ISMAGILOV TYPE THEOREMS FOR LINEAR, GEL'FAND AND BERNSTEIN n-WIDTHS

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Abstract. Using a variational principle for s-numbers, we obtain estimates for the linear, Gel'fand and Bernstein n-widths. A simple proof of some results concerned with the exact values of n-widths of diagonal operators is given. We also calculate the exact values of the Bernstein n-widths for the Hardy–Sobolev classes.

1. Introduction

Let X, Y be normed linear spaces and $T\colon X \to Y$ be a bounded linear operator. The linear λ_n, Gel'fand d^n and Bernstein b_n n-widths of the operator T are defined by

$$\lambda_n(T) := \inf_{P_n} \sup_{x \in BX} \|Tx - P_n x\|_Y, \qquad d^n(T) := \inf_{X^n} \sup_{x \in BX^n} \|Tx\|_Y,$$
$$b_n(T) := \sup_{X_{n+1}} \inf_{\substack{x \in X_{n+1} \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X},$$

where P_n is any linear operator mapping X into Y of rank at most n, BX is the closed unit ball of X, X^n runs over all n-codimensional subspaces of X and X_{n+1} runs over all $(n+1)$-dimensional subspaces of X.

In Osipenko and Stessin [1] the exact values of the linear and Gel'fand n-widths of the Hardy classes H_2 were obtained. A method of the proof was very close to the one from Ismagilov's Theorem [2] (see also [3, p.93]). After the paper [1] several results were obtained for $\lambda_n(T)$ and $d^n(T)$ where T is a map from a Hilbert space H into $C(E)$ (see [4]–[6]). Parfenov [7] solved an analogous problem for the Bernstein n-widths $b_n(T)$ where $T\colon L_\infty(E,\nu) \to H$ and ν is a probability measure on E.

In this paper we show that many of these results can be obtained, using a general principle concerned with extremal properties of s-numbers. Section 2 is devoted to this principle. In Section 3 we prove the estimates of the linear, Gel'fand and Bernstein n-widths. In Section 4 we give a simple proof of two results about the exact values of n-widths for diagonal operators in the discrete case. Finally, in Section 5 we calculate the Bernstein n-widths of the Hardy–Sobolev classes.

The research of the first author was supported in part by Russian Foundation of Fundamental Research and by the International Scientific Fund

The research of the second author was supported in part by the International Scientific Fund and by the Natural Sciences Academy of Russian Federation

2. Variational Principle for s-Numbers

Let H and H_1 be Hilbert spaces and $T\colon H \to H_1$ a bounded linear operator. Suppose that

$$T'T\varphi_k = \lambda_k\varphi_k, \quad k = 1, 2, \dots,$$

where $\lambda_1 \geq \lambda_2 \geq \dots > 0$ and $\{\varphi_k\}$ form a complete orthonormal basis for the range of $T'T$ (a sufficient condition is that T be a compact operator). The values $s_k(T) = \sqrt{\lambda_k}$ are called the s-numbers of T.

Set $\psi_k := s_k^{-1}(T)T\varphi_k$. Note that $\{\psi_k\}$ is an orthonormal system in H_1. Then there exists the Schmidt decomposition of T (see, for example, [8]) which is given by

$$T = \sum_{k=1}^{\infty} s_k(T)(\cdot, \varphi_k)\psi_k.$$

Theorem 1. *Let T be as above. Then*

$$\sum_{k=1}^{n} s_k^2(T) = \max_{\{e_k\}_1^n} \sum_{k=1}^{n} \|Te_k\|_{H_1}^2 \tag{1}$$

where the maximum is taken over all orthonormal systems $\{e_k\}_1^n$ in H. Furthemore,

$$\sum_{k=n+1}^{\infty} s_k^2(T) = \min_{\{e_k\}_1^\infty} \sum_{k=1}^{\infty} \|Te_k\|_{H_1}^2 \tag{2}$$

where the minimum is taken over all orthonormal systems $\{e_k\}_1^\infty$ in H such that $\operatorname{codim}\operatorname{span}\{e_k\}_1^\infty \leq n$.

For a compact operator T this theorem was proved in Parfenov [6]. In our case the proof is almost the same because it does not so much depend on the compactness of T as on the fact that the eigenvectors $\{\varphi_k\}$ of $T'T$ form an orthonormal basis for the range of $T'T$.

We remark that both parts in (2) are finite iff

$$\sum_{k=1}^{\infty} s_k^2(T) < \infty.$$

In Parfenov [6] Theorem 1 was the basic tool in calculating the Gel'fand n-widths of operators $T\colon H \to L_\infty(E, \nu)$. Similar results were obtained in Osipenko [5], using Ismagilov's Theorem and the duality between the Kolmogorov and Gel'fand n-widths.

In order to estimate the Bernstein n-widths we need the following properties of s-numbers.

Theorem 2. *Let T be as in Theorem 1 and* $\operatorname{Ker} T = 0$. *Then*

$$\sum_{k=1}^{n} s_k^{-2}(T) = \min_{\{f_k\}_1^n} \sum_{k=1}^{n} \|T^{-1}f_k\|_H^2 \tag{3}$$

where the minimum is taken over all orthonormal systems $\{f_k\}_1^n$ *in* $T(H)$. *Furthemore, if* $\dim H = N < \infty$, *then*

$$\sum_{k=n+1}^{N} s_k^{-2}(T) = \max_{\{f_k\}_1^{N-n}} \sum_{k=1}^{N-n} \|T^{-1} f_k\|_H^2 \tag{4}$$

where the maximum is taken over all orthonormal systems $\{f_k\}_1^{N-n}$ *in* $T(H)$.

Proof. Let $\{f_k\}_1^n$ be any orthonormal system in $T(H)$. Set $L_n := \operatorname{span}\{T^{-1} f_k\}_1^n$, $\widetilde{L}_n := \operatorname{span}\{f_k\}_1^n$, and $T_n := T\big|_{L_n}$. Suppose that the Schmidt decomposition of T_n has the form

$$T_n = \sum_{k=1}^{n} s_k(T_n)(\cdot, \varphi_k')\psi_k'.$$

The value

$$\left(\sum_{k=1}^{n} \|T_n^{-1} f_k\|_H^2\right)^{1/2}$$

is the Hilbert–Schmidt norm of T_n^{-1} and does not depend on the choice of the orthonormal basis in $\widetilde{L}_n$. Therefore

$$\sum_{k=1}^{n} \|T^{-1} f_k\|_H^2 = \sum_{k=1}^{n} \|T_n^{-1} f_k\|_H^2 = \sum_{k=1}^{n} \|T_n^{-1} \psi_k'\|_H^2 = \sum_{k=1}^{n} s_k^{-2}(T_n).$$

Let P_n be an orthoprojector in H onto L_n. Using the properties of s-numbers of bounded linear operators (see [8, p.82]) we have

$$s_k(T_n) = s_k(T \circ P_n) \leq \|P_n\| s_k(T) = s_k(T).$$

Thus

$$\sum_{k=1}^{n} \|T^{-1} f_k\|_H^2 \geq \sum_{k=1}^{n} s_k^{-2}(T).$$

If $f_k = \psi_k$, $k = 1, \ldots, n$, then

$$\sum_{k=1}^{n} \|T^{-1} \psi_k\|_H^2 = \sum_{k=1}^{n} s_k^{-2}(T).$$

The equality (3) is proved.

To prove (4) note that

$$s_k(T^{-1}) = s_{N-k+1}^{-1}(T), \quad k = 1, \ldots, N.$$

Now (4) follows from (1). The theorem is proved.

3. Estimates of Linear, Gel'fand and Bernstein n-Widths

To obtain estimates of n-widths we need the following simple result.

Lemma 1. *Let H be a Hilbert space, $\omega := \dim H$ and $T\colon H \to C(E)$ a bounded linear operator. Then*

$$\|T\|_{H\to C(E)} = \sup_{z\in E}\left(\sum_{k=1}^{\omega} |(Te_k)(z)|^2\right)^{1/2}$$

for any orthonormal basis $\{e_k\}_1^{\omega}$ in H.

Proof. We have

$$\|T\|_{H\to C(E)} = \sup_{h\in BH}\ \sup_{z\in E}|(Th)(z)| = \sup_{z\in E}\ \sup_{h\in BH}\ |(Th)(z)|$$
$$= \sup_{z\in E}\ \sup_{\{c_k\}_1^{\omega}\in Bl_2}\left|\sum_{k=1}^{\omega} c_k(Te_k)(z)\right| = \sup_{z\in E}\left(\sum_{k=1}^{\omega}|(Te_k)(z)|^2\right)^{1/2}.$$

The lemma is proved.

Let H be a Hilbert space of functions defined on some set Ω. A function $K(z,w)$ defined on $\Omega\times\Omega$ is called a reproducing kernel of H if for each $w\in\Omega$, $K(z,w)\in H$ and for all $f\in H$

$$f(w) = (f(\cdot), K(\cdot, w))_H.$$

It is a well-known fact that if the $\{\varphi_k\}_1^{\omega}$ form an orthonormal basis in H, then

$$K(z,w) = \sum_{k=1}^{\omega}\varphi_k(z)\overline{\varphi_k(w)}.$$

Suppose that Ω is a topological space, $E\subset\Omega$ and $Tf := f\big|_E$ is a bounded linear operator from H into $C(E)$. Then from Lemma 1 we obtain

$$\|T\|_{H\to C(E)} = \sup_{z\in E}(K(z,z))^{1/2}.$$

Theorem 3([5], [6])**.** *Let H be a Hilbert space, E a topological space with probability measure ν such that $\operatorname{supp}\nu = E$, and $T\colon H\to C(E)$ a bounded linear operator. Define $T_0\colon H\to L_2(E,\nu)$ by the equation $T_0h := Th$. Assume that*

$$T_0'T_0\varphi_k = s_k^2\varphi_k, \quad k = 1,2,\dots, \tag{5}$$

where $s_1\ge s_2\ge\dots>0$ and $\{\varphi_k\}$ is an orthonormal basis for the range of $T_0'T_0$. Then

$$\left(\sum_{k=n+1}^{\infty} s_k^2\right)^{1/2} \le \lambda_n(T) = d^n(T) \le \sup_{z\in E}\left(\sum_{k=n+1}^{\infty}|(T\varphi_k)(z)|^2\right)^{1/2}.$$

Proof. Since $\operatorname{supp}\nu = E$, $\operatorname{Ker} T_0'T_0 = \operatorname{Ker} T_0 = \operatorname{Ker} T$ and we can assume, without loss of generality, that $\{\varphi_k\}$ is an orthonormal basis in H. From the defenition of the Gel'fand n-width it follows that

$$d^n(T) = \inf_{H^n} \|T\|_{H^n \to C(E)}$$

where H^n runs over all n-codimensional subspaces of H. Consider $H^n = \{\varphi_k\}_{n+1}^{\infty}$. We obtain from Lemma 1

$$d^n(T) \le \sup_{z\in E} \left(\sum_{k=n+1}^{\infty} |(T\varphi_k)(z)|^2 \right)^{1/2}.$$

Let H^n be any n-codimensional subspace of H. Suppose that $\{\varphi_k'\}$ is an orthonormal basis in H^n. Using Lemma 1 and (2), we have

$$\begin{aligned} \|T\|^2_{H^n \to C(E)} = \sup_{z\in E} \sum_{k=1}^{\infty} |(T\varphi_k')(z)|^2 \ge \int_E \sum_{k=1}^{\infty} |(T\varphi_k')(z)|^2 \, d\nu(z) \\ = \sum_{k=1}^{\infty} \|T_0\varphi_k'\|^2_{L_2(E,\nu)} \ge \sum_{k=n+1}^{\infty} s_k^2. \end{aligned}$$

Thus

$$d^n(T) \ge \left(\sum_{k=n+1}^{\infty} s_k^2 \right)^{1/2}.$$

The equality $\lambda_n(T) = d^n(T)$ is the well-known fact for operators defined on Hilbert spaces (see, [3, p.33]). The theorem is proved.

Now we will obtain the similar estimates for the Bernstein n-widths.

Theorem 4. *Let H, E and ν be as above, H_1 be a subspace of $L_2(E,\nu)$ and $X_E \subset H_1$ be a subspace of $C(E)$. Assume that a bounded linear operator $T_0 \colon H_1 \to H$ satisfies the conditions (5) where $s_1 \ge s_2 \ge \dots > 0$, $\{\varphi_k\}$ is an orthonormal basis for the range of $T_0'T_0$ and $\varphi_k \in X_E$, $k = 1, 2, \dots$. Define $T \colon X_E \to H$ by the equation $Tf := T_0 f$. Then*

$$\left(\sup_{z\in E} \sum_{k=1}^{n+1} s_k^{-2} |\varphi_k(z)|^2 \right)^{-1/2} \le b_n(T) \le \left(\sum_{k=1}^{n+1} s_k^{-2} \right)^{-1/2}. \tag{6}$$

Proof. Let $L_{n+1} \subset X_E$ and $\dim L_{n+1} = n+1$. Consider the operator $T_{n+1} := T\big|_{L_{n+1}}$. If $\operatorname{Ker} T_{n+1} \ne 0$, then

$$\inf_{\substack{f\in L_{n+1} \\ f\ne 0}} \frac{\|Tf\|_H}{\|f\|_{C(E)}} = 0.$$

Suppose that $\operatorname{Ker} T_{n+1} = 0$. Then we can define the operator $T_{n+1}^{-1} \colon T(L_{n+1}) \to L_{n+1}$. Using Lemma 1 and (4), for any orthonormal system $\{e_k\}_1^{n+1}$ in $T(L_{n+1})$ we have

$$\inf_{\substack{f \in L_{n+1} \\ f \neq 0}} \frac{\|Tf\|_H}{\|f\|_{C(E)}} = \inf_{\substack{g \in T(L_{n+1}) \\ g \neq 0}} \frac{\|g\|_H}{\|T_{n+1}^{-1} g\|_{C(E)}} = \|T_{n+1}^{-1}\|_{T(L_{n+1}) \to C(E)}^{-1}$$
$$= \left(\sup_{z \in E} \sum_{k=1}^{n+1} |(T_{n+1}^{-1} e_k)(z)|^2 \right)^{-1/2} \le \left(\sum_{k=1}^{n+1} \int_E |(T_{n+1}^{-1} e_k)(z)|^2 \, d\nu(z) \right)^{-1/2}$$
$$= \left(\sum_{k=1}^{n+1} \|T_{n+1}^{-1} e_k\|_{H_1}^2 \right)^{-1/2} \le \left(\sum_{k=1}^{n+1} s_k^{-2}(T_{n+1}) \right)^{-1/2}.$$

Since $s_k(T_{n+1}) \le s_k(T_0) = s_k$, $k = 1, \ldots, n+1$, we obtain

$$b_n(T) \le \left(\sum_{k=1}^{n+1} s_k^{-2} \right)^{-1/2}.$$

Let $L_{n+1} = \operatorname{span}\{\varphi_k\}_1^{n+1}$. Then $\psi_k := s_k^{-1} T \varphi_k$, $k = 1, \ldots, n+1$, form an orthonormal system in $T(L_{n+1})$. Thus

$$b_n(T) \ge \inf_{\substack{f \in L_{n+1} \\ f \neq 0}} \frac{\|Tf\|_H}{\|f\|_{C(E)}} = \left(\sup_{z \in E} \sum_{k=1}^{n+1} |(T_{n+1}^{-1} \psi_k)(z)|^2 \right)^{-1/2}$$
$$= \left(\sup_{z \in E} \sum_{k=1}^{n+1} s_k^{-2} |\varphi_k(z)|^2 \right)^{-1/2}.$$

The theorem is proved.

4. n-Widths of Diagonal Operators

Let $T \colon l_2 \to l_\infty$ be the diagonal operator

$$T\left(\{x_k\}_1^\infty\right) := \{\lambda_k x_k\}_1^\infty \tag{7}$$

where $\lambda_1 \ge \lambda_2 \ge \cdots > 0$. Smolyak [9] (in the finite-dimensional case) proved that

$$d^n(T) = \sup_{m > n} \left(\frac{m - n}{\sum_{k=1}^m \lambda_k^{-2}} \right)^{1/2}. \tag{8}$$

In dual terms this result was obtained by Sofman [10] (see also [11]). We will show that the lower bound in (8) easily follows from Theorem 3.

Denote by $\{e_k\}_1^\infty$ the standard basis of l_2. Fix any $m > n$. Let $T_m \colon l_2^m \to l_\infty^m$ be the operator defined by

$$T_m\left(\{x_k\}_1^m\right) := \{\lambda_k x_k\}_1^m.$$

It is easy to see that $d^n(T) \geq d^n(T_m)$. Define the probability measure ν_m on the set $\{1, 2, \ldots, m\}$ as follows

$$\nu_m(\{j\}) := \lambda_j^{-2} \left(\sum_{k=1}^{m} \lambda_k^{-2} \right)^{-1} .$$

Denote by T_{m0} the operator T_m regarded as an operator from l_2^m into $l_2^m(\nu_m)$. Then

$$T'_{m0} T_{m0} e_j = \left(\sum_{k=1}^{m} \lambda_k^{-2} \right)^{-1} e_j, \quad j = 1, \ldots, m.$$

From Theorem 3

$$d^n(T_m) \geq \left(\frac{m-n}{\sum_{k=1}^{m} \lambda_k^{-2}} \right)^{1/2} .$$

Thus

$$\lambda_n(T) = d^n(T) \geq \sup_{m>n} d^n(T_m) \geq \left(\frac{m-n}{\sum_{k=1}^{m} \lambda_k^{-2}} \right)^{1/2} .$$

Note that the values (8) are also related to linear stochastic n-widths (see [12]).

Consider the operator (7) as an operator from l_∞ into l_2. Here we assume that $\{\lambda_k\}_1^\infty \in l_2$. Galeev [13] proved the equality

$$b_n(T) = \min_{0 \leq m < n+1} \left(\frac{\sum_{k=m+1}^{\infty} \lambda_k^2}{n-m+1} \right)^{1/2} .$$

We will show how the upper bound can be obtained from Theorem 4.

Let $0 < \varepsilon < 1$ and $0 \leq m < n+1$. Define the probability measure ν_m on $\mathbb{N}$ as follows

$$\nu_m(\{j\}) := \begin{cases} (1-\varepsilon) \dfrac{\lambda_j^2}{\sum_{k=m+1}^{\infty} \lambda_k^2}, & j > m \\ \dfrac{\varepsilon}{m}, & j \leq m. \end{cases}$$

Denote $T_0 \colon l_2(\nu_m) \to l_2$ by the equation $T_0 x := Tx$. It is easy to obtain that

$$T'_0 T_0 e_j = s_j^2 e_j, \quad j = 1, 2, \ldots,$$

where

$$s_j^2 = \begin{cases} (1-\varepsilon)^{-1} \sum_{k=m+1}^{\infty} \lambda_k^2, & j > m \\ \varepsilon^{-1} \lambda_j^2 m, & j \leq m. \end{cases}$$

From Theorem 4

$$b_n(T) \leq \left((1-\varepsilon) \frac{n-m+1}{\sum_{k=m+1}^{\infty} \lambda_k^2} + \frac{\varepsilon}{m} \sum_{j=1}^{m} \lambda_j^{-2} \right)^{-1/2} .$$

Letting $\varepsilon \to 0$, we obtain

$$b_n(T) \leq \min_{0 \leq m < n+1} \left(\frac{\sum_{k=m+1}^{\infty} \lambda_k^2}{n-m+1} \right)^{1/2} .$$

5. Bernstein n-Widths of Hardy–Sobolev Classes

Let A be a closed, convex centrally symmetric subset of a normed linear space Y. The Bernstein n-width of A is defined by

$$b_n(A,Y) := \sup_{Y_{n+1}} \sup\{\,\lambda : \lambda BY_{n+1} \subset A\,\}$$

where Y_{n+1} is any $(n+1)$-dimensional subspace of Y. If $\operatorname{Ker} T = 0$, then it is easily shown that

$$b_n(T) = b_n(T(BX), Y).$$

We need the following simple property of b_n.

Lemma 2. *Let H be a Hilbert space, A a closed, convex centrally symmetric subset of H and H_r an r-dimensional subspace of H such that $A \perp H_r$. Then*

$$b_{n+r}(A + H_r, H) = b_n(A, H). \tag{9}$$

Proof. Assume that $H_{n+1} \subset H$, $\dim H_{n+1} = n+1$ and

$$\sup\{\,\lambda : \lambda BH_{n+1} \subset A\,\} = \mu > 0.$$

Put $H_{n+r+1} := H_{n+1} + H_r$. Since $A \perp H_r$ it follows that $H_{n+1} \perp H_r$ and $\dim H_{n+r+1} = n+r+1$. If $x \in H_{n+r+1}$, $\|x\|_H \le \mu$, then $x = x_1 + x_2$ where $x_1 \in H_{n+1}$, $x_2 \in H_r$ and

$$\|x_1\|_H^2 \le \|x_1\|_H^2 + \|x_2\|_H^2 = \|x\|_H^2 \le \mu^2.$$

Thus $x_1 \in A$. Consequently $x \in A + H_r$. We have

$$\sup\{\,\lambda : \lambda BH_{n+r+1} \subset A + H_r\,\} \ge \mu.$$

So we proved that

$$b_{n+r}(A + H_r, H) \ge b_n(A, H). \tag{10}$$

If $b_{n+r}(A + H_r, H) = 0$, then (9) follows from (10). Suppose that $b_{n+r}(A + H_r, H) > 0$. Let $H_{n+r+1} \subset H$, $\dim H_{n+r+1} = n+r+1$ and

$$\sup\{\,\lambda : \lambda BH_{n+r+1} \subset A + H_r\,\} = \mu > 0.$$

Since $H_{n+r+1} \subset \operatorname{span} A + H_r$, $\dim(H_{n+r+1} \cap \operatorname{span} A) \ge n+1$. Hence there exists a subspace $H_{n+1} \subset H_{n+r+1} \cap \operatorname{span} A$ with $\dim H_{n+1} = n+1$. Let $x \in H_{n+1}$ and $\|x\|_H \le \mu$. Then $x \in A + H_r$. In addition $x \in \operatorname{span} A$. Thus $x \in A$ and

$$\sup\{\,\lambda : \lambda BH_{n+1} \subset A\,\} \ge \mu.$$

Therefore

$$b_n(A, H) \ge b_{n+r}(A + H_r, H).$$

The lemma is proved.

Theorem 5. *Let H, E, ν, H_1 and X_E be as in Theorem 4. Suppose that the Hilbert space H_1 has a reproducing kernel. Let $\{\varphi_k\}$ be an orthonormal basis in H_1 and $T_0\colon H_1 \to H$ a bounded linear operator. Assume that $\{T_0\varphi_k\}$ is an orthogonal system in H, $\varphi_k \in X_E$, $k = 1, 2, \dots$ and $s_k := \|T_0\varphi_k\|_H$ form a non-increasing sequence. Define $T\colon X_E \to H$ by the equation $Tf := T_0 f$. Then the inequalities (6) hold.*

Proof. Denote by $K(z,w)$ the reproducing kernel of H_1. Since $\{\varphi_k\}$ is an orthonormal basis in H_1 the representation

$$K(z,w) = \sum_{k=1}^{\infty} \varphi_k(z)\overline{\varphi_k(w)}$$

holds. We have

$$(T_0'T_0 f)(w) = \big((T_0'T_0 f)(\cdot), K(\cdot, w)\big)_{H_1} = \big((T_0 f)(\cdot), (T_0 K)(\cdot, w)\big)_H.$$

Thus

$$(T_0'T_0\varphi_j)(w) = \Big((T_0\varphi_j)(\cdot), \sum_{k=1}^{\infty} \overline{\varphi_k(w)}(T_0\varphi_k)(\cdot)\Big)_H = s_j^2\varphi_j(w).$$

Now it suffices to apply Theorem 4. The theorem is proved.

Let B_ρ be the ball of $\mathbb{C}^n$ of radius ρ

$$B_\rho := \{\, z := (z_1, \dots, z_n) \in \mathbb{C}^n : |z|^2 := \sum_{k=1}^{n} |z_k|^2 < \rho \,\},$$

$S_\rho := \partial B_\rho$, σ_ρ the probability measure on the sphere S_ρ which is invariant with respect to the orthogonal group $O(2n)$ and ν_ρ the normalized Lebesgue measure in B_ρ (if $\rho = 1$ we will write B, S, σ and ν).

The Hardy space $H_p(B)$ (H_p) is the set of holomorphic functions in B which satisfy

$$\|f\|_{H_p} := \sup_{0<\rho<1} \left(\int_S |f(z)|^p \, d\sigma(z) \right)^{1/p} < \infty, \quad 1 \le p < \infty,$$
$$\|f\|_{H_\infty} := \sup_{z \in B} |f(z)|.$$

Let $f(z)$ be a holomorphic function in B and

$$f(z) = \sum_{k=0}^{\infty} F_k(z)$$

be a homogeneous decomposition of f. The radial derivative of order r is defined by

$$\mathcal{R}^r f(z) := \sum_{k=r}^{\infty} \frac{k!}{(k-r)!} F_k(z)$$

(for $r=1$ see [14, Chap.6]). Denote by $H\mathcal{R}^r_\infty(B)$ ($H\mathcal{R}^r_\infty$) the class of holomorphic functions in B for which $\mathcal{R}^r f\in BH_\infty$.

Set

$$N_m:=\sum_{k=0}^{m-1}\binom{n+k-1}{n-1}.$$

Note that $N_m=\dim\mathcal{P}^n_{m-1}$ where $\mathcal{P}^n_m$ is the space of n-variable polynomials of degree m or less.

Theorem 6. *For all $0<\rho\le 1$ and all $m\ge r+1$*

(11)
$$b_{N_m-1}\bigl(H\mathcal{R}^r_\infty,L_2(S_\rho,\sigma_\rho)\bigr)=\left(\frac{1}{(n-1)!}\sum_{k=r}^{m-1}\frac{k!(n+k-1)!}{((k-r)!)^2}\rho^{-2k}\right)^{-1/2},$$

(12)
$$b_{N_m-1}\bigl(H\mathcal{R}^r_\infty,L_2(B_\rho,\nu_\rho)\bigr)=\left(\frac{1}{n!}\sum_{k=r}^{m-1}\frac{k!(n+k)!}{((k-r)!)^2}\rho^{-2k}\right)^{-1/2}.$$

Proof. For multiindex $\alpha:=(\alpha_1,\dots,\alpha_n)$ and $z\in\mathbb{C}^n$ set

$$z^\alpha:=z_1^{\alpha_1}\cdots z_n^{\alpha_n},\qquad |\alpha|:=\alpha_1+\dots+\alpha_n,\qquad \alpha!:=\alpha_1!\cdots\alpha_n!,$$
$$D_j:=\frac{\partial}{\partial z_j},\qquad D^\alpha:=D_1^{\alpha_1}\cdots D_n^{\alpha_n}.$$

Denote by H_p^0 the space of all functions $f\in H_p$ for which $(D^\alpha f)(0)=0$, $|\alpha|=0,\dots,r-1$. It is known (see [14]) that functions from H_p, $1\le p\le\infty$, have finite boundary values almost everywhere. Moreover, H_2 is a Hilbert space with the inner product

$$(f,g)_{H_2}:=\int_S f(z)\overline{g(z)}\,d\sigma(z).$$

The space H_2 has the reproducing kernel

$$K(z,w)=\left(1-\sum_{k=1}^n z_k\overline{w}_k\right)^{-n}.$$

Define $T_0\colon H_2^0\to L_2(S_\rho,\sigma_\rho)$ and $T\colon H_\infty^0\to L_2(S_\rho,\sigma_\rho)$ by the equations

(13)
$$(T_0f)(z):=\sum_{k=r}^\infty\frac{(k-r)!}{k!}F_k(z),\qquad Tf:=T_0f,$$

where

$$f(z)=\sum_{k=r}^\infty F_k(z).$$

It is easy to see that
$$H\mathcal{R}^r_\infty = T(BH^0_\infty) + \mathcal{P}^n_{r-1}.$$
Since monomials z^α are orthogonal in $L_2(S_\rho, \sigma_\rho)$ we obtain from Lemma 2
$$b_{N_m-1}\big(H\mathcal{R}^r_\infty, L_2(S_\rho, \sigma_\rho)\big) = b_{N_m-N_r-1}(T). \tag{14}$$
For every $0 < \rho \le 1$ (see [14])
$$\|z^\alpha\|^2_{L_2(S_\rho,\sigma_\rho)} = \frac{(n-1)!\alpha!}{(n+|\alpha|-1)!}\rho^{2|\alpha|}.$$
Thus the functions
$$\varphi_\alpha(z) := \left(\frac{(n+|\alpha|-1)!}{(n-1)!\alpha!}\right)^{1/2} z^\alpha, \quad |\alpha| \ge r,$$
form a complete orthonormal basis in H^0_2. We have
$$\|T_0\varphi_\alpha\|^2_{L_2(S_\rho,\sigma_\rho)} = \left(\frac{(|\alpha|-r)!}{|\alpha|!}\right)^2 \rho^{2|\alpha|}.$$
The number of different monomials z^α with $|\alpha| = k$ equals $\binom{n+k-1}{n-1}$. By Theorem 5
$$\left(\sup_{z\in S}\sum_{|\alpha|=r}^{m-1}\left(\frac{|\alpha|!}{(|\alpha|-r)!}\right)^2 \rho^{-2|\alpha|}|\varphi_\alpha(z)|^2\right)^{-1/2} \le b_{N_m-N_r-1}(T)$$
$$\le \left(\sum_{k=r}^{m-1}\left(\frac{k!}{(k-r)!}\right)^2\binom{n+k-1}{n-1}\rho^{-2k}\right)^{-1/2}.$$
Using the equation
$$\sum_{|\alpha|=k}\frac{|z^{2\alpha}|}{\alpha!} = \frac{|z|^{2k}}{k!},$$
we obtain
$$b_{N_m-N_r-1}(T) = \left(\frac{1}{(n-1)!}\sum_{k=r}^{m-1}\frac{k!(n+k-1)!}{((k-r)!)^2}\rho^{-2k}\right)^{-1/2}.$$
Now (11) follows from (14).

The proof of (12) is almost the same. The difference is that we have to consider the operators $T_0\colon H^0_2 \to L_2(B_\rho, \nu_\rho)$ and $T\colon H^0_\infty \to L_2(B_\rho, \nu_\rho)$ defined by (13). Then we use
$$\|T_0\varphi_\alpha\|^2_{L_2(B_\rho,\nu_\rho)} = \left(\frac{(|\alpha|-r)!}{|\alpha|!}\right)^2 \frac{n}{n+|\alpha|}\rho^{2|\alpha|}.$$
The theorem is proved.

For $n = 1$ the class $H\mathcal{R}^r_\infty$ coincides with the class BH^r_∞ defined as the set of all holomorphic functions in B for which $f^{(r)}(z) \in BH_\infty$. From Theorem 6 we obtain the following result.

Corollary 1. *For all $0 < \rho \le 1$ and all $m \ge r$*

$$b_m\big(BH^r_\infty, L_2(S_\rho,\sigma_\rho)\big) = \left(\sum_{k=r}^m \left(\frac{k!}{(k-r)!}\right)^2 \rho^{-2k}\right)^{-1/2},$$

$$b_m\big(BH^r_\infty, L_2(B_\rho,\nu_\rho)\big) = \left(\sum_{k=r}^m \frac{k!(k+1)!}{((k-r)!)^2}\rho^{-2k}\right)^{-1/2}.$$

In particular, we have for $r=0$, $m \ge 0$ and $0<\rho<1$

$$b_m\big(BH_\infty, L_2(S_\rho,\sigma_\rho)\big) = \rho^m\left(\frac{1-\rho^2}{1-\rho^{2m+2}}\right)^{1/2}, \tag{15}$$

$$b_m\big(BH_\infty, L_2(B_\rho,\nu_\rho)\big) = \rho^m\frac{1-\rho^2}{\sqrt{(m+1)(1-\rho^2)-\rho^2(1-\rho^{2m+2})}}. \tag{16}$$

The values (11) for $r=0$ and (15) were calculated in [7].

Let us compare (15) and (16) with the exact values of the Kolmogorov, linear, and Gel'fand n-widths. From [15] and [16] it follows that

$$d_m(BH_\infty,X) = \lambda_m(BH_\infty,X) = d^m(BH_\infty,X) = \begin{cases} \rho^m, & X = L_2(S_\rho,\sigma_\rho) \\ \dfrac{\rho^m}{\sqrt{m+1}}, & X = L_2(B_\rho,\nu_\rho). \end{cases}$$

Finally, we will determine exact values of the Bernstein n-widths for some classes of periodic holomorphic functions. Let $D_\beta := \{\, z \in \mathbb{C} : |\operatorname{Im} z| < \beta \,\}$. Denote by $\widetilde{H}_{p,\beta}$ the set of all 2π-periodic holomorphic functions in D_β which satisfy the conditions

$$\|f\|_{\widetilde{H}_{p,\beta}} := \sup_{0<h<\beta}\left(\frac{1}{4\pi}\int_0^{2\pi}\left(|f(x+ih)|^p + |f(x-ih)|^p\right)dx\right)^{1/p} < \infty, \quad 1 \le p < \infty,$$

$$\|f\|_{\widetilde{H}_{\infty,\beta}} := \sup_{z\in D_\beta}|f(z)| < \infty.$$

Let $B\widetilde{H}^r_{p,\beta}$ be the set of all 2π-periodic holomorphic functions in D_β for which $f^{(r)}(z) \in B\widetilde{H}_{p,\beta}$. Denote by L_2 the periodic complex-valued Lebesgue space on the real axis with the norm

$$\|f\|_{L_2} := \left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2\,dx\right)^{1/2}.$$

Theorem 7.

(1) *For all $n \ge 1$ and $r \ge 1$*

$$b_{2n}(B\widetilde{H}^r_{\infty,\beta}, L_2) = \left(2\sum_{k=1}^n k^{2r}\cosh 2k\beta\right)^{-1/2}.$$

(2) *For all $n \ge 0$*

$$b_{2n}(B\widetilde{H}_{\infty,\beta}, L_2) = \left(\frac{\sinh\beta}{\sinh(2n+1)\beta}\right)^{1/2}. \tag{17}$$

Proof. The space $\widetilde{H}_{2,\beta}$ is the Hilbert space with the inner product

$$(f,g)_{\widetilde{H}_{2,\beta}} := \frac{1}{4\pi}\int_0^{2\pi}\left(f(x+i\beta)\overline{g(x+i\beta)} + f(x-i\beta)\overline{g(x-i\beta)}\right)\,dx.$$

The functions

$$\varphi_k(z) := \frac{e^{ikz}}{\sqrt{\cosh 2k\beta}}, \quad k\in\mathbb{Z},$$

form an orthonormal basis in $\widetilde{H}_{2,\beta}$. The space $\widetilde{H}_{2,\beta}$ has the reproducing kernel

$$K(z,w) = \sum_{k\in\mathbb{Z}}\varphi_k(z)\overline{\varphi_k(w)} = 1 + 2\sum_{k=1}^{\infty}\frac{\cos k(z-\overline{w})}{\cosh 2k\beta}.$$

Let $r\ge 1$. Denote by $\widetilde{H}^0_{p,\beta}$ the space of functions $f\in\widetilde{H}_{p,\beta}$ for which

$$\int_0^{2\pi} f(x)\,dx = 0.$$

Define $T_0\colon \widetilde{H}^0_{2,\beta}\to L_2$ and $T\colon \widetilde{H}^0_{\infty,\beta}\to L_2$ by the equations

$$(T_0 f)(z) := \sum_{\substack{k\in\mathbb{Z}\\ k\ne 0}}\frac{c_k}{(ik)^r}e^{ikz}, \qquad Tf := T_0 f,$$

where

$$f(z) = \sum_{\substack{k\in\mathbb{Z}\\ k\ne 0}} c_k e^{ikz}.$$

It is easily seen that

$$B\widetilde{H}^r_{\infty,\beta} = T(B\widetilde{H}^0_{\infty,\beta}) + \mathbb{C}.$$

By Lemma 2 we obtain

$$b_{2n}(B\widetilde{H}^r_{\infty,\beta}, L_2) = b_{2n-1}(T).$$

The functions $\varphi_k(z)$, $k=\pm1,\pm2,\dots$ form a complete orthonormal basis in $\widetilde{H}^0_{2,\beta}$ and

$$\|T_0\varphi_k\|^2_{L_2} = \frac{1}{k^{2r}\cosh 2k\beta}.$$

Since for all $z\in\partial D_\beta$

$$|\varphi_k(z)|^2 + |\varphi_{-k}(z)|^2 = 2$$

we have by Theorem 5

$$b_{2n-1}(T) = \left(2\sum_{k=1}^{n} k^{2r}\cosh 2k\beta\right)^{-1/2}.$$

To obtain (2) we use the same scheme and the equality

$$1+2\sum_{k=1}^{n}\cosh 2k\beta=\frac{\sinh(2n+1)\beta}{\sinh\beta}.$$

The theorem is proved.

Denote by $B\widetilde{H}^{\mathbb{R}}_{\infty,\beta}$ the set of functions from $B\widetilde{H}_{\infty,\beta}$ that are real-valued on $\mathbb{R}$. For even n the exact values of the Kolmogorov, linear, and Gel'fand n-widths of $B\widetilde{H}^{\mathbb{R}}_{\infty,\beta}$ in L_q, $1\le q\le\infty$, were determined in [17]. In particular, for $q=2$

$$\begin{aligned}d_{2n}(B\widetilde{H}^{\mathbb{R}}_{\infty,\beta},L_2)&=\lambda_{2n}(B\widetilde{H}^{\mathbb{R}}_{\infty,\beta},L_2)=d^{2n}(B\widetilde{H}^{\mathbb{R}}_{\infty,\beta},L_2)\\&=\left(\frac{\lambda}{\Lambda}\int_0^1\frac{t^2\,dt}{\sqrt{(1-t^2)(1-\lambda^2t^2)}}\right)^{1/2}=\sqrt{2}e^{-\beta n}+O(e^{-5\beta n}),\end{aligned}$$

where Λ is the complete elliptic integral of the first kind with modulus

$$\lambda=4e^{-2\beta n}\left(\sum_{k=0}^{\infty}e^{-4\beta nk(k+1)}\right)^2\left(1+2\sum_{k=1}^{\infty}e^{-4\beta nk^2}\right)^{-2}.$$

By Theorem 5 it can be shown that (17) also holds in the real case for the class $B\widetilde{H}^{\mathbb{R}}_{\infty,\beta}$. Thus

$$b_{2n}(B\widetilde{H}^{\mathbb{R}}_{\infty,\beta},L_2)=\sqrt{1-e^{-2\beta}}e^{-\beta n}+O(e^{-5\beta n}).$$

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