ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2023, Vol. 323, Suppl. 1, pp. S211–S221.

published in Trudy Instituta Matematiki i Mekhaniki UrO RAN, 2023, Vol. 29, No. 4, pp. 229–240.

Sharp Carlson Type Inequalities with Many Weights

K. Yu. Osipenko[1,2]

Received June 15, 2023; revised September 20, 2023; accepted September 25, 2023

Abstract—The paper is concerned with sharp Carlson type inequalities of the form

$$\|w(\cdot)x(\cdot)\|_{L_q(T)} \le K\|w_0(\cdot)x(\cdot)\|^{\gamma}_{L_p(T)} \max_{1\le j\le n} \|w_j(\cdot)x(\cdot)\|^{1-\gamma}_{L_r(T)},$$

where T is a cone in $\mathbb{R}^d$ and the weight functions $w_j(\cdot)$, $j = 1,\dots,n$, are homogeneous with some symmetry property.

Keywords: Carlson type inequalities, sharp constants.

DOI: 10.1134/S0081543823060184

1. INTRODUCTION

Suppose that T is some nonempty set, Σ is a σ-algebra of subsets of T, and μ is a nonnegative σ-additive measure on Σ. Denote by $L_p(T,\mu)$ the family of all Σ-measurable functions with values in $\mathbb{R}$ or $\mathbb{C}$ for which

$$\|x(\cdot)\|_{L_p(T,\mu)} = \left(\int_T |x(t)|^p\,d\mu\right)^{1/p} < \infty, \quad 1 \le p < \infty.$$

For $T \subset \mathbb{R}^d$ and $d\mu = dt$, $t \in \mathbb{R}^d$, we write $L_p(T) = L_p(T,\mu)$.

The Carlson inequality [1]

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \le \sqrt{\pi}\|x(t)\|^{1/2}_{L_2(\mathbb{R}_+)}\|tx(t)\|^{1/2}_{L_2(\mathbb{R}_+)}, \quad \mathbb{R}_+ = [0,+\infty),$$

was generalized by many authors (see [2–9]). In [7], a sharp constant was found in the inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le K\|w_0(\cdot)x(\cdot)\|^{\gamma}_{L_p(T,\mu)}\|w_1(\cdot)x(\cdot)\|^{1-\gamma}_{L_r(T,\mu)}, \tag{1.1}$$

where T is a cone in a linear space; $w(\cdot)$, $w_0(\cdot)$, and $w_1(\cdot)$ are homogeneous functions; μ is a homogeneous measure; and $1 \le q < p, r < \infty$ (for $T = \mathbb{R}^d$, the sharp constant was obtained in [5]). Recall that a constant K is called sharp if it cannot be replaced by a smaller value. The inequality in this case is called sharp.

[1]Faculty of Mechanics and Mathematics, Moscow State University, Moscow, 119991 Russia
[2]Kharkevich Institute for Information Transmission Problems of the Russian Academy of Sciences, Moscow, 127994 Russia
e-mail: kosipenko@yahoo.com

Finding the sharp constant in inequality (1.1) is closely related to the following extremal problem:

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \to \max, \quad \|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu)} \le \delta, \quad \|w_1(\cdot)x(\cdot)\|_{L_r(T,\mu)} \le 1,$$

where $\delta > 0$. In the present paper, we study the extremal problem

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \to \max, \quad \|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu)} \le \delta, \quad \|w_j(\cdot)x(\cdot)\|_{L_r(T,\mu)} \le 1, \quad j = 1,\dots,n, \tag{1.2}$$

where the functions $w(\cdot)$, $w_0(\cdot)$, and $w_j(\cdot)$, $j = 1,\dots,n$, are homogeneous and the functions $w_j(\cdot)$, $j = 1,\dots,n$, satisfy some additional symmetry properties. The obtained results are used to derive sharp Carlson type inequalities with many weights.

A series of general results concerning problem (1.2) were obtained in [9], but there the main attention was focused on problems of optimal recovery of linear operators, and sharp Carlson type inequalities were derived as corollaries of extremal problems arising in the construction of optimal recovery methods. In the present paper, we obtain these inequality directly.

2. HOMOGENEOUS WEIGHT FUNCTIONS ON A CONE IN A LINEAR SPACE

Suppose that T is a cone in a linear space; $\mu(\cdot)$ is a homogeneous measure of order d; $|w(\cdot)|$ and $|w_0(\cdot)|$ are homogeneous functions of orders θ and θ_0, respectively; and $|w_j(\cdot)|$, $j = 1,\dots,n$, are homogeneous functions of order θ_1. We will assume that $w(t), w_0(t) \ne 0$ and $\sum_{j=1}^n |w_j(t)| \ne 0$ for almost all $t \in T$. If $1 \le q < p, r < \infty$, then, for $k \in [0,1)$, the function $k^{1/(p-q)}(1-k)^{-1/(r-q)}$ monotonically increases from 0 to $+\infty$. Consequently, there exists a function $k(\cdot)$ such that

$$\frac{k^{1/(p-q)}(t)}{(1-k(t))^{1/(r-q)}} = \left|\frac{w(t)}{w_0(t)}\right|^{\frac{q(p-r)}{(p-q)(r-q)}} \left(\sum_{j=1}^n \left|\frac{w_j(t)}{w_0(t)}\right|^r\right)^{-1/(r-q)} \tag{2.1}$$

for almost all $t \in T$. Define

$$\gamma = \frac{\theta_1 - \theta - d(1/q - 1/r)}{\theta_1 - \theta_0 + d(1/r - 1/p)}. \tag{2.2}$$

Theorem 1. *Let $1 \le q < p, r < \infty$ and $\theta_1 - \theta - d(1/q - 1/r) \ne 0$. Assume that*

$$I_1 = \int_T \left|\frac{w(z)}{w_0(z)}\right|^{pq/(p-q)} k^{p/(p-q)}(z)\, d\mu(z) < \infty,$$

$$I_{j+1} = \int_T \frac{|w(z)|^{qr/(p-q)}}{|w_0(z)|^{pr/(p-q)}} |w_j(z)|^r k^{r/(p-q)}(z)\, d\mu(z) < \infty, \quad j = 1,\dots,n,$$

and, in addition, $I_2 = \ldots = I_{n+1}$. Then, for all $x(\cdot) \ne 0$ such that $w_0(\cdot)x(\cdot) \in L_p(T,\mu)$ and $w_j(\cdot)x(\cdot) \in L_r(T,\mu)$, $j = 1,\dots,n$, we have the sharp inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le K \|w_0(\cdot)x(\cdot)\|^{\gamma}_{L_p(T,\mu)} \max_{1\le j\le n} \|w_j(\cdot)x(\cdot)\|^{1-\gamma}_{L_r(T,\mu)}, \tag{2.3}$$

where

$$K = I_1^{-\gamma/p} I_2^{-(1-\gamma)/r} (I_1 + nI_2)^{1/q}. \tag{2.4}$$

To prove this theorem, we will need two lemmas. The first of them is essentially the sufficient condition of extremum from the Karush–Kuhn–Tucker theorem (see, for example, [10, p. 39]); we give this proof because it is very simple.

Let $f_j\colon A \to \mathbb{R}$, $j = 0, 1, \ldots, k$, be functions defined on some set A. Consider the extremal problem

$$f_0(x) \to \max, \quad f_j(x) \le 0, \quad j = 1, \ldots, k, \quad x \in A, \tag{2.5}$$

and its Lagrange function

$$\mathcal{L}(x, \lambda) = -f_0(x) + \sum_{j=1}^{k} \lambda_j f_j(x), \quad \lambda = (\lambda_1, \ldots, \lambda_k).$$

Lemma 1. *Assume that there exist $\widehat{\lambda}_j \ge 0$, $j = 1, \ldots, k$, and an element $\widehat{x} \in A$ feasible in problem* (2.5) *for which*

$$\text{(a)} \quad \min_{x \in A} \mathcal{L}(x, \widehat{\lambda}) = \mathcal{L}(\widehat{x}, \widehat{\lambda}), \quad \widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_k),$$

$$\text{(b)} \quad \widehat{\lambda}_j f_j(\widehat{x}) = 0, \quad j = 1, \ldots, k.$$

Then $\widehat{x}$ is an extremal element in problem (2.5)*.*

Proof. For any element $x \in A$ feasible in problem (2.5),

$$-f_0(x) \ge \mathcal{L}(x, \widehat{\lambda}) \ge \mathcal{L}(\widehat{x}, \widehat{\lambda}) = -f_0(\widehat{x}). \qquad \square$$

Lemma 2 is a special case of Lemma 3 from [7].

Lemma 2. *For all $a, b \ge 0$ such that $a + b > 0$ and all $1 \le q < p, r < \infty$, there exists a unique solution $\widehat{u} > 0$ of the equation*

$$q + pau^{p-q} + rbu^{r-q} = 0.$$

Moreover, for all $u \ge 0$,

$$-\widehat{u}^q + a\widehat{u}^p + b\widehat{u}^r \le -u^q + au^p + bu^r.$$

Proof of Theorem 1. Define

$$\widehat{x}(t) = |w_0(t)|^{-p/(p-q)} \Big(\frac{q|w(t)|^q}{p\lambda_0}\Big)^{1/(p-q)} k^{1/(p-q)}(\xi t),$$

where the parameters $\lambda_0, \xi > 0$ are chosen so that

$$\int\limits_T |w_0(t)|^p \widehat{x}^p(t)\, d\mu(t) = \delta^p, \quad \int\limits_T |w_j(t)|^r \widehat{x}^r(t)\, d\mu(t) = 1, \quad j = 1, \ldots, n. \tag{2.6}$$

Making the change $z = \xi t$ and using the homogeneity of the functions $w(\cdot)$, $w_0(\cdot)$, and $w_j(\cdot)$, $j = 1, \ldots, n$, and of the measure $\mu(\cdot)$, we obtain

$$\int\limits_T |w_0(t)|^p \widehat{x}^p(t)\, d\mu(t) = \Big(\frac{q}{p\lambda_0}\Big)^{p/(p-q)} I_1 \xi^{(\theta_0-\theta)qp/(p-q)-d}.$$

Similarly, we find

$$\int_T |w_j(t)|^r \widehat{x}^r(t)\,d\mu(t) = \Big(\frac{q}{p\lambda_0}\Big)^{r/(p-q)} I_{j+1}\xi^{(\theta_0-\theta)qr/(p-q)+r(\theta_0-\theta_1)-d}, \quad j=1,\dots,n.$$

Thus, equalities (2.6) have the form

$$\Big(\frac{q}{p\lambda_0}\Big)^{p/(p-q)} I_1\xi^{(\theta_0-\theta)qp/(p-q)-d} = \delta^p,$$

$$\Big(\frac{q}{p\lambda_0}\Big)^{r/(p-q)} I_{j+1}\xi^{(\theta_0-\theta)qr/(p-q)+r(\theta_0-\theta_1)-d} = 1, \quad j=1,\dots,n.$$

It is easy to see that these equalities are satisfied for

$$\xi = \big(\delta I_1^{-1/p} I_2^{1/r}\big)^{1/(\theta_1-\theta_0+d(1/r-1/p))}, \quad \lambda_0 = \frac{q}{p} I_1^{1-q/p}\xi^{(\theta_0-\theta)q-d(1-q/p)}\delta^{q-p}.$$

Consider the following extremal problem, which is equivalent to (1.2):

$$\int_T |w(t)|^q |x(t)|^q\,d\mu(t) \to \max, \quad \int_T |w_0(t)|^p |x(t)|^p\,d\mu(t) \le \delta^p,$$
$$\int_T |w_j(t)|^r |x(t)|^r\,d\mu(t) \le 1, \quad j=1,\dots,n. \tag{2.7}$$

The Lagrange function for this problem has the form

$$\mathcal{L}(x(\cdot),\overline{\lambda}) = \int_T L(t,x(t),\overline{\lambda})\,d\mu(t), \quad \overline{\lambda} = (\lambda_0,\lambda_1,\dots\lambda_n),$$

where

$$L(t,x(t),\overline{\lambda}) = -|w(t)|^q|x(t)|^q + \lambda_0|w_0(t)|^p|x(t)|^p + |x(t)|^r\sum_{j=1}^n \lambda_j|w_j(t)|^r.$$

From the definition of the function $\widehat{x}(\cdot)$, we get

$$p\lambda_0|w_0(t)|^p\widehat{x}^{p-q}(t) = q|w(t)|^q k(\xi t), \tag{2.8}$$

$$r\sum_{j=1}^n |w_j(t)|^r\widehat{x}^{r-q}(t) = r\sum_{j=1}^n |w_j(t)|^r|w_0(t)|^{-p(r-q)/(p-q)}\Big(\frac{q|w(t)|^q}{p\lambda_0}\Big)^{(r-q)/(p-q)} k^{(r-q)/(p-q)}(\xi t).$$

It follows from (2.1) and the homogeneity of the functions $|w(\cdot)|$, $|w_0(\cdot)|$, and $w_j(\cdot)$, $j=1,\dots,n$, that

$$k^{(r-q)/(p-q)}(\xi t) = \Big|\frac{w(\xi t)}{w_0(\xi t)}\Big|^{q(p-r)/(p-q)}\Big(\sum_{j=1}^n\Big|\frac{w_j(\xi t)}{w_0(\xi t)}\Big|^r\Big)^{-1}(1-k(\xi t))$$

$$= \xi^{(\theta-\theta_0)q(p-r)/(p-q)-(\theta_1-\theta_0)r}\Big|\frac{w(t)}{w_0(t)}\Big|^{q(p-r)/(p-q)}\Big(\sum_{j=1}^n\Big|\frac{w_j(t)}{w_0(t)}\Big|^r\Big)^{-1}(1-k(\xi t)).$$

Thus,

$$r\sum_{j=1}^{n}|w_j(t)|^r\widehat{x}^{r-q}(t)=r\Big(\frac{q}{p\lambda_0}\Big)^{(r-q)/(p-q)}\xi^{(\theta-\theta_0)q(p-r)/(p-q)-(\theta_1-\theta_0)r}|w(t)|^q(1-k(\xi t)).$$

Define

$$\lambda=\frac{q}{r}\Big(\frac{q}{p\lambda_0}\Big)^{-(r-q)/(p-q)}\xi^{(\theta_0-\theta)q(p-r)/(p-q)+(\theta_1-\theta_0)r}.$$

Then

$$r\lambda\sum_{j=1}^{n}|w_j(t)|^r\widehat{x}^{r-q}(t)=q|w(t)|^q(1-k(\xi t)). \tag{2.9}$$

Adding (2.8) and (2.9), we obtain

$$p\lambda_0|w_0(t)|^p\widehat{x}^{p-q}(t)+r\lambda\sum_{j=1}^{n}|w_j(t)|^r\widehat{x}^{r-q}(t)=q|w(t)|^q. \tag{2.10}$$

It follows from Lemma 2 that, for all functions $x(\cdot)$ feasible in (2.7) and almost all $t\in T$, the following inequality holds with $\overline{\lambda}=(\lambda_0,\lambda,\dots,\lambda)$:

$$L(t,\widehat{x}(t),\overline{\lambda})\le L(t,x(t),\overline{\lambda}).$$

Consequently,

$$\mathcal{L}(\widehat{x}(\cdot),\overline{\lambda})\le \mathcal{L}(x(\cdot),\overline{\lambda}).$$

Since equalities (2.6) are satisfied, we find from Lemma 1 that $\widehat{x}(\cdot)$ is an extremal function in problem (2.7). Using (2.10) and (2.6), we get

$$\sup_{\substack{\|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu)}\le\delta\\ \|w_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\le 1,\ j=1,\dots,n}}\|w(\cdot)x(\cdot)\|^q_{L_q(T,\mu)}=\int_T|w(t)|^q|\widehat{x}(t)|^q\,d\mu(t)$$

$$=q^{-1}\int_T\Big(p\lambda_0|w_0(t)|^p\widehat{x}^p(t)+r\lambda\sum_{j=1}^{n}|w_j(t)|^r\widehat{x}^r(t)\Big)\,d\mu(t)=\frac{p\lambda_0\delta^p+nr\lambda}{q}$$

$$=I_1^{1-q/p}\xi^{(\theta_0-\theta)q-d(1-q/p)}\delta^q+n\left(\frac{p\lambda_0}{q}\right)^{(r-q)/(p-q)}\xi^{(\theta_0-\theta)q(p-r)/(p-q)+(\theta_1-\theta_0)r}$$

$$=I_1^{1-q/p}\xi^{(\theta_0-\theta)q-d(1-q/p)}\delta^q+nI_1^{r/p-q/p}\xi^{(\theta_0-\theta)q-d(r/p-q/p)+(\theta_1-\theta_0)r}\delta^{q-r}=\delta^{q\gamma}K^q. \tag{2.11}$$

Let $x(\cdot)\ne 0$, $w_0(\cdot)x(\cdot)\in L_p(T,\mu)$, and $w_j(\cdot)x(\cdot)\in L_r(T,\mu)$ for $j=1,\dots,n$. Define

$$A=\max_{1\le j\le n}\|w_j(\cdot)x(\cdot)\|_{L_r(T,\mu)},\quad \delta=A^{-1}\|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu)}.$$

Then it follows from (2.11) that

$$A^{-q}\|w(\cdot)x(\cdot)\|^q_{L_q(T,\mu)}\le\delta^{q\gamma}K^q.$$

This implies inequality (2.3). If we assume that there exists a constant $K_1 < K$ for which (2.3) also holds, then

$$\sup_{\substack{\|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu)}\le\delta \\ \|w_j(\cdot)x(\cdot)\|_{L_r(T,\mu)}\le 1,\ j=1,\dots,n}} \|w(\cdot)x(\cdot)\|^q_{L_q(T,\mu)} \le K_1^q\delta^{q\gamma} < K^q\delta^{q\gamma},$$

which contradicts (2.11).

The theorem is proved. □

The statement of Theorem 1 for $n=1$ was proved in [7].

3. HOMOGENEOUS WEIGHT FUNCTIONS ON A CONE IN $\mathbb{R}^d$

Assume that T is a cone in $\mathbb{R}^d$, $d\mu(t) = dt$, $|w(\cdot)|$ and $|w_0(\cdot)|$ are homogeneous functions of orders θ and θ_0, and $|w_j(\cdot)|$, $j=1,\dots,n$, are homogeneous functions of order θ_1. As before, we will assume that $w(t), w_0(t)\ne 0$ and $\sum_{j=1}^n |w_j(t)|\ne 0$ for almost all $t\in T$. Consider the spherical coordinate system

$$\begin{aligned} t_1 &= \rho\cos\omega_1,\\ t_2 &= \rho\sin\omega_1\cos\omega_2,\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ t_{d-1} &= \rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\cos\omega_{d-1},\\ t_d &= \rho\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\sin\omega_{d-1}. \end{aligned}$$

Define $\omega = (\omega_1,\dots,\omega_{d-1})$. For any function $f(\cdot)$ given on $\mathbb{R}^d$, we introduce the notation

$$\widetilde{f}(\omega) = |f(\cos\omega_1,\dots,\sin\omega_1\sin\omega_2\dots\sin\omega_{d-2}\sin\omega_{d-1})|.$$

Note that if the function $|f(\cdot)|$ is homogeneous of order κ, then $\widetilde{f}(\omega) = \rho^{-\kappa}|f(t)|$. Denote by Ω the range of ω when $t\in T$. Since T is a cone, it follows that Ω is independent of ρ.

Assume that $\gamma\in(0,1)$, where γ is given by (2.2). Define a number q^* by the formula

$$\frac{1}{q^*} = \frac{1}{q} - \frac{\gamma}{p} - \frac{1-\gamma}{r}.$$

It is easy to see that $q^* > q \ge 1$. In addition,

$$q^* = \frac{pqr(\theta_1-\theta_0+d(1/r-1/p))}{(\theta_1-\theta_0)r(p-q)-(\theta-\theta_0)q(p-r)}.$$

Define

$$J(\omega) = \sin^{d-2}\omega_1\sin^{d-3}\omega_2\dots\sin\omega_{d-2}.$$

Theorem 2. *Let $1\le q<p, r<\infty$ and $\gamma\in(0,1)$. Assume that*

$$I = \int\limits_{\Omega} \frac{\widetilde{w}^{q^*}(\omega)}{\widetilde{w}_0^{q^*\gamma}(\omega)\left(\sum_{k=1}^n \widetilde{w}_k^r(\omega)\right)^{q^*(1-\gamma)/r}} J(\omega)\,d\omega < \infty$$

and $I_1' = \ldots = I_n'$, where

$$I_j' = \int\limits_{\Omega} \frac{\widetilde{w}^{q^*}(\omega)\widetilde{w}_j^r(\omega)}{\widetilde{w}_0^{q^*\gamma}(\omega)\left(\sum_{k=1}^n \widetilde{w}_k^r(\omega)\right)^{q^*(1-\gamma)/r+1}} J(\omega)\,d\omega, \quad j=1,\dots,n.$$

Then, for all $x(\cdot)\neq 0$ such that $w_0(\cdot)x(\cdot)\in L_p(T)$ and $w_j(\cdot)x(\cdot)\in L_r(T)$, $j=1,\dots,n$, the following sharp inequality holds:

$$\|w(\cdot)x(\cdot)\|_{L_q(T)}\le \widetilde{K}\|w_0(\cdot)x(\cdot)\|^{\gamma}_{L_p(T)}\max_{1\le j\le n}\|w_j(\cdot)x(\cdot)\|^{1-\gamma}_{L_r(T)},$$

where

$$\widetilde{K}=\gamma^{-\gamma/p}\Big(\frac{1-\gamma}{n}\Big)^{-(1-\gamma)/r}\Big(\frac{B\left(q^*\gamma/p,q^*(1-\gamma)/r\right)I}{|\theta_1-\theta_0+d(1/p-1/r)|(\gamma r+(1-\gamma)p)}\Big)^{1/q^*};\tag{3.1}$$

here $B(\cdot,\cdot)$ is the Euler B-function.

Proof. We compute the quantity I_1 from Theorem 1 by passing to spherical coordinates. We have

$$I_1=\int\limits_T\Big|\frac{w(z)}{w_0(z)}\Big|^{pq/(p-q)}k^{p/(p-q)}(z)\,dz$$

$$=\int\limits_\Omega\Big(\frac{\widetilde{w}(\omega)}{\widetilde{w}_0(\omega)}\Big)^{qp/(p-q)}J(\omega)\,d\omega\int\limits_0^{+\infty}\rho^{(\theta-\theta_0)qp/(p-q)+d-1}k^{p/(p-q)}(\rho,\omega)\,d\rho.$$

Passing to spherical coordinates, we obtain the following equality for the function $k(\cdot)$:

$$\frac{k^{1/(p-q)}(\rho,\omega)}{(1-k(\rho,\omega))^{1/(r-q)}}=\rho^{\frac{(\theta-\theta_0)q(p-r)-(\theta_1-\theta_0)r(p-q)}{(p-q)(r-q)}}\frac{\widetilde{w}^{\frac{q(p-r)}{(p-q)(r-q)}}(\omega)\widetilde{w}_0^{p/(p-q)}(\omega)}{\Big(\sum_{j=1}^n\widetilde{w}_j^r(\omega)\Big)^{1/(r-q)}}.$$

Therefore,

$$\rho^{(\theta_1-\theta_0)r(p-q)-(\theta-\theta_0)q(p-r)}=\frac{(1-k(\rho,\omega))^{p-q}}{k^{r-q}(\rho,\omega)}\frac{\widetilde{w}^{q(p-r)}(\omega)\widetilde{w}_0^{p(r-q)}(\omega)}{\Big(\sum_{j=1}^n\widetilde{w}_j^r(\omega)\Big)^{p-q}}.$$

Fix $\omega\in\Omega$. Then

$$d\rho^{(\theta-\theta_0)qp/(p-q)+d}=\left(\frac{\widetilde{w}^{q(p-r)}(\omega)\widetilde{w}_0^{p(r-q)}(\omega)}{\big(\sum_{j=1}^n\widetilde{w}_j^r(\omega)\big)^{p-q}}\right)^{\zeta}d\frac{(1-k)^{(p-q)\zeta}}{k^{(r-q)\zeta}}$$

$$=-\zeta\left(\frac{\widetilde{w}^{q(p-r)}(\omega)\widetilde{w}_0^{p(r-q)}(\omega)}{\big(\sum_{j=1}^n\widetilde{w}_j^r(\omega)\big)^{p-q}}\right)^{\zeta}\frac{(1-k)^{(p-q)\zeta-1}}{k^{(r-q)\zeta+1}}(r-q+(p-r)k)\,dk,$$

where

$$\zeta=\frac{(\theta-\theta_0)qp+d(p-q)}{(p-q)((\theta_1-\theta_0)r(p-q)-(\theta-\theta_0)q(p-r))}=\frac{q^*(1-\gamma)}{r(p-q)}.$$

If ρ changes from 0 to $+\infty$, then k changes from 0 to 1 for $(\theta_1-\theta_0)r(p-q)-(\theta-\theta_0)q(p-r)<0$ and from 1 to 0 for $(\theta_1-\theta_0)r(p-q)-(\theta-\theta_0)q(p-r)>0$. Therefore,

$$\int\limits_0^{+\infty}\rho^{(\theta-\theta_0)qp/(p-q)+d-1}k^{p/(p-q)}(\rho,\omega)\,d\rho=\frac{p-q}{(\theta-\theta_0)qp+d(p-q)}\int\limits_0^{+\infty}k^{p/(p-q)}(\rho,\omega)\,d\rho^{(\theta-\theta_0)qp/(p-q)+d}$$

$$=\frac{1}{|(\theta_1-\theta_0)r(p-q)-(\theta-\theta_0)q(p-r)|}\left(\frac{\widetilde{w}^{q(p-r)}(\omega)\widetilde{w}_0^{p(r-q)}(\omega)}{\big(\sum_{j=1}^n\widetilde{w}_j^r(\omega)\big)^{p-q}}\right)^{\zeta}$$

$$\times \int_0^1 k^{p/(p-q)} \frac{(1-k)^{(p-q)\zeta-1}}{k^{(r-q)\zeta+1}}(r-q+(p-r)k)\,dk$$

$$= \frac{1}{|(\theta_1-\theta_0)r(p-q)-(\theta-\theta_0)q(p-r)|}\left(\frac{\widetilde{w}^{q(p-r)}(\omega)\widetilde{w}_0^{p(r-q)}(\omega)}{\bigl(\sum_{j=1}^n \widetilde{w}_j^r(\omega)\bigr)^{p-q}}\right)^{\zeta}(K_1+K_2),$$

where

$$K_1 = (r-q)\int_0^1 k^{\widehat{p}}(1-k)^{\widehat{q}-1}\,dk = (r-q)B(\widehat{p}+1,\widehat{q}),$$

$$K_2 = (p-r)\int_0^1 k^{\widehat{p}+1}(1-k)^{\widehat{q}-1}\,dk = (p-r)B(\widehat{p}+2,\widehat{q}) = (p-r)\frac{\widehat{p}+1}{\widehat{p}+\widehat{q}+1}B(\widehat{p}+1,\widehat{q}),$$

$$\widehat{p} = \frac{(\theta_1-\theta)qr-d(r-q)}{(\theta_1-\theta_0)r(p-q)-(\theta-\theta_0)q(p-r)} = q^*\frac{\gamma}{p},$$

$$\widehat{q} = \frac{(\theta-\theta_0)qp+d(p-q)}{(\theta_1-\theta_0)r(p-q)-(\theta-\theta_0)q(p-r)} = q^*\frac{1-\gamma}{r}.$$

Thus,

$$K_1+K_2 = p\frac{(\theta_1-\theta_0)r(p-q)-(\theta-\theta_0)q(p-r)}{(\theta_1-\theta_0)pr+d(p-r)}B(\widehat{p}+1,\widehat{q}) = \frac{pq}{q^*}B(\widehat{p}+1,\widehat{q})$$

$$= \frac{q\gamma}{q^*}\Bigl(\frac{\gamma}{p}+\frac{1-\gamma}{r}\Bigr)^{-1}B(\widehat{p},\widehat{q}).$$

Hence,

$$I_1 = \frac{\gamma}{pr|\theta_1-\theta_0+d(1/r-1/p)|}\Bigl(\frac{\gamma}{p}+\frac{1-\gamma}{r}\Bigr)^{-1}B(\widehat{p},\widehat{q})I.$$

Let us find I_2. We have

$$I_2 = \int_T \frac{|w(z)|^{qr/(p-q)}}{|w_0(z)|^{pr/(p-q)}}|w_1(z)|^r\,k^{r/(p-q)}(z)\,dz$$

$$= \int_\Omega \frac{\widetilde{w}^{qr/(p-q)}(\omega)}{\widetilde{w}_0^{pr/(p-q)}(\omega)}\,\widetilde{w}_1^r(\omega)J(\omega)\,d\omega\int_0^{+\infty}\rho^{(\theta-\theta_0)qr/(p-q)+(\theta_1-\theta_0)r+d-1}\,k^{r/(p-q)}(\rho,\omega)\,d\rho.$$

Fix $\omega\in\Omega$. Then

$$d\rho^{(\theta-\theta_0)qr/(p-q)+(\theta_1-\theta_0)r+d} = \left(\frac{\widetilde{w}^{q(p-r)}(\omega)\widetilde{w}_0^{p(r-q)}(\omega)}{\bigl(\sum_{j=1}^n \widetilde{w}_j^r(\omega)\bigr)^{p-q}}\right)^{\zeta_1} d\,\frac{(1-k)^{(p-q)\zeta_1}}{k^{(r-q)\zeta_1}}$$

$$= -\zeta_1\left(\frac{\widetilde{w}^{q(p-r)}(\omega)\widetilde{w}_0^{p(r-q)}(\omega)}{\bigl(\sum_{j=1}^n \widetilde{w}_j^r(\omega)\bigr)^{p-q}}\right)^{\zeta_1}\frac{(1-k)^{(p-q)\zeta_1-1}}{k^{(r-q)\zeta_1+1}}\,(r-q+(p-r)k)\,dk,$$

where

$$\zeta_1 = \frac{(\theta-\theta_0)qr+((\theta_1-\theta_0)r+d)(p-q)}{(p-q)((\theta_1-\theta_0)r(p-q)-(\theta-\theta_0)q(p-r))} = \frac{q^*(1-\gamma)}{r(p-q)}+\frac{1}{p-q}.$$

We have

$$\int\limits_0^{+\infty} \rho^{(\theta-\theta_0)qr/(p-q)+(\theta_1-\theta_0)r+d-1} k^{r/(p-q)}(\rho,\omega)\,d\rho$$

$$= \frac{p-q}{(\theta-\theta_0)qr+((\theta_1-\theta_0)r+d)(p-q)} \int\limits_0^{+\infty} k^{r/(p-q)}(\rho,\omega)\,d\rho^{(\theta-\theta_0)qr/(p-q)+(\theta_1-\theta_0)r+d}$$

$$= \frac{1}{|(\theta_1-\theta_0)r(p-q)-(\theta-\theta_0)q(p-r)|}\left(\frac{\widetilde{w}^{q(p-r)}(\omega)\widetilde{w}_0^{p(r-q)}(\omega)}{\left(\sum_{j=1}^n \widetilde{w}_j^r(\omega)\right)^{p-q}}\right)^{\zeta_1}(L_1+L_2);$$

here

$$L_1 = (r-q)\int\limits_0^1 k^{\widehat{p}-1}(1-k)^{\widehat{q}}\,dk = (r-q)B(\widehat{p},\widehat{q}+1),$$

$$L_2 = (p-r)\int\limits_0^1 k^{\widehat{p}}(1-k)^{\widehat{q}}\,dk = (p-r)B(\widehat{p}+1,\widehat{q}+1) = (p-r)\frac{\widehat{p}}{\widehat{p}+\widehat{q}+1}B(\widehat{p},\widehat{q}+1).$$

Thus,

$$L_1+L_2 = r\frac{(\theta_1-\theta_0)r(p-q)-(\theta-\theta_0)q(p-r)}{(\theta_1-\theta_0)pr+d(p-r)}B(\widehat{p},\widehat{q}+1) = \frac{qr}{q^*}B(\widehat{p},\widehat{q}+1)$$

$$= \frac{q(1-\gamma)}{q^*}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q}).$$

Consequently,

$$I_2 = \frac{1-\gamma}{pr|\theta_1-\theta_0+d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q})I_1'.$$

Since $I_1'+\ldots+I_n' = I$, we have $I_j' = I/n$, $j=1,\ldots,n$. Thus,

$$I_2 = \frac{1-\gamma}{pr|\theta_1-\theta_0+d(1/r-1/p)|}\left(\frac{\gamma}{p}+\frac{1-\gamma}{r}\right)^{-1}B(\widehat{p},\widehat{q})\frac{I}{n}.$$

It remains to substitute the expressions for I_1 and I_2 into (2.4).

Theorem 2 is proved. □

For $n=1$, the statement of Theorem 2 was proved in [5].

Let us give an example of weights that satisfy the conditions of Theorem 2. Let $T=\mathbb{R}_+^d$,

$$w(t) = (t_1^2+\ldots+t_d^2)^{\theta/2}, \quad w_0(t) = (t_1^2+\ldots+t_d^2)^{\theta_0/2}, \quad w_j(t) = t_j^{\theta_1},\ j=1,\ldots,d. \tag{3.2}$$

Assume that $\gamma\in(0,1)$. This is equivalent to the fact that $\theta_1+d(1/r-1/q)>\theta>\theta_0+d(1/p-1/q)$ or $\theta_1+d(1/r-1/q)<\theta<\theta_0+d(1/p-1/q)$.

It is easy to see that $\widetilde{w}(\cdot)=\widetilde{w}_0(\cdot)=1$ and $\widetilde{w}_j(\omega)=\widetilde{t}_j^{\,\theta_1}(\omega)$, $j=1,\ldots,d$, where

$$\begin{aligned}
\widetilde{t}_1(\omega) &= \cos\omega_1,\\
\widetilde{t}_2(\omega) &= \sin\omega_1\cos\omega_2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
\widetilde{t}_{d-1}(\omega) &= \sin\omega_1\sin\omega_2\ldots\sin\omega_{d-2}\cos\omega_{d-1},\\
\widetilde{t}_d(\omega) &= \sin\omega_1\sin\omega_2\ldots\sin\omega_{d-2}\sin\omega_{d-1}.
\end{aligned}$$

Note that $\sum\limits_{k=1}^{d}\widetilde{t}_k^{\,2}(\omega)=1$.

For the quantity I from Theorem 2, we have

$$I=\int\limits_{\Pi_+^{d-1}}\frac{J(\omega)\,d\omega}{\big(\sum_{k=1}^d\widetilde{t}_k^{\,r\theta_1}(\omega)\big)^{q^*(1-\gamma)/r}},\quad \Pi_+^{d-1}=[0,\pi/2]^{d-1}. \tag{3.3}$$

If $r\theta_1\le 2$, then

$$\sum_{k=1}^{d}\widetilde{t}_k^{\,r\theta_1}(\omega)\ge\sum_{k=1}^{d}\widetilde{t}_k^{\,2}(\omega)=1. \tag{3.4}$$

In the case where $r\theta_1>2$, by Hölder's inequality,

$$1=\sum_{k=1}^{d}\widetilde{t}_k^{\,2}(\omega)\le\Big(\sum_{k=1}^{d}\widetilde{t}_k^{\,r\theta_1}(\omega)\Big)^{2/(r\theta_1)}d^{1-2/(r\theta_1)}.$$

Thus,

$$\sum_{k=1}^{d}\widetilde{t}_k^{\,r\theta_1}(\omega)\ge d^{1-r\theta_1/2}. \tag{3.5}$$

It follows from (3.4) and (3.5) that $I<\infty$.

For I_j' we have

$$I_j'=\int\limits_{\Pi_+^{d-1}}\frac{\widetilde{t}_j^{\,r\theta_1}J(\omega)\,d\omega}{\big(\sum_{k=1}^d\widetilde{t}_k^{\,r\theta_1}(\omega)\big)^{q^*(1-\gamma)/r+1}},\quad j=1,\dots,d.$$

Consider the integrals

$$M_j=\int\limits_{\mathbb{R}_+^d\cap\mathbb{B}^d}\frac{\big(\sum_{k=1}^d t_k^2\big)^{\theta_1q^*(1-\gamma)/2}t_j^{r\theta_1}}{\big(\sum_{k=1}^d t_k^{r\theta_1}\big)^{q^*(1-\gamma)/r+1}}\,dt,\quad j=1,\dots,d,$$

where $\mathbb{B}^d$ is the unit ball in $\mathbb{R}^d$. If make a change of variables in the integral M_j, swapping the variables t_j and t_k, then the integral M_j turns into the integral M_k. Consequently, $M_1=\ldots=M_d$. Passing to spherical coordinates, we get $M_j=I_j'/d$, $j=1,\dots,d$. Thus, $I_1'=\ldots=I_d'$.

For the case under consideration from Theorem 2, we obtain the following statement.

Corollary 1. *Assume that* $1\le q<p,r<\infty$ *and one of the following inequalities holds*: $\theta_1+d(1/r-1/q)>\theta>\theta_0+d(1/p-1/q)$ *or* $\theta_1+d(1/r-1/q)<\theta<\theta_0+d(1/p-1/q)$. *Then, for the weights* (3.2) *and all* $x(\cdot)$ *for which* $w_0(\cdot)x(\cdot)\in L_p(\mathbb{R}_+^d)$ *and* $w_j(\cdot)x(\cdot)\in L_r(\mathbb{R}_+^d)$, $j=1,\dots,d$, *the following sharp inequality holds*:

$$\|w(\cdot)x(\cdot)\|_{L_q(\mathbb{R}_+^d)}\le\widetilde{K}\|w_0(\cdot)x(\cdot)\|^{\gamma}_{L_p(\mathbb{R}_+^d)}\max_{1\le j\le d}\|w_j(\cdot)x(\cdot)\|^{1-\gamma}_{L_r(\mathbb{R}_+^d)},$$

where the quantity $\widetilde{K}$ *is defined by equality* (3.1) *with* I *from* (3.3).

We give one more result for the weights (3.2).

Corollary 2. *Let* $1 \le q < p, r < \infty$, *and let the weights* (3.2) *be such that* $\theta = d(1-1/q)$, $\theta_0 = d - (\lambda + d)/p$, *and* $\theta_1 = d + (\mu - d)/r$, *where* $\lambda, \mu > 0$. *Define*

$$\alpha = \frac{\mu}{p\mu + r\lambda}, \quad \beta = \frac{\lambda}{p\mu + r\lambda}.$$

Then, for all $x(\cdot)$ *such that* $w_0(\cdot)x(\cdot) \in L_p(\mathbb{R}^d_+)$ *and* $w_j(\cdot)x(\cdot) \in L_r(\mathbb{R}^d_+)$, $j = 1, \dots, d$, *the following sharp inequality holds*:

$$\|w(\cdot)x(\cdot)\|_{L_q(\mathbb{R}^d_+)} \le C \|w_0(\cdot)x(\cdot)\|^{p\alpha}_{L_p(\mathbb{R}^d_+)} \max_{1\le j\le d} \|w_j(\cdot)x(\cdot)\|^{r\beta}_{L_r(\mathbb{R}^d_+)},$$

where

$$C = \frac{d^\beta}{(p\alpha)^\alpha (r\beta)^\beta} \Big(\frac{I}{\lambda+\mu} B\Big(\frac{\alpha}{1/q-\alpha-\beta}, \frac{\beta}{1/q-\alpha-\beta}\Big)\Big)^{1/q-\alpha-\beta},$$

$$I = \int\limits_{\Pi^{d-1}_+} \frac{J(\omega)\,d\omega}{\big(\sum_{k=1}^d \widetilde{t}_k^{\,r(d-1)+\mu}(\omega)\big)^{\beta/(1/q-\alpha-\beta)}}.$$

For $d = 1$ and $q = 1$, the statement of Corollary 2 was obtained in [2].

FUNDING

This work was supported by ongoing institutional funding. No additional grants to carry out or direct this particular research were obtained.

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

REFERENCES

1. F. Carlson, “Une inégalité,” Ark. Mat. Astr. Fys. **25B**, 1–5 (1934).
2. V. I. Levin, “Sharp constants in Carlson type inequalities,” Dokl. AN SSSR **59**, 635–638 (1948).
3. F. I. Andrianov, “Multidimensional analogs of Carlson inequalities and its generalizations,” Izv. Vyssh. Uchebn. Zaved. Ser. Mat., No. 1, 3–7 (1967).
4. V. V. Arestov, “Approximation of linear operators and related extremal problems,” Proc. Steklov Inst. Math. **138**, 31–44 (1977).
5. S. Barza, V. Burenkov, J. Pečarić, and L.-E. Persson, “Sharp multidimensional multiplicative inequalities for weighted L_p spaces with homogeneous weights,” Math. Ineq. Appl. **1** (1), 53–67 (1998). https://doi.org/10.7153/mia-01-04
6. M.-J. Luo and R. K. Raina, “A new extension of Carlson’s inequality,” Math. Ineq. Appl. **19** (2), 417–424 (2016). https://doi.org/10.7153/mia-19-33
7. K. Yu. Osipenko, “Optimal recovery of operators and multidimensional Carlson type inequalities,” J. Complexity **32** (1), 53–73 (2016). https://doi.org/10.1016/j.jco.2015.07.004
8. K. Yu. Osipenko, “Inequalities for derivatives with the Fourier transform,” Appl. Comp. Harm. Anal. **53**, 132–150 (2021). https://doi.org/10.1016/j.acha.2021.02.001
9. K. Yu. Osipenko, *Optimal Recovery and Generalized Carlson Inequality for Weights with Symmetry Properties* (2023). https://arxiv.org/pdf/2303.10355.pdf
10. K. Yu. Osipenko, *Convex Analysis* (URSS, Moscow, 2022) [in Russian].

Translated by E. Vasil’eva