)}\quad \sum_{j=0}^{n-1} C_j a_j\left(\frac{\partial h_M}{\partial t_m}\Big|_{M_0}\right) = \frac{\partial h_M(0)}{\partial t_m}\Big|_{M_0} = \sqrt{\lambda}\operatorname{ctn}\frac{2m-1}{2n}K, \qquad m = 1, \ldots, n.$$

We have

$$\begin{aligned} a_j\left(\frac{\partial h_M}{\partial t_m}\Big|_{M_0}\right) &= -\frac{1}{\pi}\int_{\mathbb{T}} h_{M_0}(z)\operatorname{ctn}\left(\frac{K}{\pi}z - t_m^0\right)\cos jz\,dz \\ &= -\frac{1}{\pi}\int_{\mathbb{T}} h_{M_0}\left(z + \frac{2m-1}{2n}\pi\right)\operatorname{ctn}\frac{K}{\pi}z\cos j\left(z + \frac{2m-1}{2n}\pi\right)dz \\ &= (-1)^{m+1}\sqrt{\lambda}a_j(\sigma)\cos j\frac{2m-1}{2n}\pi. \end{aligned}$$

From the fact that the determinant of the system (2) does not vanish it follows that $a_j(\sigma) \neq 0$, $j = 0, \dots, n-1$. Thus the system (2) is equivalent to the system

$$(3)\quad \sum_{j=0}^{n-1} C_j a_j(\sigma) \cos j\frac{2m-1}{2n}\pi = (-1)^{m+1} \operatorname{ctn} \frac{2m-1}{2n} K, \qquad m = 1, \dots, n.$$

In view of the orthogonality of the system $1, \cos x, \dots, \cos(n-1)x$ at the points $\frac{2m-1}{2n}\pi$, $m = 1, \dots, n$, we obtain that $C_0 = d_0/2$, $C_j = d_j$, $j = 1, \dots, n-1$.

Now let $r \geq 1$. Consider the function

$$h_{P,r}(z) := \begin{cases} \frac{t_0}{2} + (D_r * h_M)(z), & r = 2l, \\ \frac{t_0}{2} + (D_r * h_M)\left(z - \frac{\pi}{2n}\right), & r = 2l+1, \end{cases}$$

where $P = (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1}$. For all P such that

$$\psi(P) := a_0(h_M) = 0,$$

$h_{P,r} \in H^{r,\mathbb{R}}_{\infty,\beta}$, and furthermore, for $P_0 := (0, t_1^0, \dots, t_n^0)$, $h_{P_0,r} = \varphi^{\beta}_{n,r}$. By Theorem 1 (by now with $s = 1$) it follows that for finding an optimal method of recovery one have to solve the system

$$(4)\quad \sum_{j=0}^{n-1} C_j \frac{\partial a_j(h_{P,r})}{\partial t_m}\Big|_{P_0} + C_n \frac{\partial \psi}{\partial t_m} = \frac{\partial h_{P,r}(0)}{\partial t_m}\Big|_{P_0}, \quad m = 0, \dots, n.$$

We have (all partial derivatives are calculated at the point P_0)

$$\frac{\partial a_0(h_{P,r})}{\partial t_0} = 1, \quad \frac{\partial a_j(h_{P,r})}{\partial t_0} = 0,\ j = 1, \dots, n-1, \quad \frac{\partial \psi}{\partial t_0} = 0,$$
$$\frac{\partial a_0(h_{P,r})}{\partial t_m} = 0, \quad \frac{\partial \psi}{\partial t_m} = (-1)^{m+1}\sqrt{\lambda} a_0(\sigma), \quad m = 1, \dots, n,$$

$$\frac{\partial a_j(h_{P,r})}{\partial t_m} = \begin{cases} (-1)^{r/2+m+1} \frac{\sqrt{\lambda} a_j(\sigma)}{j^r} \cos j\frac{2m-1}{2n}\pi, & r = 2l, \\ (-1)^{r/2+m-1/2} \frac{\sqrt{\lambda} a_j(\sigma)}{j^r} \sin j\frac{m}{n}\pi, & r = 2l+1, \end{cases}$$
$$j = 1, \dots, n-1,\ m = 1, \dots, n.$$

For $r=2l$ the system (4) takes the form: $C_0=1/2$,

$$\sum_{j=1}^{n-1}(-1)^{r/2}\frac{a_j(\sigma)}{j^r}C_j\cos j\frac{2m-1}{2n}\pi+a_0(\sigma)C_n$$
$$=(D_r*\sigma)\left(\frac{2m-1}{2n}\pi\right),\quad m=1,\dots,n,$$

and is solved similarly to the system (2).

If $r=2l+1$, then the system (4) is reduced to the following: $C_0=1/2$,

$$\sum_{j=1}^{n-1}(-1)^{(r-1)/2}\frac{a_j(\sigma)}{j^r}C_j\sin j\frac{m}{n}\pi-a_0(\sigma)C_n$$
$$=(D_r*\sigma)\left(\frac{m}{n}\pi\right),\quad m=1,\dots,n.$$

Using the orthogonality of the system $\sin x,\dots,\sin(n-1)x$ at the points $m\pi/n$, $m=1,\dots,n-1$, we obtain the solution of this system.

Let us prove that the constructed method (denote it by S) is optimal for the class $H^r_{\infty,\beta}$. Assume that there exists a function $f_0\in H^r_{\infty,\beta}$ for which

$$|f_0(0)-S(I_0f_0)|>e(H^r_{\infty,\beta},I_0). \tag{5}$$

Then for $\overline{f_0(\overline{z})}\in H^r_{\infty,\beta}$ the inequality (5) also holds. Without loss of generality we may assume that $f_0(0)-S(I_0f_0)>0$. Consequently, for the function

$$g(z):=\frac{f_0(z)+\overline{f_0(\overline{z})}}{2}\in H^{r,\mathbb{R}}_{\infty,\beta}$$

we have

$$g(0)-S(I_0g)>e(H^r_{\infty,\beta},I_0)\ge e(H^{r,\mathbb{R}}_{\infty,\beta},I_0),$$

which is impossible in view of optimality of the method S on the class $H^{r,\mathbb{R}}_{\infty,\beta}$.

To find an optimal method of recovery of $f(\xi)$ it is sufficient to consider an optimal method of recovery at zero for the function $F(z)=f(z+\xi)$. $\square$

Corollary 1. *For all $\xi\in\mathbb{T}$ the method*

$$f(\xi)\approx\frac{1}{n}\sum_{|j|\le n-1}\left(\sum_{m=1}^{n}(-1)^{m+1}\operatorname{ctn}\frac{2m-1}{2n}K\cos j\frac{2m-1}{2n}\pi\right)\frac{e^{ij\xi}}{c_j(\sigma)}c_j(f)$$

is an optimal method of recovery on the class $H_{\infty,\beta}$ by the Fourier coefficients

$$c_j(f)=\frac{1}{2\pi}\int_{\mathbb{T}}f(t)e^{-ijt}\,dt,\quad |j|\le n-1.$$

For $r \geq 1$ and all $\xi \in \mathbb{T}$ the method

$$f(\xi) \approx c_0(f) + \frac{1}{n} \sum_{\substack{|j|\leq n-1 \\ j\neq 0}} |j|^r d_{|j|^r} \frac{e^{ij\xi}}{c_j(\sigma)} c_j(f)$$

is an optimal method on the class $H^r_{\infty,\beta}$.

3. Recovery by function values

Consider now the problem of optimal recovery of the value $f(\xi)$, $f \in H^r_{\infty,\beta}$, $\xi \in \mathbb{T}$, by the values of the information operator

$$I_\tau := (f(\tau_1), \dots , f(\tau_{2n})),$$

where $\tau = (\tau_1, \dots , \tau_{2n})$, $0 \leq \tau_1 < \dots < \tau_{2n} < 2\pi$. Assume that $r \geq 1$ (for $r = 0$ the solution of considered problem was obtained in [8]).

First we prove a number of auxiliary statements.

Lemma 2. *For all $0 \leq \tau_1 < \dots < \tau_{2n} < 2\pi$ there exist such $0 \leq \theta_1 < \dots < \theta_{2n} < 2\pi$ that for the function $f_0 \in H^r_{\infty,\beta}$ which has the form*

$$f_0 = c + D_r * B_0,$$

where $c \in \mathbb{R}$ and

$$B_0(t) = k^n \prod_{j=1}^{2n} \operatorname{sn}\left(\frac{K}{\pi}(t - \theta_j)\right),$$

$a_0(B_0) = 0$ and $f_0(\tau_1) = \dots = f_0(\tau_{2n}) = 0$.

Proof. Recall that a Blaschke product of degree m for a domain $\Omega \subset \mathbb{C}$ is a function of the form

$$B(z) = \varepsilon \exp\left(- \sum_{j=1}^{m} P(z, \zeta_j)\right),$$

where $\zeta_1, \dots , \zeta_m$ are points from Ω, $|\varepsilon| = 1$, $P(z, \zeta) = u(z, \zeta) + iv(z, \zeta)$, $u(z, \zeta)$ is the Green's function for Ω with singularity at ζ, and $v(z, \zeta)$ is the harmonic conjugate of $u(z, \zeta)$ (which, in general, is multiple valued). Denote by $\mathcal{B}_{2n}$ the set of single valued Blaschke products of degree at most $2n$ for the annulus $\Omega_\beta = \{z \in \mathbb{C} : e^{-\beta} < |z| < e^\beta\}$ real on the unite circle $E = \{z \in \mathbb{C} : |z| = 1\}$. Using the generalized Pick–Nevanlinna problem in [11] an odd and continuous in the topology of uniform convergence on compacts from Ω_β mapping $\sigma \colon S^{2n} \to \mathcal{B}_{2n}$, where

$$S^{2n} = \{\, x = (x_0, \dots , x_{2n}) \in \mathbb{R}^{2n+1} : \sum_{j=0}^{2n+1} |x_j|^2 = 1 \,\},$$

was constructed.

For $x \in S^{2n}$ we set

$$\sigma_0(x) = (a_0(B), f_B(\tau_2) - f_B(\tau_1), \dots , f_B(\tau_{2n}) - f_B(\tau_1)),$$

where $B(t)=\sigma(x)(e^{it})$ and $f_B=D_r*B$. Then the mapping $\sigma_0\colon S^{2n}\to\mathbb{R}^{2n}$ is odd and continuous and consequently, by Borsuk's theorem there exists $x_0\in S^{2n}$ such that $\sigma_0(x_0)=0$. Thus the function $\widehat{B}(t):=\sigma(x_0)(e^{it})$ for which $a_0(\widehat{B})=0$ and the function $\widehat{f}:=a+D_r*\widehat{B}$ for which for $a=-(D_r*\widehat{B})(\tau_1)$

$$\widehat{f}(\tau_1)=\ldots=\widehat{f}(\tau_{2n})=0$$

are constructed. Since $\widehat{f}^{(r)}=\widehat{B}$, by Rolle's theorem follows that $\widehat{B}$ has at least $2n$ distinct zeros on $\mathbb{T}$ and in view of the fact that $\sigma(x_0)\in\mathcal{B}_{2n}$, $\widehat{B}$ has exactly $2n$ zeros on $\mathbb{T}$. It remains to note that from the form of Blaschke products for the annulus with zeros on the unite circle (see [9]) the equality

$$\widehat{B}(t)=\varepsilon B_0(t)$$

follows, where $\varepsilon=1$ or -1. Putting $f_0=\varepsilon\widehat{f}$, we obtain the assertion of the lemma. □

Proposition 1. *Let $0\le\tau_1<\ldots<\tau_{2n}<2\pi$ and f_0 be the function from Lemma 2. Then for any $\xi\in\mathbb{T}$ and any function $f\in H^r_{\infty,\beta}$ such that $f(\tau_1)=\ldots=f(\tau_{2n})=0$ the inequality*

$$|f(\xi)|\le|f_0(\xi)|$$

holds.

Proof. Suppose that for some $\xi\in\mathbb{T}\setminus\{\tau_1,\ldots,\tau_{2n}\}$ there exists a function $g\in H^r_{\infty,\beta}$ for which $g(\tau_1)=\ldots=g(\tau_{2n})=0$ and $|g(\xi)|>|f_0(\xi)|$. Since the function $\widehat{g}(z)=g(z)\exp(-i\arg g(\xi))$ satisfies the same conditions without loss of generality we may assume that $g(\xi)>0$. Set

$$g_0(z):=\frac{g(z)+\overline{g(\overline{z})}}{2}.$$

Evidently that $g_0\in H^{r,\mathbb{R}}_{\infty,\beta}$ and $g_0(\xi)=g(\xi)$. Consider the function

$$F:=f_0-\rho g_0,\quad \rho=\frac{f_0(\xi)}{g_0(\xi)}.$$

This function has zeros at the points $\tau_1,\ldots,\tau_{2n},\xi$. Consequently, by Rolle's theorem $F^{(r)}$ has at least $2n+1$ zeros on $\mathbb{T}$. Hence it follows that the single valued and analytic in the annulus Ω_β function $F^{(r)}\left(\frac{1}{i}\ln w\right)$ has on this annulus at least $2n+1$ zeros. Since on the boundary Ω_β a Blaschke product satisfies the condition

$$\left|B_0\left(\frac{1}{i}\ln w\right)\right|=1,\quad w\in\partial\Omega_\beta,$$

and $f_0^{(r)} = B_0$, for $w \in \partial\Omega_\beta$ we have

$$\left| f_0^{(r)}\left(\frac{1}{i}\ln w\right) - F^{(r)}\left(\frac{1}{i}\ln w\right)\right| = \left|\rho g_0^{(r)}\left(\frac{1}{i}\ln w\right)\right| \le |\rho| < 1 = \left| f_0^{(r)}\left(\frac{1}{i}\ln w\right)\right|.$$

As $B_0\left(\frac{1}{i}\ln w\right)$ has exactly $2n$ zeros on the domain Ω_β, by Rouche's theorem the function $F^{(r)}\left(\frac{1}{i}\ln w\right)$ must have the same number of zeros. The contradiction so obtained completes the proof of the theorem. □

Theorem 3. *For all $\xi \in \mathbb{T}$ the method*

$$f(\xi) \approx \sum_{j=1}^{2n} C_j(\xi) f(\tau_j),$$

in which $C_1(\xi), \ldots, C_{2n}(\xi)$ are the solutions of the system

$$\begin{pmatrix} 1 & \ldots & 1 & 0 \\ f_1(\tau_1) & \ldots & f_1(\tau_{2n}) & a_0(g_1) \\ \cdots & \cdots & \cdots & \cdots \\ f_{2n}(\tau_1) & \ldots & f_{2n}(\tau_{2n}) & a_0(g_{2n}) \end{pmatrix} \begin{pmatrix} C_1(\xi) \\ C_2(\xi) \\ \vdots \\ C_{2n+1}(\xi) \end{pmatrix} = \begin{pmatrix} 1 \\ f_1(\xi) \\ \vdots \\ f_{2n}(\xi) \end{pmatrix}, \tag{6}$$

where

$$f_m = (D_r * g_m), \quad g_m(t) = B_0(t)\,\mathrm{ctn}\left(\frac{K}{\pi}(t-\theta_m)\right), \quad m = 1, \ldots, 2n,$$

and the function B_0 with the zeros θ_m is defined by Lemma 2, is optimal on the class $H^r_{\infty,\beta}$. Moreover, for the error of optimal recovery the equality

$$e(\xi, H^r_{\infty,\beta}, I_\tau) = |(D_r * B_0)(\xi) - (D_r * B_0)(\tau_1)|$$

holds.

Proof. Set $\theta_0 := -(D_r * B_0)(\tau_1)$ and

$$B_P(t) := k^n \prod_{j=1}^{2n} \mathrm{sn}\left(\frac{K}{\pi}(t - t_j)\right), \quad f_P := t_0 + D_r * B_P,$$

where $P = (t_0, t_1, \ldots, t_{2n})$. Then in view of duality (1) and Proposition 1 for $P = P_0 := (\theta_0, \theta_1, \ldots, \theta_{2n})$ the function f_{P_0} is extremal in the problem of optimal recovery of $f(\xi)$ on the class $H^r_{\infty,\beta}$ and on the class $H^{r,\mathbb{R}}_{\infty,\beta}$, too. Put $\tau_0 := \xi$,

$$\varphi_j(P) := f_P(\tau_j),\ j = 0, \ldots, 2n, \quad \psi(P) := a_0(B_P).$$

We have
$$\frac{\partial\varphi_j}{\partial t_0}=1,\ j=0,\dots,2n,\quad \frac{\partial\psi}{\partial t_0}=0,$$
and for all $m=1,\dots,2n$
$$\frac{\partial\varphi_j}{\partial t_m}\Big|_{P_0}=-\frac{K}{\pi}f_m(\tau_j),\quad j=0,\dots,2n,\quad \frac{\partial\psi}{\partial t_m}\Big|_{P_0}=-\frac{K}{\pi}a_0(g_m).$$
From Theorem 1 it follows that the coefficients of optimal recovery method are determined from the system (6) under the condition that the determinant of this system does not vanish. If we assume the converse, then there should exist real $\alpha_0,\alpha_1,\dots,\alpha_{2n}$ not all equal zero for which the function
$$g=\alpha_0+D_r*\left(\sum_{j=1}^{2n}\alpha_jg_j\right)$$
vanishes at the points $\tau_1,\dots,\tau_{2n}$ and furthermore
$$a_0\left(\sum_{j=1}^{2n}\alpha_jg_j\right)=0.$$
Let $\tau_0\in\mathbb{T}\setminus\{\tau_1,\dots,\tau_{2n}\}$. Consider the function
$$F:=g-\rho f_{P_0},\quad \rho=\frac{g(\tau_0)}{f_{P_0}(\tau_0)}.$$
The function F has zeros at the points $\tau_0,\tau_1,\dots,\tau_{2n}$. Consequently, by Rolle's theorem $F^{(r)}$ has at least $2n+1$ zeros on $\mathbb{T}$. We have
$$F^{(r)}(t)=\sum_{j=1}^{2n}\alpha_jg_j(t)-\rho B_0(t)=B_0(t)\left(\sum_{j=1}^{2n}\alpha_j\operatorname{ctn}\left(\frac{K}{\pi}(t-\theta_j)\right)-\rho\right).$$
From Lemma 1 it follows that $F^{(r)}$ can not have more than $2n$ zeros on $\mathbb{T}$. The contradiction so obtained proves that the determinant of the system (6) does not vanish.

To prove the optimality of constructed method on the class $H^r_{\infty,\beta}$ one may use the same arguments which were realized in the proof of Theorem 2. $\square$

Denote by A the matrix of the system (6). Then the optimal method constructed in Theorem 3 will have the form
$$f(\xi)\approx(A^{-1}\mathbf{G}(\xi),\mathbf{f})=(\mathbf{G}(\xi),(A^*)^{-1}\mathbf{f}),$$
where $\mathbf{G}(\xi)=(1,f_1(\xi),\dots,f_{2n}(\xi))$, $\mathbf{f}=(f(\tau_1),\dots,f(\tau_{2n}),0)$ and A^* is the matrix conjugated to A (here $(\cdot,\cdot)$ is the standard scalar product in $\mathbb{R}^{2n+1}$). Thus the optimal method can be written in the form
$$f(\xi)\approx d_0+\sum_{m=1}^{2n}d_m(D_r*g_m)(\xi),$$

where $d_0, d_1, \dots, d_{2n}$ are the solutions of the system

$$\left\{\begin{aligned} d_0 + \sum_{m=1}^{2n} d_m (D_r * g_m)(\tau_j) = f(\tau_j), \quad j = 1, \dots, 2n, \\ \sum_{m=1}^{2n} d_m a_0(g_m) = 0. \end{aligned}\right. \tag{7}$$

Let X_{2n}^{θ} is the set of functions of the form

$$c_0 + \sum_{n=1}^{2n} c_m (D_r * g_m),$$

where $c_1, \dots, c_{2n}$ satisfy the condition

$$\sum_{n=1}^{2n} c_m a_0(g_m) = 0.$$

Then, taking into account (7), we obtain

Corollary 2. *Let $0 \le \tau_1 < \dots < \tau_{2n} < 2\pi$ and $0 \le \theta_1 < \dots < \theta_{2n} < 2\pi$ be defined by Lemma 2. Then the function $g(\xi) \in X_{2n}^{\theta}$ interpolated f at the points $\tau_1, \dots, \tau_{2n}$ is an optimal method of recovery of $f(\xi)$, $\xi \in \mathbb{T}$, on the class $H_{\infty,\beta}^r$ by the values at the points $\tau_1, \dots, \tau_{2n}$.*

Consider the problem of optimal recovery for the equidistant points

$$\tau_m^0 := \frac{m-1}{n}\pi, \quad m = 1, \dots, 2n.$$

Put

$$t_{mr} := \begin{cases} \tau_m^0, & r = 2l, \\ \tau_m^0 + \dfrac{\pi}{2n}, & r = 2l+1, \end{cases} \quad m = 1, \dots, 2n.$$

Theorem 4. *For all $\xi \in \mathbb{T}$ the method*

$$f(\xi) \approx \widehat{f}_1 + \frac{1}{2n} \sum_{m=1}^{2n} \left(\sum_{j=2}^{2n} \frac{\widehat{f}_j}{\widehat{c}_j} e^{-i(m-1)(j-1)\pi/n} \right) (D_r * \sigma)(\xi - t_{mr}),$$

where

$$\left.\begin{aligned} \widehat{f}_j &= \frac{1}{2n} \sum_{m=1}^{2n} f(\tau_m^0) e^{i(m-1)(j-1)\pi/n}, \\ \widehat{c}_j &= \frac{1}{2n} \sum_{m=1}^{2n} (D_r * \sigma)(t_{mr}) e^{i(m-1)(j-1)\pi/n}, \end{aligned}\right. \quad j = 1, \dots, 2n,$$

is an optimal method of recovery on the class $H_{\infty,\beta}^r$ by the information about function values at the equidistant points τ_m^0, $m = 1, \dots, 2n$.

Proof. Let $r=2l$. Then $\Phi^\beta_{n,r}(\tau^0_m)=0$, $m=1,\dots,2n$. Since

$$\Phi^\beta_{n,r}=(D_r*\varphi^\beta_{n,0})\left(z-\frac{\pi}{2n}\right)$$

and

$$\varphi^\beta_{n,0}\left(z-\frac{\pi}{2n}\right)=-k^n\prod_{m=1}^{2n}\operatorname{sn}\left(\frac{k}{\pi}(z-\tau^0_m)\right),$$

for the system of points τ^0_m the points θ_m from Lemma 2 coincide with τ^0_m. Thus

$$g_m(z)=(-1)^{m+1}\sqrt{\lambda}\sigma(z-\tau^0_m).$$

The system (7) in the considered case will have the form

$$\begin{cases} d_0+\sqrt{\lambda}\displaystyle\sum_{m=1}^{2n}(-1)^{m+1}d_m(D_r*\sigma)(\tau^0_j-\tau^0_m)=f(\tau_j), \quad j=1,\dots,2n,\\ \displaystyle\sum_{m=1}^{2n}(-1)^{m+1}d_m=0.\end{cases}$$

Putting $x_0=d_0$, $x_m=\sqrt{\lambda}(-1)^{m+1}d_m$, $c_m=(D_r*\sigma)(\tau^0_m)$, and using the periodicity and evenness of the function $D_r*\sigma$, we arrive at the system

$$\begin{pmatrix}1 & c_1 & c_2 & \dots & c_{2n}\\ 1 & c_{2n} & c_1 & \dots & c_{2n-1}\\ \hdotsfor{5}\\ 1 & c_2 & c_3 & \dots & c_1\\ 0 & 1 & 1 & \dots & 1\end{pmatrix}\begin{pmatrix}x_0\\ x_1\\ \vdots\\ x_{2n-1}\\ x_{2n}\end{pmatrix}=\begin{pmatrix}f(\tau^0_1)\\ f(\tau^0_2)\\ \vdots\\ f(\tau^0_{2n})\\ 0\end{pmatrix}.$$

Summing the first $2n$ equalities of this system and using the last equality, we find that $x_0=\widehat{f}_1$. Further, the solution may be easily found using the equality

$$U^*CU=\begin{pmatrix}\widehat{c}_1 & & & 0\\ & \widehat{c}_2 & & \\ & & \ddots & \\ 0 & & & \widehat{c}_{2n}\end{pmatrix},$$

where

$$U=\left\{\frac{1}{\sqrt{2n}}e^{i(j-1)(m-1)\pi/n}\right\}_{j,m=1}^{2n},\quad C=\begin{pmatrix}c_1 & c_2 & \dots & c_{2n}\\ c_{2n} & c_1 & \dots & c_{2n-1}\\ \hdotsfor{4}\\ c_2 & c_3 & \dots & c_1\end{pmatrix}.$$

If $r=2l+1$, then

$$\Phi^\beta_{n,r}\left(\tau^0_m+\frac{\pi}{2n}\right)=0,\quad m=1,\dots,2n.$$

Since

$$\Phi^{\beta}_{n,r}\left(z+\frac{\pi}{2n}\right)=(D_r*\varphi^{\beta}_{n,0})(z)$$

and

$$\varphi^{\beta}_{n,0}(z)=k^n\prod_{m=1}^{2n}\operatorname{sn}\left(\frac{K}{\pi}\left(z-\left(\tau^0_m+\frac{\pi}{2n}\right)\right)\right),$$

the points θ_m from Lemma 2 coincide in this case with the points $\tau^0_m+\dfrac{\pi}{2n}$. Consequently,

$$g_m(z)=(-1)^m\sqrt{\lambda}\sigma\left(z-\left(\tau^0_m+\frac{\pi}{2n}\right)\right).$$

Further arguments are carried out similarly to the even case. □

4. The non-periodic case

In the non-periodic case we call the Hardy–Sobolev class H^r_∞ the set of functions analytic in the unit disk $D:=\{z\in\mathbb{C}:|z|<1\}$ and satisfying the condition $|f^{(r)}(z)|<1$, $z\in D$. Denote by $H^{r,\mathbb{R}}_\infty$ the set of functions from H^r_∞ real on the interval $(-1,1)$. For $r=0$ we denote the class H^r_∞ by H_∞.

Consider the problem of optimal recovery of $f(\xi)$, $\xi\in(-1,1)$ by the values of the information operator

$$I_\tau f=(f(\tau_1),\dots,f(\tau_{n+r})),$$

where $-1<\tau_1<\ldots<\tau_{n+r}<1$. In the case $r=0$ the solution of the considered problem was obtained in [2], therefore we assume that $r\geq 1$.

For functions f analytic in the unit disk we put

$$(T_rf)(z):=\int_0^z\frac{(z-\zeta)^{r-1}}{(r-1)!}f(\zeta)\,d\zeta.$$

Obviously, $(T_rf)^{(r)}=f$ and consequently, $T_rf\in H^r_\infty$ for all $f\in H_\infty$.

From [12] the following result follows.

Proposition 2. *For all $-1<\tau_1<\ldots<\tau_{n+r}<1$ there exist such $\tau_1<z_1<\ldots<z_n<\tau_{n+r}$ that for the function $f_0\in H^r_\infty$ of the form*

$$f_0=P_{r-1}+T_rB_0,$$

where P_{r-1} is a polynomial of degree $r-1$ and B_0 is a Blaschke product of degree n

$$B_0(z)=\prod_{j=1}^{n}\frac{z-z_j}{1-z_jz},$$

the equalities

$$f_0(\tau_j)=0,\quad j=1,\dots,n+r,$$

hold. Moreover, for all $\xi \in (-1,1)$

$$\sup_{\substack{f\in H_\infty^r \\ f(\tau_1)=\ldots=f(\tau_{n+r})=0}} |f(\xi)| = |f_0(\xi)|. \tag{8}$$

By the equality (8) it immediately follows that for the error of optimal recovery of $f(\xi)$ on the class H_∞^r the equality

$$e(\xi, H_\infty^r, I_\tau) = |f_0(\xi)|$$

holds. An optimal method of recovery may be also obtained in this case from Theorem 1.

Theorem 5. *For all* $\xi \in (-1,1)$ *the method*

$$f(\xi) \approx \sum_{j=1}^{n+r} C_j(\xi) f(\tau_j)$$

in which $C_1(\xi), \ldots, C_{n+r}(\xi)$ *are the solutions of the system*

$$\begin{pmatrix} 1 & \ldots & 1 \\ \tau_1 & \ldots & \tau_{n+r} \\ \ldots & \ldots & \ldots \\ \tau_1^{r-1} & \ldots & \tau_{n+r}^{r-1} \\ (T_r g_1)(\tau_1) & \ldots & (T_r g_1)(\tau_{n+r}) \\ \ldots & \ldots & \ldots \\ (T_r g_n)(\tau_1) & \ldots & (T_r g_n)(\tau_{n+r}) \end{pmatrix} \begin{pmatrix} C_1(\xi) \\ C_2(\xi) \\ \vdots \\ C_{n+r}(\xi) \end{pmatrix} = \begin{pmatrix} 1 \\ \xi \\ \vdots \\ \xi^{r-1} \\ (T_r g_1)(\xi) \\ \vdots \\ (T_r g_n)(\xi) \end{pmatrix}, \tag{9}$$

where

$$g_m(z) = B_0(z)\frac{1-z^2}{(z-z_m)(1-z_m z)}, \quad m = 1, \ldots, n,$$

and the function B_0 *with the zeros* z_m *is defined by Proposition 2, is optimal on the class* H_∞^r.

Proof. For $P = (a_0, a_1, \ldots, a_{r-1}, t_1, \ldots, t_n) \in \mathbb{R}^{n+r}$ we set

$$f_P(z) := \sum_{j=0}^{r-1} a_j z^j + (T_r B_P)(z),$$

where

$$B_P(z) = \prod_{j=1}^{n} \frac{z-t_j}{1-t_j z}.$$

Let the polynomial P_{r-1} from Proposition 2 has the form

$$P_{r-1} = \sum_{j=0}^{r-1} a_j^0 z^j.$$

Then for $P = P_0 := (a_0^0, a_1^0, \dots, a_{r-1}^0, z_1, \dots, z_n)$ the function f_{P_0} is extremal in the problem of optimal recovery of $f(\xi)$ on the classes H_∞^r and $H_\infty^{r,\mathbb{R}}$. Put $\tau_0 := \xi$,

$$\varphi_j(P) := f_P(\tau_j), \quad j = 0, \dots, n+r.$$

We have for all $j = 0, \dots, n+r$

$$\frac{\partial \varphi_j}{\partial a_m} = \tau_j^m, \quad m = 0, \dots, r-1,$$
$$\frac{\partial \varphi_j}{\partial t_m} = (T_r g_m)(\tau_j), \quad m = 1, \dots, n+r.$$

To obtain an optimal method on the class $H_\infty^{r,\mathbb{R}}$ it remains to use Theorem 1, checking previously that the determinant of the system (9) does not vanish. If we assume that this determinant vanishes, than there exist $C_1, \dots, C_{n+r}$ not all equal zero such that the function

$$F(z) := \sum_{j=0}^{r-1} C_{j+1} z^j + \sum_{j=1}^{n} C_{j+r} (T g_j)(z)$$

vanishes at the points $\tau_1, \dots, \tau_{n+r}$. Then by Rolle's theorem there exist points $\tau_1 < \xi_1 < \dots < \xi_n < \tau_{n+r}$ at which $F^{(r)}$ vanishes. Thus

$$F^{(r)} = \sum_{j=1}^{n} C_{j+r} g_j(\xi_m) = 0, \quad m = 1, \dots, n.$$

It was proved in [13] that the system of functions

$$\frac{1}{(z-\xi_m)(1-\xi_m z)}, \quad m = 1, \dots, n,$$

is a Chebyshev system on the set $(-1,1) \setminus \{\xi_1, \dots, \xi_n\}$. Consequently, $g_1, \dots, g_m$ is a Chebyshev system on the set $(-1,1)$ and $C_{r+1} = \dots = C_{n+r} = 0$. Hence it follows that $C_1 = \dots = C_r = 0$.

The proof of optimality of the constructed method on the class H_∞^r is carrying out by the same scheme which was used in Theorem 2. $\square$

For fixed $-1 < z_1 < \dots < z_n < 1$ set

$$X_{n+r}^{\mathbf{z}} := \operatorname{span}\{1, z, \dots, z^{r-1}, (T_r g_1)(z), \dots, (T_r g_n)(z)\}.$$

Analogously to Corollary 2 we get

Corollary 3. *Let $-1 < \tau_1 < \dots < \tau_{n+r} < 1$ and $\tau_1 < z_1 < \dots < z_n < \tau_{n+r}$ are defined by Proposition 2. Then the function $g(\xi) \in X_{n+r}^{\mathbf{z}}$ interpolated f at the points $\tau_1, \dots, \tau_{n+r}$ is an optimal method of recovery of $f(\xi)$, $\xi \in (-1,1)$, on the class H_∞^r by the values at the points $\tau_1, \dots, \tau_{n+r}$.*

References

[1] S. A. Smolyak, *On optimal restoration of functions and functionals of them*, Candidate dissertation, Moscow State University, Moscow 1965. (Russian)

[2] K. Yu. Osipenko, "Best approximation of analytic functions from their values at a finite number of points", *Mat. Zametki* **19** (1976), 29–40; English transl. in *Math. Notes* **19** (1976), 17–23.

[3] G. G. Magaril-Il'yaev, K. Yu. Osipenko, "Optimal recovery of functionals based on inaccurate data", *Mat. Zametki* **50** (1991), 85–93; English transl. in *Math. Notes* **50** (1991), 1274–1279.

[4] S. D. Fisher, C. A. Micchelli, "The n-width of sets of analytic functions, *Duke Math. J.* **47**:4 (1980), 789–801.

[5] N. I. Akhiezer, *Elements of the theory of elliptic functions*, Nauka, Moscow 1970; English transl., Amer. Math. Soc., Providence, RI 1990.

[6] K. Yu. Osipenko, "Exact values of n-widths of Hardy-Sobolev classes", *Constr. Approx.* **13** (1997), 17–27.

[7] N. P. Korneichuk, *Exact constants in approximation theory*, Nauka, Moscow 1987; English transl., Cambridge University Press, 1991.

[8] K. Yu. Osipenko, K. Wilderotter, "Optimal information for approximating periodic functions", *Math. Comput.* **66**:220 (1997), 1579–1592.

[9] K. Yu. Osipenko, "On n-widths, optimal quadrature formulas, and optimal recovery of functions analytic in a strip", *Izv. Ross. Akad. Nauk. Ser. Mat.* **58**:4 (1994), 55–79; English transl. in *Russian Acad. Sci. Izv. Math.* **45** (1995), 55–78.

[10] A. Pinkus, *n-widths in approximation theory*, Springer-Verlag, Berlin 1985.

[11] K. Wilderotter, "Optimal approximation of periodic analytic functions with integrable boundary values", *J. Approx. Theory.* **84**:2 (1996), 236–246.

[12] S. D. Fisher, "Envelopes, widths, and Landau problems for analytic functions", *Constr. Approx.* **5**:2 (1989), 171–187.

[13] B. D. Bojanov, G. R. Grozev, "A note on the optimal recovery of functions in H^∞", *J. Approx. Theory.* **53**:1 (1988), 67–77.

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