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How Best to Recover a Function from Its Inaccurately Given Spectrum?

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Abstract—Consider the problem of optimal recovery of a function and its derivatives on the line from the Fourier transform of the function known approximately on a set of finite measure. We find an optimal recovery method and an optimal set on which we must measure the Fourier transform with given error.

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1. INTRODUCTION

The question initially stimulating the writing of the present paper was as follows: "How best to recover a signal from the measurements of a fixed number of its harmonics with fixed error?"

We give an answer to this question in the following case. Suppose we are given a function $x(\,\cdot\,) \in W_2^n(\mathbb{R})$ belonging to the Sobolev class of functions $x(\,\cdot\,) \in L_2(\mathbb{R})$ whose $(n-1)$th derivative is locally absolutely continuous and $\|x^{(n)}(\,\cdot\,)\|_{L_2(\mathbb{R})} \le 1$; the Fourier transform of this function is known on a measurable set M_σ of measure not greater than 2σ with accuracy up to $\delta > 0$ in the metric of $L_p(M_\sigma)$, $1 \le p \le \infty$. We pose the problem of the optimal recovery of a function from $W_2^n(\mathbb{R})$ and its kth derivative $(k \le n-1)$ in the metric of $L_2(\mathbb{R})$ from this given data. The gist of the answer to the question posed above is that it is best to measure the Fourier transform on a set which is a symmetric (with respect to zero) closed interval of length $2\sigma_0$, where $\sigma_0 = \min(\sigma, \widehat{\sigma})$ and $\widehat{\sigma}$ is a positive number (depending on n, k, p, and δ). Further, outside the closed interval $[-\sigma_0, \sigma_0]$, information about the Fourier transform turns out to be superfluous. We must "smooth out" the remaining (useful) data in a suitable way, take its inverse Fourier transform, and then differentiate k times (if $k \ge 1$). This procedure fully corresponds to what occurs in practice (the high frequencies are neglected and, in view of unavoidable measurement errors, the remaining frequencies are filtered out in one way or another).

2. STATEMENT OF THE PROBLEM AND FORMULATION OF RESULTS

Suppose that n is a natural number, $W_2^n(\mathbb{R})$ is the Sobolev class of functions on $\mathbb{R}$ defined above, $\sigma > 0$, and $\mathscr{M}_\sigma$ is the set of measurable subsets of the line whose measures are not greater than 2σ. Assume that we know the Fourier transform $Fx(\,\cdot\,)$ of a function $x(\,\cdot\,) \in W_2^n(\mathbb{R})$ on a set $M_\sigma \in \mathscr{M}_\sigma$ with accuracy up to $\delta > 0$ in the metric of $L_p(M_\sigma)$, $1 \le p \le \infty$, i.e., we know a function $y(\,\cdot\,) \in L_p(M_\sigma)$ such that $\|Fx(\,\cdot\,) - y(\,\cdot\,)\|_{L_p(M_\sigma)} \le \delta$. By the problem of the optimal recovery of a function from the class $W_2^n(\mathbb{R})$ or of its kth derivative $(0 \le k \le n-1)$ in the metric of $L_2(\mathbb{R})$ from given data we mean the determination of the quantity, called the *optimal recovery error*,

$$E(k,\sigma,p,\delta) = \inf_{M_\sigma} \inf_{m} \sup_{\substack{x(\,\cdot\,) \in W_2^n(\mathbb{R}),\, y(\,\cdot\,) \in L_p(M_\sigma) \\ \|Fx(\,\cdot\,) - y(\,\cdot\,)\|_{L_p(M_\sigma)} \le \delta}} \|x^{(k)}(\,\cdot\,) - m(y(\,\cdot\,))(\,\cdot\,)\|_{L_2(\mathbb{R})},$$

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where the first infimum is taken over all sets $M_\sigma \in \mathscr{M}_\sigma$ and the second infimum, over all mappings (recovery methods) $m\colon L_p(M_\sigma) \to L_2(\mathbb{R})$ and the determination of $\widehat{M}_\sigma$ and $\widehat{m}$ called the *optimal set* and the *optimal method* at which the infimum are attained.

The determination of the optimal recovery error and the optimal method for a class of elements dates back to Kolmogorov's paper on the widths of function classes [1]. The statement of the problem of optimal recovery (but in a considerably simpler case) is due to Smolyak [2]. One can get an idea of the subsequent development of problems related to those of optimal recovery from [3]–[7]. The problem stated above for the case in which $\mathscr{M}_\sigma$ consists of one closed interval $[-\sigma,\sigma]$ and $p=2$ and ∞, was studied in [8]. There it was proved that, in that case, if $1 \le p < 2$, then the supremum in the definition of $E(k,\sigma,p,\delta)$ is equal to infinity, so that such a case is of no interest and any method is optimal. The case in which $\mathscr{M}_\sigma$ consists of all closed intervals of length 2σ, and $p=2$, was studied in [9]. In the present paper, we consider the general case in which $2<p<\infty$. The extreme cases $p=2$ and ∞ can be obtained by passing to the limit, but we will not go into this.

Suppose that $2<p<\infty$. Set

$$\widehat{\sigma} = \left(\frac{\sqrt{2\pi}(n-k)^{1-1/p}}{\delta\sqrt{k+1/2-1/p}\,B^{1/2-1/p}}\right)^{1/(n+1/2-1/p)},$$

where

$$B = B\left(\frac{k+1/2-1/p}{(n-k)(1-2/p)},\, 2\frac{1-1/p}{1-2/p}\right) \tag{2.1}$$

is the B-Euler function.

Theorem 1. *Suppose that k and n are integers, $0 \le k \le n-1$, $\sigma>0$, $\delta>0$, $2<p<\infty$, and $\sigma_0 = \min(\sigma,\widehat{\sigma})$. Then*

$$E(k,\sigma,p,\delta) = \begin{cases} \sqrt{\dfrac{\delta^2}{2\pi}\left(\dfrac{B}{n-k}\right)^{1-2/p}\sigma^{2k+1-2/p} + \dfrac{1}{\sigma^{2(n-k)}}}\,, & \sigma \le \widehat{\sigma}, \\ \sqrt{\dfrac{n+1/2-1/p}{k+1/2-1/p}}\,\widehat{\sigma}^{-(n-k)}, & \sigma \ge \widehat{\sigma}. \end{cases}$$

The optimal set is the closed interval $[-\sigma_0,\sigma_0]$. The optimal method is of the form

$$\widehat{m}(y(\,\cdot\,))(t) = \frac{1}{2\pi}\int_{|\xi|\le\sigma_0}(i\xi)^k\left(1-\left(\frac{\xi}{\sigma_0}\right)^{2(n-k)}\right)y(\xi)e^{i\xi t}\,d\xi.$$

As is seen from the statements of the theorem, the knowledge of the Fourier transform outside the closed interval $[-\widehat{\sigma},\widehat{\sigma}]$ is superfluous, i.e., the optimal recovery error does not decrease. The useful data (on the closed interval $[-\sigma_0,\sigma_0]$) is smoothed.

3. PROOF OF THE THEOREM

Below we shall deal with extremal problems that have no solutions; therefore, we begin with the proof of a statement dealing with the determination of the value of the problem in such a case. Suppose that X is an arbitrary nonempty set, $f_i\colon X \to \mathbb{R}$, $i=0,1,\dots,N$, $\alpha_i \in \mathbb{R}$, $i=1,\dots,N$, and A is a nonempty subset X. Consider the problem

$$f_0(x) \to \max, \qquad f_i(x) \le \alpha_i, \quad i=1,\dots,N, \qquad x \in A, \tag{3.1}$$

of finding admissible (i.e., satisfying the constraints of the problem) elements at which f_0 attains its maximum. The supremum of $f_0(x)$ over all admissible x is called the *value of problem* (3.1).

With problem (3.1) we associate the following Lagrange function:

$$\mathscr{L}(x,\lambda) = -f_0(x) + \sum_{i=1}^{N}\lambda_i f_i(x),$$

where $\lambda = (\lambda_1,\dots,\lambda_N)$ is the collection of Lagrange multipliers.

Lemma 1. *Suppose that there exists a collection* $\widehat{\lambda}=(\widehat{\lambda}_1,\dots,\widehat{\lambda}_N)$ *of nonnegative Lagrange multipliers, a number* $\widehat{\mathscr{L}}$*, and a sequence of admissible elements* $\{x_m\}$ *in* (3.1) *such that*

$$\text{(a)}\quad \mathscr{L}(x,\widehat{\lambda})\ge\widehat{\mathscr{L}}\qquad \text{for all}\quad x\in A,$$

$$\text{(b)}\quad \lim_{m\to\infty}\mathscr{L}(x_m,\widehat{\lambda})=\widehat{\mathscr{L}},$$

$$\text{(c)}\quad \lim_{m\to\infty}\widehat{\lambda}_i(f_i(x_m)-\alpha_i)=0,\qquad i=1,\dots,N.$$

Then

$$\sum_{i=1}^N\widehat{\lambda}_i\alpha_i-\widehat{\mathscr{L}}$$

is the value of problem (3.1).

Proof of the lemma. Let S denote the value of problem (3.1). For any admissible element x in (3.1), taking into account the fact that the $\widehat{\lambda}_i$ $i=1,\dots,N$, are nonnegative and using condition (a), we obtain

$$-f_0(x)\ge -f_0(x)+\sum_{i=1}^N\widehat{\lambda}_i(f_i(x)-\alpha_i)=\mathscr{L}(x,\widehat{\lambda})-\sum_{i=1}^N\widehat{\lambda}_i\alpha_i\ge\widehat{\mathscr{L}}-\sum_{i=1}^N\widehat{\lambda}_i\alpha_i,$$

i.e.,

$$S\le\sum_{i=1}^N\widehat{\lambda}_i\alpha_i-\widehat{\mathscr{L}}.$$

On the other hand, in view of conditions (b) and (c), we find that

$$\begin{aligned}\widehat{\mathscr{L}}&=\lim_{m\to\infty}\mathscr{L}(x_m,\widehat{\lambda})=-\lim_{m\to\infty}f_0(x_m)+\sum_{i=1}^N\lim_{m\to\infty}\widehat{\lambda}_if_i(x_m)\\&=-\lim_{m\to\infty}f_0(x_m)+\sum_{i=1}^N\lim_{m\to\infty}\widehat{\lambda}_i(f_i(x_m)-\alpha_i)+\sum_{i=1}^N\widehat{\lambda}_i\alpha_i\ge -S+\sum_{i=1}^N\widehat{\lambda}_i\alpha_i,\end{aligned}$$

and hence

$$S\ge\sum_{i=1}^N\widehat{\lambda}_i\alpha_i-\widehat{\mathscr{L}}.$$

The lemma is proved. □

Proof of Theorem 1. 1. *The infimum for* $E(k,\sigma,p,\delta)$. Choose $M_\sigma\in\mathscr{M}_\sigma$ and, for a given M_σ, let $E(k,M_\sigma,p,\delta)$ denote the quantity under the sign of the first infimum in the definition of $E(k,\sigma,p,\delta)$. Let us show that $E(k,M_\sigma,p,\delta)$ is not less than the value of the problem:

$$\|x^{(k)}(\,\cdot\,)\|_{L_2(\mathbb{R})}\to\max,\qquad \|Fx(\,\cdot\,)\|_{L_p(M_\sigma)}\le\delta,\quad \|x^{(n)}(\,\cdot\,)\|_{L_2(\mathbb{R})}\le 1. \tag{3.2}$$

Indeed, suppose that $x(\,\cdot\,)$ is an admissible function in (3.2) (i.e., $x(\,\cdot\,)$ satisfies the constraints of the problem). Then, obviously, the function $-x(\,\cdot\,)$ is also admissible and, for any $m\colon L_p(M_\sigma)\to L_2(\mathbb{R})$, we have

$$\begin{aligned}2\|x^{(k)}(\,\cdot\,)\|_{L_2(\mathbb{R})}&\le\|x^{(k)}(\,\cdot\,)-m(0)(\,\cdot\,)\|_{L_2(\mathbb{R})}+\|-x^{(k)}(\,\cdot\,)-m(0)(\,\cdot\,)\|_{L_2(\mathbb{R})}\\&\le 2\sup_{x(\,\cdot\,)\in W_2^n(\mathbb{R}),\,\|Fx(\,\cdot\,)\|_{L_p(M_\sigma)}\le\delta}\|x^{(k)}(\,\cdot\,)-m(0)(\,\cdot\,)\|_{L_2(\mathbb{R})}\\&\le 2\sup_{\substack{x(\,\cdot\,)\in W_2^n(\mathbb{R}),\,y(\,\cdot\,)\in L_p(M_\sigma)\\ \|Fx(\,\cdot\,)-y(\,\cdot\,)\|_{L_p(M_\sigma)}\le\delta}}\|x^{(k)}(\,\cdot\,)-m(y(\,\cdot\,))(\,\cdot\,)\|_{L_2(\mathbb{R})}.\end{aligned}$$

Passing, on the left, to the supremum over all admissible functions in (3.2) and, on the right, to the infimum over all methods m, we obtain the required assertion.

In Fourier images, denoting $u(\,\cdot\,) = (2\pi)^{-1/2}|Fx(\,\cdot\,)|$ and using Plancherel's theorem, we find that the square of the value of problem (3.2) is equal to the value of the following problem:

$$\int_{\mathbb{R}} \xi^{2k} u^2(\xi)\,d\xi \to \max, \qquad \int_{M_\sigma} u^p(\xi)\,d\xi \le \frac{\delta^p}{(2\pi)^{p/2}}, \quad \int_{\mathbb{R}} \xi^{2n} u^2(\xi)\,d\xi \le 1, \qquad u(\,\cdot\,) \ge 0. \tag{3.3}$$

Set

$$\widehat{a} = \sup\{a \ge 0 : \operatorname{mes}\{M_\sigma \cap [-a,a]\} = 2a\}.$$

Obviously, zero belongs to the set given in the braces. Let us show that if $\widehat{a} = 0$, then the value of problem (3.3) (and hence also of (3.2)) is equal to infinity. Indeed, in this case, $\operatorname{mes}\{M_\sigma \cap [-\varepsilon,\varepsilon]\} < 2\varepsilon$ for any $\varepsilon > 0$, and hence $\operatorname{mes}\Omega_\varepsilon = \{(\mathbb{R}\setminus M_\sigma) \cap [-\varepsilon,\varepsilon]\} > 0$. Set

$$u_\varepsilon(\xi) = \begin{cases} \left(\displaystyle\int_{\Omega_\varepsilon} \tau^{2n}\,d\tau\right)^{-1/2}, & \xi \in \Omega_\varepsilon \\ 0, & \xi \notin \Omega_\varepsilon. \end{cases}$$

This function is admissible in problem (3.3) and

$$\int_{\mathbb{R}} \xi^{2k} u_\varepsilon^2(\xi)\,d\xi = \frac{\int_{\Omega_\varepsilon} \xi^{2k}\,d\xi}{\int_{\Omega_\varepsilon} \tau^{2n}\,d\tau} = \frac{\int_{\Omega_\varepsilon} \xi^{2n}\xi^{-2(n-k)}\,d\xi}{\int_{\Omega_\varepsilon} \tau^{2n}\,d\tau} \ge \varepsilon^{-2(n-k)};$$

hence, since ε is arbitrary, it follows that the value of the maximized functional in (3.3) can be made arbitrarily large.

Now suppose that $\widehat{a} > 0$. Let us find the value of problem (3.3) in this case using the lemma proved above. Problem (3.3) has the same form as problem (3.1) (X is the set of all measurable functions $u(\,\cdot\,)$ on $\mathbb{R}$ and A is the subset of nonnegative functions). Let us write the Lagrange function (3.3) as

$$\begin{aligned} \mathscr{L}(u(\,\cdot\,),\lambda_1,\lambda_2) = \int_{M_\sigma} &(-\xi^{2k}u^2(\xi) + \lambda_1 u^p(\xi) + \lambda_2 \xi^{2n} u^2(\xi))\,d\xi \\ &+ \int_{\mathbb{R}\setminus M_\sigma} (-\xi^{2k} + \lambda_2 \xi^{2n}) u^2(\xi)\,d\xi. \end{aligned} \tag{3.4}$$

Set $a_0 = \min(\widehat{\sigma}, \widehat{a})$ and $\widehat{\lambda}_2 = a_0^{-2(n-k)}$. Then, for any $\lambda_1 > 0$, for $\xi \in [-a_0, a_0]$, the function

$$u \mapsto f(u) = -\xi^{2k}u^2 + \lambda_1 u^p + a_0^{-2(n-k)} \xi^{2n} u^2 \qquad \text{on} \quad [0,\infty)$$

attains an absolute minimum at the point

$$\widetilde{u}(\xi) = \left(\frac{2}{\lambda_1 p}\right)^{1/(p-2)} \xi^{2k/(p-2)} \left(1 - \left(\frac{\xi}{a_0}\right)^{2(n-k)}\right)^{1/(p-2)}$$

and, for $|\xi| > a_0$, at zero.

Now let us choose λ_1; we denote it by $\widehat{\lambda}_1$ if it satisfies the condition

$$\int_{-a_0}^{a_0} \widetilde{u}^p(\xi)\,d\xi = \left(\frac{2}{\widehat{\lambda}_1 p}\right)^{p/(p-2)} \int_{-a_0}^{a_0} \xi^{2pk/(p-2)} \left(1 - \left(\frac{\xi}{a_0}\right)^{2(n-k)}\right)^{p/(p-2)} d\xi = \frac{\delta^p}{(2\pi)^{p/2}}. \tag{3.5}$$

Replacing $\eta = (\xi/a_0)^{2(n-k)}$ in the integral and performing simple calculations, we obtain

$$\widehat{\lambda}_1 = \left(\frac{\sqrt{2\pi}}{\delta}\right)^{p-2} \frac{2B^{1-2/p}}{p(n-k)^{1-2/p}}\, a_0^{2(k+1/2-1/p)},$$

where B is defined by (2.1). Obviously, $\widehat{\lambda}_1 > 0$.

Since $\operatorname{mes}\{M_\sigma \cap [-\widehat{a}, \widehat{a}]\} = 2\widehat{a}$ and the function f is nonnegative outside the closed interval $[-a_0, a_0]$, it follows that, for all $u(\,\cdot\,) \ge 0$, we have

$$\int_{M_\sigma} f(u(\xi))\,d\xi \ge \int_{M_\sigma \cap [-\widehat{a},\widehat{a}]} f(u(\xi))\,d\xi = \int_{-\widehat{a}}^{\widehat{a}} f(u(\xi))\,d\xi$$
$$\ge \int_{-a_0}^{a_0} f(u(\xi))\,d\xi \ge \int_{-a_0}^{a_0} f(\widetilde{u}(\xi))\,d\xi.$$

Further, since $\mathbb{R} \setminus M_\sigma \subset \mathbb{R} \setminus [-\widehat{a}, \widehat{a}]$ up to a set of zero measure and the function

$$\xi \mapsto -\xi^{2k} + a_0^{-2(n-k)}\xi^{2n}$$

is positive for $|\xi| > a_0$, it follows that, for all $u(\,\cdot\,)$,

$$\int_{\mathbb{R}\setminus M_\sigma} (-\xi^{2k} + a_0^{-2(n-k)}\xi^{2n})u^2(\xi)\,d\xi \ge 0.$$

These relations imply that, for all $u(\,\cdot\,) \ge 0$, the following inequality holds:

$$\mathscr{L}(u(\,\cdot\,), \widehat{\lambda}_1, \widehat{\lambda}_2) \ge \int_{-a_0}^{a_0} f(\widetilde{u}(\xi))\,d\xi. \tag{3.6}$$

Let us consider separately the following two cases: $\widehat{a} < \widehat{\sigma}$ and $\widehat{a} \ge \widehat{\sigma}$.

Suppose that $\widehat{a} < \widehat{\sigma}$. Then $a_0 = \widehat{a}$. For each $m \in \mathbb{N}$, we set

$$\Omega_m = (\mathbb{R} \setminus M_\sigma) \cap \left(\left(-\widehat{a} - \frac{1}{m}, \widehat{a}\right) \cup \left(\widehat{a}, \widehat{a} + \frac{1}{m}\right)\right).$$

It follows from the definition of $\widehat{a}$ that $\operatorname{mes}\Omega_m > 0$ for all m. Set

$$u_m(\xi) = \begin{cases} \widetilde{u}(\xi), & \xi \in [-\widehat{a}, \widehat{a}], \\ \gamma_m, & \xi \in \Omega_m, \\ 0 & \text{otherwise}, \end{cases}$$

while γ_m is chosen so that

$$\int_{\mathbb{R}} \xi^{2n} u_m^2(\xi)\,d\xi = \int_{-\widehat{a}}^{\widehat{a}} \xi^{2n}\widetilde{u}^2(\xi)\,d\xi + \gamma_m^2 \int_{\Omega_m} \xi^{2n}\,d\xi = 1. \tag{3.7}$$

Let us show that this is possible. Indeed, just as above, replacing $\eta = (\xi/\widehat{a})^{2(n-k)}$ in the expression for $\widetilde{u}(\,\cdot\,)$ and using well-known properties of the B-function, we obtain the relation

$$\int_{-\widehat{a}}^{\widehat{a}} \xi^{2n}\widetilde{u}^2(\xi)\,d\xi = \frac{\delta^2(k + 1/2 - 1/p)B^{1-2/p}}{2\pi(n-k)^{2-2/p}}\widehat{a}^{2n+1-2/p}.$$

It follows from the definition of $\widehat{\sigma}$ that, for $\widehat{a} = \widehat{\sigma}$, the quantity on the right is equal to 1 and since we have $\widehat{a} < \widehat{\sigma}$, this quantity is less than 1. Denoting it by C and using (3.7), we obtain

$$\gamma_m = (1 - C)^{1/2}\left(\int_{\Omega_m} \xi^{2n}\,d\xi\right)^{-1/2}.$$

Now let us verify that

$$\lim_{m\to\infty} \mathscr{L}(u_m(\,\cdot\,), \widehat{\lambda}_1, \widehat{\lambda}_2) = \lim_{m\to\infty} \int_{\Omega_m} (-\xi^{2k} + \widehat{a}^{2(n-k)}\xi^{2n})u_m^2(\xi)\,d\xi + \int_{-\widehat{a}}^{\widehat{a}} f(\widetilde{u}(\xi))\,d\xi$$
$$= \int_{-\widehat{a}}^{\widehat{a}} f(\widetilde{u}(\xi))\,d\xi. \tag{3.8}$$

Indeed, using the definitions of $u_m(\,\cdot\,)$ and γ_m, we obtain

$$\int_{\Omega_m}(-\xi^{2k}+\widehat{a}^{2(n-k)}\xi^{2n})u_m^2(\xi)\,d\xi=\gamma_m^2\int_{\Omega_m}(-\xi^{2k}+\widehat{a}^{-2(n-k)}\xi^{2n})\,d\xi$$
$$\le\biggl(\widehat{a}^{-2(n-k)}-\biggl(\widehat{a}+\frac{1}{m}\biggr)^{-2(n-k)}\biggr)\gamma_m^2\int_{\Omega_m}\xi^{2n}\,d\xi$$
$$=(1-C)\biggl(\widehat{a}^{-2(n-k)}-\biggl(\widehat{a}+\frac{1}{m}\biggr)^{-2(n-k)}\biggr)\to 0,\qquad m\to\infty,$$

which proves (3.8).

Now we can find the value of problem (3.3) using the lemma (whose conditions (a), (b), and (c) follow, respectively, from (3.6), (3.8), (3.5), and (3.7)) in the case where $\widehat{a}<\widehat{\sigma}$. This value is

$$\frac{\delta^2}{2\pi}\biggl(\frac{B}{n-k}\biggr)^{1-2/p}\widehat{a}^{2k+1-2/p}+\frac{1}{\widehat{a}^{2(n-k)}}\,. \tag{3.9}$$

Let us pass to the case $\widehat{a}\ge\widehat{\sigma}$. Then $a_0=\widehat{\sigma}$. Suppose that $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ are defined just as above (but with $a_0=\widehat{\sigma}$). The sequence $\{u_m(\,\cdot\,)\}$ is chosen constant, namely,

$$u_m(\xi)=\begin{cases}\widetilde{u}(\xi), & \xi\in[-\widehat{\sigma},\widehat{\sigma}],\\ 0 & \text{otherwise.}\end{cases}$$

It follows from the definition of $\widehat{\sigma}$ that

$$\int_{\mathbb{R}}\xi^{2n}u_m^2(\xi)\,d\xi=\int_{-\widehat{\sigma}}^{\widehat{\sigma}}\xi^{2n}\widetilde{u}^2(\xi)\,d\xi=1.$$

Applying the lemma (whose other conditions are easily verified), we find that, in this case, the value of problem (3.3) is

$$\frac{n+1/2-1/p}{k+1/2-1/p}\,\widehat{\sigma}^{-2(n-k)}.$$

Suppose that $\sigma<\widehat{\sigma}$. Then, obviously, $\widehat{a}\le\sigma<\widehat{\sigma}$. Expression (3.9), just as the function $\widehat{a}$, decreases on $(0,\widehat{\sigma}]$ and, therefore, the value of problem (3.2) is not less than

$$\sqrt{\frac{\delta^2}{2\pi}\biggl(\frac{B}{n-k}\biggr)^{1-2/p}\sigma^{2k+1-2/p}+\frac{1}{\sigma^{2(n-k)}}}\,. \tag{3.10}$$

Then, by what has proved above, the quantity $E(k,M_\sigma,p,\delta)$ is not less than the number (3.10), whci is independent of the structure of the set M_σ. Therefore, for $\sigma<\widehat{\sigma}$,

$$E(k,\sigma,p,\delta)\ge\sqrt{\frac{\delta^2}{2\pi}\biggl(\frac{B}{n-k}\biggr)^{1-2/p}\sigma^{2k+1-2/p}+\frac{1}{\sigma^{2(n-k)}}}\,.$$

Suppose that $\sigma\ge\widehat{\sigma}$. If $\widehat{a}<\widehat{\sigma}$, then, as proved above, the value of problem (3.2) is, necessarily, not less than the value of expression (3.10) at the point $\sigma=\widehat{\sigma}$; namely, it is easy to verify that it is

$$\sqrt{\frac{n+1/2-1/p}{k+1/2-1/p}}\,\widehat{\sigma}^{-(n-k)}. \tag{3.11}$$

But if $\widehat{a}\ge\widehat{\sigma}$, then, as has already been, the value of problem (3.2) is equal to the quantity (3.11). Thus, for $\sigma\ge\widehat{\sigma}$,

$$E(k,\sigma,p,\delta)\ge\sqrt{\frac{n+1/2-1/p}{k+1/2-1/p}}\,\widehat{\sigma}^{-(n-k)}.$$

2. *Proof of the optimality of the set* $\Delta_{\sigma_0} = [-\sigma_0, \sigma_0]$ *and of the method* $\widehat{m}$. The optimality of Δ_{σ_0} and $\widehat{m}$ implies that the value of the problem (the value of the supremum in the definition of $E(k, \sigma, p, \delta)$)

$$\begin{gathered}\|x^{(k)}(\,\cdot\,) - \widehat{m}(y(\,\cdot\,))(\,\cdot\,)\|_{L_2(\mathbb{R})} \to \max, \qquad \|Fx(\,\cdot\,) - y(\,\cdot\,)\|_{L_p(\Delta_{\sigma_0})} \le \delta, \\ \|x^{(n)}(\,\cdot\,)\|_{L_2(\mathbb{R})} \le 1, \qquad y(\,\cdot\,) \in L_p(\Delta_{\sigma_0})\end{gathered} \tag{3.12}$$

coincides with $E(k, \sigma, p, \delta)$.

Denoting $z(\,\cdot\,) = Fx(\,\cdot\,) - y(\,\cdot\,)$ and, for brevity, $\gamma(\xi) = (\xi/\sigma_0)^{2(n-k)}$, and using Plancherel's theorem, we find that the square of the value of problem (3.12) is equal to the value of the following problem:

$$\begin{gathered}\frac{1}{2\pi}\int_{\Delta_{\sigma_0}} \xi^{2k}|(1-\gamma(\xi))z(\xi) + \gamma(\xi)Fx(\xi)|^2\,d\xi + \frac{1}{2\pi}\int_{\mathbb{R}\setminus\Delta_{\sigma_0}} \xi^{2k}|Fx(\xi)|^2\,d\xi \to \max, \\ \int_{\Delta_{\sigma_0}} |z(\xi)|^p\,d\xi \le \delta^p, \qquad \frac{1}{2\pi}\int_{\mathbb{R}} \xi^{2n}|Fx(\xi)|^2\,d\xi \le 1.\end{gathered} \tag{3.13}$$

Setting $u(\xi) = (2\pi)^{-1/2}|z(\xi)|$ and $v(\xi) = (2\pi)^{-1/2}|Fx(\xi)|$, we put (3.13) in correspondence with the problem

$$\begin{gathered}\int_{\Delta_{\sigma_0}} \xi^{2k}((1-\gamma(\xi))u(\xi) + \gamma(\xi)v(\xi))^2\,d\xi + \int_{\mathbb{R}\setminus\Delta_{\sigma_0}} \xi^{2k}v^2(\xi)\,d\xi \to \max, \\ \int_{\Delta_{\sigma_0}} u^p(\xi)\,d\xi \le \frac{\delta^p}{(2\pi)^{p/2}}, \qquad \int_{\mathbb{R}} \xi^{2n}v^2(\xi)\,d\xi \le 1, \qquad u(\xi) \ge 0, \qquad v(\xi) \ge 0,\end{gathered} \tag{3.14}$$

whose value is, obviously, not less than that of problem (3.13). In order to find the value of problem (3.14) again, we use the lemma. The Lagrange function of this problem is of the form

$$\begin{aligned}\mathscr{L}(u(\,\cdot\,), v(\,\cdot\,), \lambda_1, \lambda_2) = \int_{\Delta_{\sigma_0}} &(-\xi^{2k}((1-\gamma(\xi))u(\xi) + \gamma(\xi)v(\xi))^2 \\ &+ \lambda_1 u^p(\xi) + \lambda_2\xi^{2n}v^2(\xi))\,d\xi + \int_{\mathbb{R}\setminus\Delta_{\sigma_0}} (-\xi^{2k} + \lambda_2\xi^{2n})v^2(\xi)\,d\xi.\end{aligned}$$

Suppose that $\xi \in \Delta_{\sigma_0}$. Set $\widehat{\lambda}_2 = \sigma_0^{-2(n-k)}$ and, for a fixed $\lambda_1 > 0$, consider the function

$$(u, v) \mapsto g(u, v) = -\xi^{2k}((1-\gamma(\xi))u + \gamma(\xi)v)^2 + \lambda_1 u^p + \sigma^{-2(n-k)}\xi^{2n}v^2$$

on $[0, \infty) \times [0, \infty)$. It is easy to verify that, for each $u \ge 0$, the function $v \mapsto g(u, v)$ attains its absolute minimum on $[0, \infty)$ at the point $v = u$, and hence $g(u, v) \ge g(u, u)$ for all $u \ge 0$ and $v \ge 0$. But $g(u, u) = f(u)$, where the function f was defined above, and the minimum of f is attained at the point $\widetilde{u}(\xi)$ for $a_0 = \sigma_0$.

Specifying the sequence $\{u_m(\,\cdot\,)\}$ just as above, taking $v_m(\,\cdot\,) = u_m(\,\cdot\,)$, and using the lemma, we find, in exactly the same way, that the value of problem (3.14) is

$$\frac{\delta^2}{2\pi}\left(\frac{B}{n-k}\right)^{1-2/p}\sigma_0^{2k+1-2/p} + \frac{1}{\sigma_0^{2(n-k)}}.$$

Therefore,

$$E(k, \sigma, p, \delta) \ge \sqrt{\frac{\delta^2}{2\pi}\left(\frac{B}{n-k}\right)^{1-2/p}\sigma_0^{2k+1-2/p} + \frac{1}{\sigma_0^{2(n-k)}}}\,,$$

which proves the optimality of the closed interval Δ_{σ_0} and the optimality of the method $\widehat{m}$. The theorem is proved. $\square$

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