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Optimal recovery in weighted spaces with homogeneous weights

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Abstract. The paper concerns problems of the recovery of operators from noisy information in weighted L_q-spaces with homogeneous weights. A number of general theorems are proved and applied to problems of the recovery of differential operators from a noisy Fourier transform. In particular, optimal methods are obtained for the recovery of powers of the Laplace operator from a noisy Fourier transform in the L_p-metric.

Bibliography: 30 titles.

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§ 1. General statement

Let T be some nonempty set, Σ be a σ-algebra of subsets of T, and μ be a nonnegative σ-additive measure on Σ. We let $L_p(T,\mu)$ denote the class of all Σ-measurable functions with values in $\mathbb{R}$ or in $\mathbb{C}$ such that

$$\|x(\,\cdot\,)\|_{L_p(T,\mu)}=\begin{cases}\displaystyle\biggl(\int_T|x(t)|^p\,d\mu\biggr)^{1/p}<\infty, & 1\leqslant p<\infty,\\ \operatorname*{vraisup}_{t\in T}|x(t)|<\infty, & p=\infty.\end{cases}$$

We set

$$\mathcal{W}=\bigl\{x(\,\cdot\,)\in L_p(T,\mu)\colon \|\varphi(\,\cdot\,)x(\,\cdot\,)\|_{L_r(T,\mu)}<\infty\bigr\}$$

and

$$W=\bigl\{x(\,\cdot\,)\in \mathcal{W}\colon \|\varphi(\,\cdot\,)x(\,\cdot\,)\|_{L_r(T,\mu)}\leqslant 1\bigr\},$$

where $1\leqslant p,r\leqslant\infty$ and $\varphi(\,\cdot\,)$ is some function on T.

Consider the problem of the recovery of the operator $\Lambda\colon \mathcal{W}\to L_q(T,\mu)$, $1\leqslant q\leqslant\infty$, given by $\Lambda x(\,\cdot\,)=\psi(\,\cdot\,)x(\,\cdot\,)$, where $\psi(\,\cdot\,)$ is some function on T. The recovery is effected on the class W from a function $x(\,\cdot\,)\in W$ known with error on T (we assume that the functions $\varphi(\,\cdot\,)$ and $\psi(\,\cdot\,)$ are such that the operator Λ maps the space $\mathcal{W}$ into $L_q(T,\mu)$).

It is assumed that, for each function $x(\,\cdot\,)\in W$, one knows a function $y(\,\cdot\,)\in L_p(T,\mu)$ such that $\|x(\,\cdot\,)-y(\,\cdot\,)\|_{L_p(T,\mu)}\leqslant\delta$, $\delta>0$. It is required to recover the function $\Lambda x(\,\cdot\,)$ from $y(\,\cdot\,)$. As methods of recovery, we consider all possible mappings $m\colon L_p(T,\mu)\to L_q(T,\mu)$.

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The *error of a method* m is defined by

$$e_{pqr}(m) = \sup_{\substack{x(\,\cdot\,)\in W,\ y(\,\cdot\,)\in L_p(T,\mu)\\ \|x(\,\cdot\,)-y(\,\cdot\,)\|_{L_p(T,\mu)}\leqslant\delta}} \|\Lambda x(\,\cdot\,) - m(y)(\,\cdot\,)\|_{L_q(T,\mu)}.$$

The quantity

$$E_{pqr} = \inf_{m\colon L_p(T,\mu)\to L_q(T,\mu)} e_{pqr}(m) \tag{1.1}$$

is known as the *error of optimal recovery*; a method on which the infimum is attained is called an *optimal method*.

The above problem is a particular case of the general problem of the recovery of a linear operator Λ, which acts from a linear space X into a normed linear space Z, on a set $W \subset X$ from the values of a linear operator I acting from X into a normed linear space Y and given with some error δ. In problem (1.1),

$$X = \mathcal{W}, \qquad Z = L_q(T,\mu), \qquad Y = L_p(T,\mu)$$

and the operator $I\colon \mathcal{W} \to L_p(T,\mu)$ is defined by $Ix(\,\cdot\,) = x(\,\cdot\,)$.

The original general recovery problem, which appeared as a generalization of Kolmogorov's problem of the best quadrature formula (see [1]), was posed by Smolyak (see [2]). In this statement, Λ is a linear functional, I consists of a finite number of precisely given ($\delta = 0$) linear functionals, and, in contrast to the problem of best quadrature formulae, the class of recovery methods consists of all possible (not necessarily linear) methods of approximation. Smolyak proved that for convex symmetric sets W the set of optimal methods contains a linear method.

Bakhvalov proposed to extend this setting to the case when linear functionals are not known exactly, but only approximately, with some error. It was found out that a similar result also holds in this case (see [3]).

The problem of recovery was given its most general form in the paper [4], which was concerned with the recovery of linear operators in the infinite-dimensional setting. Problems of the existence of a linear optimal method in the problem of recovery of a linear functional were examined in [4]–[6]. The most general result in this direction was obtained in [7], and a final (in a certain sense) criterion for the existence of a linear optimal method, in [8].

A linear method in the problem of the recovery of linear operators may fail to exist — this is a difference from the problem of the recovery of linear functionals. For a corresponding example, see [9], where, in particular, conditions are given for a set of optimal methods to contain linear ones and for the error of optimal recovery to be equal to that of the dual extremal problem

$$\sup\bigl\{\|\Lambda x\|_Z\colon x \in W,\ \|Ix\|_Y \leqslant \delta\bigr\}. \tag{1.2}$$

The quantity (1.2) is frequently called the modulus of continuity of the operator Λ on the class W (with respect to the operator I). The study of this quantity has great value to the derivation of a number of sharp inequalities like Carlson's inequalities, Landau-Kolmogorov type inequalities for derivatives, in the Stechkin problem and so on. Carlson's inequality itself was found to be closely related to inequalities for derivatives; for example, Taikov's inequality (see [10]) can be easily derived from the generalized Carlson inequality (see [11]), which was put forward by Levin [12] already in 1948. For more details on the relations of the quantity (1.2) to the Stechkin problem and to recovery problems, see [13], [14].

We also point out a series of papers by Arestov [15]–[18] dealing with differentiation operators on the real line. His results are pretty close to those considered in our paper. For similar multivariate problems, see [19] and [20].

The method of construction of an optimal method of recovery for linear operators, as proposed in [9], can only be used in the case when all the metrics in problem (1.1) are Euclidean. For non-Euclidean metrics, under the condition that at least two of them are the same, a method was proposed in [21], which is also used in the present paper. This method consists of two stages. In the first stage, the error of optimal recovery is estimated from below in terms of the value of the extremal problem (1.2). Note that, in order to abbreviate the proof, there is no need to write down the solution (1.2), because (since we are concerned with estimates from below) it suffices to show a 'correctly' chosen admissible function (though, as a rule, this 'correctly' chosen function is found as a result of solving the extremal problem (1.2) itself). The upper estimate is given in the second stage. To this end, one considers a method consisting of a recovery operator, which would be applied with precise information and which involves some smoothing factor. The error of this method is estimated by employing either the Cauchy-Bunyakovskii-Schwarz inequality or Hölder's inequality with some weights. Next, weights and a smoothing factor are chosen so that the upper estimate coincides with the lower one.

The case when all three parameters p, q and r in problem (1.1) are distinct was considered in [11], where the scheme of construction of an optimal recovery method also involves upper and lower estimates, but first one requires a more subtle analysis of the Lagrange functions for the extremal problem (1.2) and for the extremal problem for the error of the estimated recovery method. This approach, which was implemented in [11], was shown to be capable of not merely delivering an optimal recovery method, but also of producing a sharp Carlson-type inequality in a fairly general form. In the case of homogeneous weights, the inequality thus obtained implies the one derived previously in [22].

In this paper we solve problem (1.1) for homogeneous weights $\varphi(\,\cdot\,)$ and $\psi(\,\cdot\,)$ with $(p,q,r)\in P_1\cup P_2$, where

$$P_1=\bigl\{(p,q,r)\colon 1\leqslant q=r<p<\infty\bigr\}\quad\text{and}\quad P_2=\bigl\{(p,q,r)\colon 1\leqslant q=p<r<\infty\bigr\}.$$

The case when $(p,q,r)\in P=\{(p,q,r)\colon 1\leqslant q<p,r<\infty\}$ was considered in [11]. The main results in this paper, which are based on the solution of problem (1.2), provide, in the multivariate case, optimal recovery methods for linear operators defined, in terms of Fourier images, by multiplication by a homogeneous weight, on classes of functions defined in terms of similar-type operators, in the $L_2(\mathbb{R}^d)$- and $L_\infty(\mathbb{R}^d)$-metrics from information about a noisy Fourier transform in $L_p(\mathbb{R}^d)$ (Theorems 3 and 5). Based on these general results, we obtain methods for the optimal recovery of powers of the Laplace operator $(-\Delta)^{k/2}$ and the differentiation operators D^α of orders $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{R}^d_+$. Similar results were obtained previously only for powers of the Laplace operator with $p=2,\infty$ in the $L_2(\mathbb{R}^d)$-metric (see [23] and [24]) and $p=\infty$ in the $L_\infty(\mathbb{R}^d)$-metric (see [25]). In our paper, in the first case we obtain results for $2<p<\infty$, and in the second case, for $1\leqslant p<\infty$.

§ 2. Optimal recovery with homogeneous weights for two equal metrics

Denote

$$a_+ = \begin{cases} a, & a \geqslant 0, \\ 0, & a < 0. \end{cases}$$

We require the following result from [21].

Theorem 1. 1. *Let $(p,q,r) \in P_1$. If $\widehat{\lambda}_2$ is a solution of the equation*

$$\begin{gathered}\biggl(\int_T \bigl(|\psi(t)|^q - \widehat{\lambda}_2|\varphi(t)|^q\bigr)_+^{p/(p-q)}\,d\mu(t)\biggr)^{1/p} \\ = \delta\biggl(\int_T |\varphi(t)|^q\bigl(|\psi(t)|^q - \widehat{\lambda}_2|\varphi(t)|^q\bigr)_+^{q/(p-q)}\,d\mu(t)\biggr)^{1/q} > 0, \qquad (2.1) \\ \widehat{\lambda}_1 = \frac{q}{p}\delta^{q-p}\biggl(\int_T \bigl(|\psi(t)|^q - \widehat{\lambda}_2|\varphi(t)|^q\bigr)_+^{p/(p-q)}\,d\mu(t)\biggr)^{(p-q)/p},\end{gathered}$$

then

$$E_{pqq} = \biggl(\frac{p}{q}\widehat{\lambda}_1\delta^p + \widehat{\lambda}_2\biggr)^{1/q},$$

and the method

$$\widehat{m}(y)(t) = \biggl(1 - \widehat{\lambda}_2\frac{|\varphi(t)|^q}{|\psi(t)|^q}\biggr)_+\psi(t)y(t) \qquad (2.2)$$

is optimal.

2. *Let $(p,q,r) \in P_2$. If $\widehat{\lambda}_1$ is the solution of the equation*

$$\begin{gathered}\biggl(\int_T |\varphi(t)|^{pr/(p-r)}\bigl(|\psi(t)|^p - \widehat{\lambda}_1\bigr)_+^{p/(r-p)}\,d\mu(t)\biggr)^{1/p} \\ = \delta\biggl(\int_T |\varphi(t)|^{pr/(p-r)}\bigl(|\psi(t)|^p - \widehat{\lambda}_1\bigr)_+^{r/(r-p)}\,d\mu(t)\biggr)^{1/r} > 0, \qquad (2.3) \\ \widehat{\lambda}_2 = \frac{p}{r}\delta^{p-r}\biggl(\int_T |\varphi(t)|^{pr/(p-r)}\bigl(|\psi(t)|^p - \widehat{\lambda}_1\bigr)_+^{p/(r-p)}\,d\mu(t)\biggr)^{(r-p)/p},\end{gathered}$$

then

$$E_{ppr} = \biggl(\widehat{\lambda}_1\delta^p + \frac{r}{p}\widehat{\lambda}_2\biggr)^{1/p},$$

and the method

$$\widehat{m}(y)(t) = \alpha(t)\psi(t)y(t),$$

where

$$\alpha(t) = \min\biggl\{1, \frac{\widehat{\lambda}_1}{|\psi(t)|^p}\biggr\},$$

is optimal.

We apply this result to the case when T is a cone in a linear space, $|\psi(\,\cdot\,)|$ and $|\varphi(\,\cdot\,)|$ are homogeneous functions of orders $k \geqslant 0$ and $n > 0$, respectively (k and n are not necessarily integers) and $\mu(\,\cdot\,)$ is a homogeneous measure of order $d > 0$.

Corollary 1. 1. *Let* $(p,q,r)\in P_1$, $k\geqslant 0$, $n>k$, *and let*

$$I_1=\int_T\big(|\psi(\xi)|^q-|\varphi(\xi)|^q\big)_+^{p/(p-q)}\,d\mu(\xi)<\infty$$

and

$$I_2=\int_T|\varphi(\xi)|^q\big(|\psi(\xi)|^q-|\varphi(\xi)|^q\big)_+^{q/(p-q)}\,d\mu(\xi)<\infty.$$

Then

$$E_{pqq}=I_1^{-\frac{1}{p}\frac{n-k}{n+d(1/q-1/p)}}I_2^{-\frac{1}{q}\frac{k+d(1/q-1/p)}{n+d(1/q-1/p)}}(I_1+I_2)^{1/q}\delta^{\frac{n-k}{n+d(1/q-1/p)}},$$

and the method

$$\widehat{m}(y)(t)=\left(1-\left(\delta\frac{I_2^{1/q}}{I_1^{1/p}}\right)^{\frac{(n-k)q}{n+d(1/q-1/p)}}\frac{|\varphi(t)|^q}{|\psi(t)|^q}\right)_+\psi(t)y(t) \tag{2.4}$$

is optimal.

2. *Let* $(p,q,r)\in P_2$, $k>0$, $n>k+d(1/p-1/r)$, *and let*

$$J_1=\int_T|\varphi(\xi)|^{pr/(p-r)}\big(|\psi(\xi)|^p-1\big)_+^{p/(r-p)}\,d\mu(\xi)<\infty$$

and

$$J_2=\int_T|\varphi(\xi)|^{pr/(p-r)}\big(|\psi(\xi)|^p-1\big)_+^{r/(r-p)}\,d\mu(\xi)<\infty.$$

Then

$$E_{ppr}=J_1^{-\frac{1}{p}\frac{n-k-d(1/p-1/r)}{n-d(1/p-1/r)}}J_2^{-\frac{1}{r}\frac{k}{n-d(1/p-1/r)}}(J_1+J_2)^{1/p}\delta^{\frac{n-k-d(1/p-1/r)}{n-d(1/p-1/r)}},$$

and the method

$$\widehat{m}(y)(t)=\min\left\{1,\left(\frac{J_1^{1/p}}{\delta J_2^{1/r}}\right)^{\frac{kp}{n-d(1/p-1/r)}}\frac{1}{|\psi(t)|^p}\right\}\psi(t)y(t) \tag{2.5}$$

is optimal.

Proof. 1. Consider equation (2.1). We seek $\widehat{\lambda}_2$ in the form $\widehat{\lambda}_2=a^{(k-n)q}$, $a>0$. Substituting $t=a\xi$ into (2.1) we obtain

$$a^{kq/(p-q)+d/p}I_1^{1/p}=\delta a^{n+kq/(p-q)+d/q}I_2^{1/q}.$$

As a result,

$$a=\left(\frac{I_1^{1/p}}{\delta I_2^{1/q}}\right)^{\frac{1}{n+d(1/q-1/p)}}.$$

Using the same substitution we obtain

$$\widehat{\lambda}_1=\frac{q}{p}\delta^{q-p}a^{q(k+d(1/q-1/p))}I_1^{(p-q)/p}.$$

It remains to plug the resulting quantities into the expressions for the error of optimal recovery and for the optimal method.

2. Consider equation (2.3). We seek $\widehat{\lambda}_1$ in the form $\widehat{\lambda}_1 = a^{kp}$, $a > 0$. Substituting $t = a\xi$ into (2.3) we obtain

$$a^{nr/(p-r)+d/p} J_1^{1/p} = \delta a^{np/(p-r)+d/r} J_2^{1/r}.$$

Hence

$$a = \left(\frac{J_1^{1/p}}{\delta J_2^{1/r}}\right)^{\frac{1}{n+d(1/r-1/p)}}.$$

The same substitution gives

$$\widehat{\lambda}_2 = \frac{p}{r}\delta^{p-r} a^{r(-n+kp/r+d(1/p-1/r))} J_1^{(r-p)/p}.$$

Now the required result follows if we plug the resulting quantities into the expressions for the density of optimal recovery and for the optimal method.

§ 3. Homogeneous weights in $\mathbb{R}^d$

Let T be a cone in $\mathbb{R}^d$, $d\mu(t) = dt$, and let $|\psi(\,\cdot\,)|$ and $|\varphi(\,\cdot\,)|$ be homogeneous functions of orders $k \geqslant 0$ and $n > 0$, respectively, $\varphi(t) \neq 0$ and $\psi(t) \neq 0$ for almost all $t \in T$. Consider the spherical coordinate system

$$\begin{aligned} t_1 &= \rho\cos\omega_1, \\ t_2 &= \rho\sin\omega_1\cos\omega_2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ t_{d-1} &= \rho\sin\omega_1\sin\omega_2\cdots\sin\omega_{d-2}\cos\omega_{d-1}, \\ t_d &= \rho\sin\omega_1\sin\omega_2\cdots\sin\omega_{d-2}\sin\omega_{d-1}. \end{aligned}$$

We set $\omega = (\omega_1, \dots, \omega_{d-1})$,

$$\begin{aligned} \widetilde{\psi}(\omega) &= \rho^{-k}|\psi(\rho\cos\omega_1, \dots, \rho\sin\omega_1\sin\omega_2\cdots\sin\omega_{d-2}\sin\omega_{d-1})|, \\ \widetilde{\varphi}(\omega) &= \rho^{-n}|\varphi(\rho\cos\omega_1, \dots, \rho\sin\omega_1\sin\omega_2\cdots\sin\omega_{d-2}\sin\omega_{d-1})|. \end{aligned} \tag{3.1}$$

Let Ω be the range of ω as t runs over T. Since T is a cone, it follows that Ω is independent of ρ. We set

$$J(\omega) = \sin^{d-2}\omega_1\sin^{d-3}\omega_2\cdots\sin\omega_{d-2}.$$

For $1 \leqslant q < p, r$, the function $\kappa^{r-q}(1-\kappa)^{-(p-q)}$ is monotone increasing from 0 to $+\infty$ for $\kappa \in [0, 1)$. Hence for all $t \in T$ we can define the function $\kappa(t)$ by

$$\frac{\kappa^{r-q}(t)}{(1-\kappa(t))^{p-q}} = \frac{|\psi(t)|^{q(p-r)}}{|\varphi(t)|^{r(p-q)}}.$$

For $q = r$ we set

$$\kappa(t) = \left(1 - \frac{|\varphi(t)|^q}{|\psi(t)|^q}\right)_+,$$

and for $q = p$ we define

$$\kappa(t) = \min\{1, |\psi(t)|^{-p}\}.$$

Consider the quantity

$$\gamma = \frac{n-k-d(1/q-1/r)}{n+d(1/r-1/p)}.$$

Let $k > d(1/p-1/q)$ and $n > k+d(1/q-1/r)$. It is easily checked that $\gamma \in (0,1)$. In this case we define the number q^* by

$$\frac{1}{q^*} = \frac{1}{q} - \frac{\gamma}{p} - \frac{1-\gamma}{r}.$$

Theorem 2. *Let $k > d(1/p-1/q)$, $n > k+d(1/q-1/r)$ and $(p,q,r) \in P \cup P_1 \cup P_2$. Assume that*

$$I = \int_\Omega \frac{\widetilde{\psi}^{q^*}(\omega)}{\widetilde{\varphi}^{q^*(1-\gamma)}(\omega)} J(\omega)\, d\omega < \infty.$$

Then $E_{pqr} = C\delta^\gamma$, where

$$C = \gamma^{-\gamma/p}(1-\gamma)^{-(1-\gamma)/r}\left(\frac{B\big(q^*\gamma/p+1, q^*(1-\gamma)/r\big)I}{r(n-k-d(1/q-1/r))}\right)^{1/q^*},$$

and $B(\,\cdot\,,\cdot\,)$ is the Euler beta function. Moreover, the method

$$\widehat{m}(y)(t) = \kappa\big(\xi_1^{\frac{1}{n+d(1/r-1/p)}} t\big)\psi(t)y(t),$$

where

$$\xi_1 = \delta\big(\gamma^{q-r}(1-\gamma)^{p-q}C^{(p-r)q}\big)^{q^*/(pqr)},$$

is optimal.

Proof. The case $(p,q,r) \in P$ is covered by Theorem 3 in [11] (in that paper, the answer is given in terms of the beta function with arguments $q^*\gamma/p$ and $q^*(1-\gamma)/r$, but it is more convenient for our purposes to change to $q^*\gamma/p+1$ and $q^*(1-\gamma)/r$; this can easily be effected by using properties of the beta function). It remains to consider the following two cases: $(p,q,r) \in P_1$ and $(p,q,r) \in P_2$.

1. Let $(p,q,r) \in P_1$. We use Corollary 1 and change to the spherical coordinates in the integral I_1. Then we have

$$\begin{aligned} I_1 &= \int_0^{+\infty} \rho^{d-1}\, d\rho \int_\Omega \big(\rho^{kq}\widetilde{\psi}^q(\omega) - \rho^{nq}\widetilde{\varphi}^q(\omega)\big)_+^{p/(p-q)} J(\omega)\, d\omega \\ &= \int_\Omega \widetilde{\psi}^{qp/(p-q)}(\omega)J(\omega)\, d\omega \int_0^{+\infty} \rho^{kqp/(p-q)+d-1}\left(1-\rho^{(n-k)q}\frac{\widetilde{\varphi}^q(\omega)}{\widetilde{\psi}^q(\omega)}\right)_+^{p/(p-q)} d\rho. \end{aligned}$$

For a fixed ω, substituting

$$t = \rho^{(n-k)q}\frac{\widetilde{\varphi}^q(\omega)}{\widetilde{\psi}^q(\omega)} \tag{3.2}$$

into the second integral, we have

$$\begin{aligned} I_1 &= \frac{1}{(n-k)q}\int_\Omega \widetilde{\psi}^{qp/(p-q)}(\omega)\left(\frac{\widetilde{\psi}(\omega)}{\widetilde{\varphi}(\omega)}\right)^{\frac{kqp}{(p-q)(n-k)}+\frac{d}{n-k}} J(\omega)\,d\omega \\ &\quad\times\int_0^1 t^{\frac{kp}{(p-q)(n-k)}+\frac{d}{(n-k)q}-1}(1-t)^{p/(p-q)}\,dt \\ &= \frac{I}{(n-k)q}B\left(\frac{q^*\gamma}{p}+2,\frac{q^*(1-\gamma)}{q}\right). \end{aligned}$$

A similar analysis for I_2 shows that

$$\begin{aligned} I_2 &= \int_0^{+\infty}\rho^{nq+d-1}\,d\rho\int_\Omega \widetilde{\varphi}^q(\omega)\big(\rho^{kq}\widetilde{\psi}^q(\omega)-\rho^{nq}\widetilde{\varphi}^q(\omega)\big)_+^{q/(p-q)}J(\omega)\,d\omega \\ &= \int_\Omega \widetilde{\varphi}^q(\omega)\widetilde{\psi}^{q^2/(p-q)}(\omega)J(\omega)\,d\omega \\ &\quad\times\int_0^{+\infty}\rho^{nq+kq^2/(p-q)+d-1}\left(1-\rho^{(n-k)q}\frac{\widetilde{\varphi}^q(\omega)}{\widetilde{\psi}^q(\omega)}\right)_+^{q/(p-q)}\,d\rho. \end{aligned}$$

Making the same change (3.2) we obtain

$$\begin{aligned} I_2 &= \frac{1}{(n-k)q}\int_\Omega \widetilde{\varphi}^q(\omega)\widetilde{\psi}^{q^2/(p-q)}(\omega)\left(\frac{\widetilde{\psi}(\omega)}{\widetilde{\varphi}(\omega)}\right)^{\frac{nq}{n-k}+\frac{kq^2}{(p-q)(n-k)}+\frac{d}{n-k}} J(\omega)\,d\omega \\ &\quad\times\int_0^1 t^{\frac{n}{n-k}+\frac{kq}{(p-q)(n-k)}+\frac{d}{(n-k)q}-1}(1-t)^{q/(p-q)}\,dt \\ &= \frac{I}{(n-k)q}B\left(\frac{q^*\gamma}{p}+1,\frac{q^*(1-\gamma)}{q}+1\right). \end{aligned}$$

We set

$$B_1 = B\left(\frac{q^*\gamma}{p}+1,\frac{q^*(1-\gamma)}{r}\right).$$

Hence, from the properties of the beta function we have

$$I_1 = \frac{q^*\gamma/p+1}{q(n-k)(q^*\gamma/p+1+q^*(1-\gamma)/q)}B_1 I$$

and

$$I_2 = \frac{q^*(1-\gamma)/q}{q(n-k)(q^*\gamma/p+1+q^*(1-\gamma)/q)}B_1 I.$$

In the case under consideration (for $r=q$)

$$\frac{1}{q^*} = \gamma\left(\frac{1}{q}-\frac{1}{p}\right) \quad\text{and}\quad \gamma = \frac{n-k}{n+d(1/q-1/p)}.$$

Hence $q^*\gamma/p+1 = q^*\gamma/q$. As a result,

$$I_1 = \gamma\frac{B_1 I}{q(n-k)} \quad\text{and}\quad I_2 = (1-\gamma)\frac{B_1 I}{q(n-k)}.$$

From Corollary 1 we find that

$$E_{pqq} = I_1^{-\gamma/p} I_2^{-(1-\gamma)/q} (I_1+I_2)^{1/q}\delta^\gamma$$
$$= \gamma^{-\gamma/p}(1-\gamma)^{-(1-\gamma)/q}\left(\frac{B_1 I}{q(n-k)}\right)^{\gamma(1/q-1/p)}\delta^\gamma = C\delta^\gamma.$$

The method (2.4) can be written as

$$\widehat{m}(y)(t) = \kappa\big(b^{\frac{1}{n+d(1/r-1/p)}} t\big)\psi(t)y(t),$$

where

$$b = \delta\frac{I_2^{1/q}}{I_1^{1/p}} = \delta\gamma^{-1/p}(1-\gamma)^{1/q}\left(\frac{B_1 I}{q(n-k)}\right)^{1/q-1/p}$$
$$= \delta\gamma^{-1/p}(1-\gamma)^{1/q}C^{1/\gamma}\gamma^{1/p}(1-\gamma)^{(1-\gamma)/(q\gamma)}$$
$$= \delta(1-\gamma)^{1/(q\gamma)}C^{1/\gamma} = \delta\big((1-\gamma)C^q\big)^{1/(q\gamma)} = \delta\big((1-\gamma)C^q\big)^{\frac{q^*}{q}(\frac{1}{q}-\frac{1}{p})}$$
$$= \delta\big((1-\gamma)C^q\big)^{q^*(p-q)/(pq^2)} = \xi_1.$$

2. Let $(p,q,r)\in P_2$. We employ Corollary 1 again. Changing to the spherical coordinates in J_1 we obtain

$$J_1 = \int_0^{+\infty}\rho^{d-1}\,d\rho\int_\Omega \rho^{npr/(p-r)}\widetilde{\varphi}^{pr/(p-r)}(\omega)\big(\rho^{kp}\widetilde{\psi}^p(\omega)-1\big)_+^{p/(r-p)}J(\omega)\,d\omega$$
$$= \int_\Omega \widetilde{\varphi}^{pr/(p-r)}(\omega)J(\omega)\,d\omega\int_0^{+\infty}\rho^{npr/(p-r)+d-1}\big(\rho^{kp}\widetilde{\psi}^p(\omega)-1\big)_+^{p/(r-p)}\,d\rho.$$

For a fixed ω, substituting

$$t = \rho^{kp}\widetilde{\psi}^p(\omega) \tag{3.3}$$

into the second integral, we have

$$J_1 = \frac{1}{kp}\int_\Omega \widetilde{\varphi}^{\frac{pr}{p-r}}(\omega)\widetilde{\psi}^{-\frac{npr}{(p-r)k}-\frac{d}{k}}(\omega)J(\omega)\,d\omega$$
$$\times\int_1^{+\infty} t^{\frac{nr}{(p-r)k}+\frac{d}{kp}-1}(t-1)^{p/(r-p)}\,dt$$
$$= \frac{I}{kp}\int_0^1 s^{\frac{nr}{(r-p)k}-\frac{p}{r-p}-\frac{d}{kp}-1}(1-s)^{p/(r-p)}\,ds = \frac{I}{kp}B\left(\frac{q^*\gamma}{p}+1, \frac{q^*(1-\gamma)}{r}+1\right).$$

A similar analysis for J_2 shows that

$$J_2 = \int_0^{+\infty}\rho^{d-1}\,d\rho\int_\Omega \rho^{npr/(p-r)}\widetilde{\varphi}^{pr/(p-r)}(\omega)\big(\rho^{kp}\widetilde{\psi}^p(\omega)-1\big)_+^{r/(r-p)}J(\omega)\,d\omega$$
$$= \int_\Omega \widetilde{\varphi}^{pr/(p-r)}(\omega)J(\omega)\,d\omega\int_0^{+\infty}\rho^{npr/(p-r)+d-1}\big(\rho^{kp}\widetilde{\psi}^p(\omega)-1\big)_+^{r/(r-p)}\,d\rho.$$

Making the same change (3.3) yields

$$J_2 = \frac{1}{kp}\int_\Omega \widetilde{\varphi}^{pr/(p-r)}(\omega)\widetilde{\psi}^{-\frac{npr}{(p-r)k}-\frac{d}{k}}(\omega)J(\omega)\,d\omega\int_1^{+\infty} t^{\frac{nr}{(p-r)k}+\frac{d}{kp}-1}(t-1)^{r/(r-p)}\,dt$$
$$= \frac{I}{kp}\int_0^1 s^{\frac{nr}{(r-p)k}-\frac{r}{r-p}-\frac{d}{kp}-1}(1-s)^{r/(r-p)}\,ds = \frac{I}{kp}B\biggl(\frac{q^*\gamma}{p}, \frac{q^*(1-\gamma)}{r}+2\biggr).$$

We set

$$B_2 = B\biggl(\frac{q^*\gamma}{p}, \frac{q^*(1-\gamma)}{r}+1\biggr).$$

Hence, from the properties of the beta function we obtain

$$J_1 = \frac{q^*\gamma/p}{kp(q^*\gamma/p+q^*(1-\gamma)/r+1)}B_2I$$

and

$$J_2 = \frac{q^*(1-\gamma)/r+1}{kp(q^*\gamma/p+q^*(1-\gamma)/r+1)}B_2I.$$

In the case under consideration (for $q=p$),

$$\frac{1}{q^*} = (1-\gamma)\biggl(\frac{1}{p}-\frac{1}{r}\biggr) \quad \text{and} \quad \gamma = \frac{n-k-d(1/p-1/r)}{n-d(1/p-1/r)}.$$

Hence $q^*(1-\gamma)/r+1 = q^*(1-\gamma)/p$. As a result,

$$J_1 = \gamma\frac{B_2I}{kp}, \qquad J_2 = (1-\gamma)\frac{B_2I}{kp}.$$

From Corollary 1 we have

$$E_{ppr} = J_1^{-\gamma/p}J_2^{-(1-\gamma)/r}(J_1+J_2)^{1/p}\delta^\gamma$$
$$= \gamma^{-\gamma/p}(1-\gamma)^{-(1-\gamma)/r}\biggl(\frac{B_2I}{kp}\biggr)^{(1-\gamma)(1/p-1/r)}\delta^\gamma.$$

From the properties of the beta function we find that

$$B_2 = \frac{q^*(1-\gamma)/r}{q^*\gamma/p}B_1 = \frac{kpB_1}{r(n-k-d(1/q-1/r))}.$$

Therefore,

$$E_{ppr} = \gamma^{-\gamma/p}(1-\gamma)^{-(1-\gamma)/r}\biggl(\frac{B_1I}{r(n-k-d(1/q-1/r))}\biggr)^{1/q^*}\delta^\gamma = C\delta^\gamma.$$

The method (2.5) can be written as

$$\widehat{m}(y)(t) = \kappa\bigl(c^{\frac{1}{n-d(1/p-1/r)}}t\bigr)\psi(t)y(t),$$

where

$$c=\delta\frac{J_2^{1/r}}{J_1^{1/p}}=\delta\gamma^{-1/p}(1-\gamma)^{1/r}\biggl(\frac{B_2I}{kp}\biggr)^{1/r-1/p}$$
$$=\delta\gamma^{-1/p}(1-\gamma)^{1/r}C^{-1/(1-\gamma)}\gamma^{-\gamma/(p(1-\gamma))}(1-\gamma)^{-1/r}$$
$$=\delta\gamma^{-1/(p(1-\gamma))}C^{-1/(1-\gamma)}=\delta(\gamma C^p)^{-1/(p(1-\gamma))}$$
$$=\delta(\gamma C^p)^{(p-r)q^*/(p^2r)}=\xi_1.$$

It follows from [21] and [11] that, in all cases under consideration,

$$E_{pqr}=\sup_{\substack{x(\,\cdot\,)\in W\\ \|x(\,\cdot\,)\|_{L_p(T,\mu)}\leqslant\delta}}\|\Lambda x(\,\cdot\,)\|_{L_q(T,\mu)}. \tag{3.4}$$

As a result, we easily obtain the following sharp inequality:

$$\|\Lambda x(\,\cdot\,)\|_{L_q(T,\mu)}\leqslant C\|x(\,\cdot\,)\|^{\gamma}_{L_p(T,\mu)}\|\varphi(\,\cdot\,)x(\,\cdot\,)\|^{1-\gamma}_{L_r(T,\mu)}.$$

§ 4. Recovery of differential operators from a noisy Fourier transform

Let $T=\mathbb{R}^d$, $d\mu(t)=dt$, and let, as before, $|\psi(\,\cdot\,)|$ and $|\varphi(\,\cdot\,)|$ be homogeneous functions of orders $k\geqslant 0$ and $n>0$, respectively, $\varphi(t)\neq 0$ and $\psi(t)\neq 0$ for almost all $t\in\mathbb{R}^d$. We set

$$X_p=\bigl\{x(\,\cdot\,)\in L_2(\mathbb{R}^d)\colon \varphi(\,\cdot\,)Fx(\,\cdot\,)\in L_2(\mathbb{R}^d),\ Fx(\,\cdot\,)\in L_p(\mathbb{R}^d)\bigr\},$$

where $Fx(\,\cdot\,)$ is the Fourier transform of $x(\,\cdot\,)$,

$$Fx(\xi)=\int_{\mathbb{R}^d}x(t)e^{-i\langle\xi,t\rangle}\,dt,\qquad \langle\xi,t\rangle=\xi_1t_1+\dots+\xi_dt_d.$$

Let the operator D be defined by

$$Dx(\,\cdot\,)=F^{-1}\bigl(\varphi(\,\cdot\,)Fx(\,\cdot\,)\bigr)(\,\cdot\,).$$

Assume that $\psi(\,\cdot\,)x(\,\cdot\,)\in L_2(\mathbb{R}^d)$ for all $x(\,\cdot\,)\in X_p$. We set

$$\Lambda x(\,\cdot\,)=F^{-1}\bigl(\psi(\,\cdot\,)Fx(\,\cdot\,)\bigr)(\,\cdot\,).$$

Consider the problem of the optimal recovery of values of the operator Λ on the class

$$W_p=\bigl\{x(\,\cdot\,)\in X_p\colon \|Dx(\,\cdot\,)\|_{L_2(\mathbb{R}^d)}\leqslant 1\bigr\}$$

from the noisy Fourier transform of the function $x(\,\cdot\,)$. We assume that, for each $x(\,\cdot\,)\in W_p$, one knows a function $y(\,\cdot\,)\in L_p(\mathbb{R}^d)$ such that $\|Fx(\,\cdot\,)-y(\,\cdot\,)\|_{L_p(\mathbb{R}^d)}\leqslant\delta$, $\delta>0$. It is required to recover the function $\Lambda x(\,\cdot\,)$ from $y(\,\cdot\,)$. Assume that $\Lambda x(\,\cdot\,)\in L_q(\mathbb{R}^d)$ for all $x(\,\cdot\,)\in X_p$. As recovery methods we consider all possible mappings $m\colon L_p(\mathbb{R}^d)\to L_q(\mathbb{R}^d)$. The *error of a method* m is defined by

$$e_{pq}(\Lambda,D,m)=\sup_{\substack{x(\,\cdot\,)\in W_p,\ y(\,\cdot\,)\in L_p(\mathbb{R}^d)\\ \|Fx(\,\cdot\,)-y(\,\cdot\,)\|_{L_p(\mathbb{R}^d)}\leqslant\delta}}\|\Lambda x(\,\cdot\,)-m(y)(\,\cdot\,)\|_{L_q(\mathbb{R}^d)}.$$

The quantity
$$E_{pq}(\Lambda, D) = \inf_{m\colon L_p(\mathbb{R}^d)\to L_q(\mathbb{R}^d)} e_{pq}(\Lambda, D, m) \tag{4.1}$$
is called the *error of optimal recovery*, and the method on which the infimum is attained, an *optimal method*.

4.1. Recovery in the metric $L_2(\mathbb{R}^d)$. By Plancherel's theorem,
$$\|\Lambda x(\,\cdot\,) - m(y)(\,\cdot\,)\|_{L_2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{d/2}}\|\widetilde{\Lambda} x(\,\cdot\,) - F(m(y))(\,\cdot\,)\|_{L_2(\mathbb{R}^d)},$$
where
$$\widetilde{\Lambda} x(\,\cdot\,) = \psi(\,\cdot\,)Fx(\,\cdot\,).$$
Moreover,
$$\|Dx(\,\cdot\,)\|_{L_2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{d/2}}\|\varphi(\,\cdot\,)Fx(\,\cdot\,)\|_{L_2(\mathbb{R}^d)}.$$

So, the problem under consideration coincides, up to a factor of $(2\pi)^{-d/2}$, with problem (1.1) for $q = r = 2$ with $\varphi(\,\cdot\,)$ replaced by $(2\pi)^{-d/2}\varphi(\,\cdot\,)$.

We set
$$\widetilde{\gamma} = \frac{n-k}{n + d(1/2 - 1/p)}, \qquad \widetilde{q} = \frac{1}{\widetilde{\gamma}(1/2 - 1/p)}$$
and
$$C_p(n,k) = \widetilde{\gamma}^{-\widetilde{\gamma}/p}(1-\widetilde{\gamma})^{-(1-\widetilde{\gamma})/2}\left(\frac{B(\widetilde{q}\widetilde{\gamma}/p + 1, \widetilde{q}(1-\widetilde{\gamma})/2)}{2(n-k)}\right)^{1/\widetilde{q}}.$$

Theorem 3. *Let $k \geqslant 0$, $n > k$, $2 < p \leqslant \infty$,*
$$I = \int_{\Pi_{d-1}} \frac{\widetilde{\psi}^{\widetilde{q}}(\omega)}{\widetilde{\varphi}^{\widetilde{q}(1-\widetilde{\gamma})}(\omega)} J(\omega)\, d\omega < \infty \quad \textit{and} \quad \Pi_{d-1} = [0,\pi]^{d-2} \times [0, 2\pi].$$
Then
$$E_{p2}(\Lambda, D) = \frac{1}{(2\pi)^{d\widetilde{\gamma}/2}} C_p(n,k) I^{1/\widetilde{q}} \delta^{\widetilde{\gamma}}.$$
Moreover, the method
$$\widehat{m}(y)(\,\cdot\,) = F^{-1}\left(\left(1 - \beta\left|\frac{\varphi(\xi)}{\psi(\xi)}\right|^2\right)_+ \psi(\xi) y(\xi)\right)(\,\cdot\,), \tag{4.2}$$
where
$$\beta = \frac{k + d(1/2 - 1/p)}{n + d(1/2 - 1/p)} C_p^2(n,k)\left(\frac{\delta I^{1/2 - 1/p}}{(2\pi)^{d/2}}\right)^{\frac{2(n-k)}{n + d(1/2 - 1/p)}},$$
is optimal.

Proof. The case $2 < p < \infty$ is secured by Theorem 2. Consider the case $p = \infty$. From a well-known upper estimate (see, for example, [21]) we have
$$E_{\infty 2}(\Lambda, D) \geqslant \sup_{\substack{x(\,\cdot\,)\in W_\infty \\ \|Fx(\,\cdot\,)\|_{L_\infty(\mathbb{R}^d)} \leqslant \delta}} \|\Lambda x(\,\cdot\,)\|_{L_2(\mathbb{R}^d)}. \tag{4.3}$$

Let $\widehat{x}(\,\cdot\,)$ be such that

$$F\widehat{x}(\xi)=\begin{cases}\delta, & |\psi(\xi)|>\lambda|\varphi(\xi)|,\\ 0, & |\psi(\xi)|\leqslant\lambda|\varphi(\xi)|,\end{cases}$$

where $\lambda>0$ is selected from the condition

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\varphi(\xi)|^2\,|F\widehat{x}(\xi)|^2\,d\xi=1.$$

Now $\lambda>0$ should be chosen from the condition

$$\delta^2\int_{|\psi(\xi)|>\lambda|\varphi(\xi)|}|\varphi(\xi)|^2\,d\xi=(2\pi)^d.$$

Changing to the spherical coordinates we obtain

$$\delta^2\int_{\Pi_{d-1}}\widetilde{\varphi}^2(\omega)J(\omega)\,d\omega\int_0^{\Phi_1(\omega)}\rho^{2n+d-1}\,d\rho=(2\pi)^d,$$

where

$$\Phi_1(\omega)=\left(\frac{\widetilde{\psi}(\omega)}{\lambda\widetilde{\varphi}(\omega)}\right)^{1/(n-k)}.$$

Hence

$$\frac{\delta^2}{2n+d}\lambda^{-(2n+d)/(n-k)}I=(2\pi)^d.$$

Therefore,

$$\lambda=\left(\frac{\delta^2 I}{(2\pi)^d(2n+d)}\right)^{(n-k)/(2n+d)}.$$

It is easily checked that

$$C_\infty^2(n,k)=\frac{1}{2k+d}(2n+d)^{(k+d/2)/(n+d/2)}.$$

As a result, $\lambda^2=\beta$. Hence by (4.3),

$$\begin{aligned}E_{\infty 2}^2(\Lambda,D)\geqslant\|\Lambda\widehat{x}(\,\cdot\,)\|_{L_2(\mathbb{R}^d)}^2&=\frac{\delta^2}{(2\pi)^d}\int_{|\psi(\xi)|>\lambda|\varphi(\xi)|}|\psi(\xi)|^2\,d\xi\\&=\frac{\delta^2}{(2\pi)^d}\int_{\Pi_{d-1}}\widetilde{\psi}^2(\omega)J(\omega)\,d\omega\int_0^{\Phi_1(\omega)}\rho^{2k+d-1}\,d\rho\\&=\frac{\delta^2}{(2k+d)(2\pi)^d}\lambda^{-(2k+d)/(n-k)}I=\frac{1}{(2\pi)^{d\widetilde{\gamma}}}C_\infty^2(n,k)I^{2/\widetilde{q}}\delta^{2\widetilde{\gamma}}.\end{aligned}\tag{4.4}$$

We estimate the error of the method (4.2). We set

$$a(\xi)=\left(1-\beta\frac{|\varphi(\xi)|^2}{|\psi(\xi)|^2}\right)_+.$$

Taking the Fourier transform we obtain

$$\|\Lambda x(\,\cdot\,)-\widehat{m}(y)(\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\psi(\xi)|^2\left|Fx(\xi)-a(\xi)y(\xi)\right|^2d\xi.$$

We set $z(\,\cdot\,)=Fx(\,\cdot\,)-y(\,\cdot\,)$ and note that

$$\|z(\,\cdot\,)\|_{L_\infty(\mathbb{R}^d)}\leqslant\delta\quad\text{and}\quad\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\varphi(\xi)|^2\,|Fx(\xi)|^2\,d\xi\leqslant 1.$$

Hence

$$\|\Lambda x(\,\cdot\,)-\widehat{m}(y)(\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\psi(\xi)|^2\left|(1-a(\xi))Fx(\xi)+a(\xi)z(\xi)\right|^2d\xi.$$

The integrand can be written as

$$\left|\frac{|\psi(\xi)|(1-a(\xi))\sqrt{\beta}|\varphi(\xi)|Fx(\xi)}{\sqrt{\beta}|\varphi(\xi)|}+\sqrt{a(\xi)}\sqrt{a(\xi)}\,|\psi(\xi)|z(\xi)\right|^2.$$

Using the Cauchy-Bunyakovskii-Schwarz inequality

$$|ab+cd|^2\leqslant(|a|^2+|c|^2)(|b|^2+|d|^2)$$

we obtain the estimate

$$\begin{aligned}&\|\Lambda x(\,\cdot\,)-\widehat{m}(y)(\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}\\&\qquad\leqslant\operatorname*{vraisup}_{\xi\in\mathbb{R}^d}S(\xi)\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\beta|\varphi(\xi)|^2\,|Fx(\xi)|^2+a(\xi)|\psi(\xi)|^2\,|z(\xi)|^2\right)d\xi,\end{aligned}$$

where

$$S(\xi)=\frac{|\psi(\xi)|^2\,|(1-a(\xi))^2}{\beta|\varphi(\xi)|^2}+a(\xi).$$

If $|\psi(\xi)|^2\leqslant\beta|\varphi(\xi)|^2$, then $a(\xi)=0$ and $S(\xi)\leqslant 1$. If $|\psi(\xi)|^2>\beta|\varphi(\xi)|^2$, then $S(\xi)=1$. So we have

$$\begin{aligned}e^2_{\infty 2}(\Lambda,D,\widehat{m})&\leqslant\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\beta|\varphi(\xi)|^2\,|Fx(\xi)|^2+a(\xi)|\psi(\xi)|^2\,|z(\xi)|^2\right)d\xi\\&\leqslant\beta+\frac{\delta^2}{(2\pi)^d}\int_{|\psi(\xi)|>\lambda|\varphi(\xi)|}\left(|\psi(\xi)|^2-\beta|\varphi(\xi)|^2\right)d\xi\\&=\beta+\frac{\delta^2}{(2\pi)^d}\int_{|\psi(\xi)|>\lambda|\varphi(\xi)|}|\psi(\xi)|^2\,d\xi-\beta\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\varphi(\xi)|^2\,|F\widehat{x}(\xi)|^2\,d\xi\\&=\frac{\delta^2}{(2\pi)^d}\int_{|\psi(\xi)|>\lambda|\varphi(\xi)|}|\psi(\xi)|^2\,d\xi\leqslant E^2_{\infty 2}(\Lambda,D).\end{aligned}$$

It follows that the method $\widehat{m}(y)(\,\cdot\,)$ is optimal. Moreover, by (4.4) we have

$$E^2_{\infty 2}(\Lambda,D)=\frac{\delta^2}{(2\pi)^d}\int_{|\psi(\xi)|>\lambda|\varphi(\xi)|}|\psi(\xi)|^2\,d\xi=\frac{1}{(2\pi)^{d\widetilde{\gamma}}}C^2_\infty(n,k)I^{2/\widetilde{q}}\delta^{2\widetilde{\gamma}}.$$

For $d = 1$ (in this case $I = 2$), $D = d^n/dt^n$ and $\Lambda = d^k/dt^k$, the conclusion of Theorem 3 was obtained in [26].

We define the operator $(-\Delta)^{n/2}$, $n \geqslant 0$, by

$$(-\Delta)^{n/2}x(\,\cdot\,) = F^{-1}\big(|\xi|^n Fx(\xi)\big)(\,\cdot\,), \qquad |\xi| = \sqrt{\xi_1^2 + \dots + \xi_d^2}.$$

We set

$$I_0 = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{4.5}$$

Corollary 2. *Let $k \geqslant 0$, $n > k$ and $2 < p \leqslant \infty$. Then*

$$E_{p2}\big((-\Delta)^{k/2}, (-\Delta)^{n/2}\big) = \frac{1}{(2\pi)^{d\widetilde{\gamma}/2}} C_p(n,k) I_0^{1/\widetilde{q}} \delta^{\widetilde{\gamma}}.$$

The method

$$\widehat{m}(y)(\,\cdot\,) = F^{-1}\big((1 - \beta|\xi|^{2(n-k)})_+ |\xi|^k y(\xi)\big)(\,\cdot\,), \tag{4.6}$$

where

$$\beta = \frac{k + d(1/2 - 1/p)}{n + d(1/2 - 1/p)} C_p^2(n,k) \left(\frac{\delta I_0^{1/2-1/p}}{(2\pi)^{d/2}}\right)^{\frac{2(n-k)}{n+d(1/2-1/p)}},$$

is optimal.

Proof. In the case under consideration, $\widetilde{\psi}(\omega) = \widetilde{\varphi}(\omega) = 1$, and so

$$I = \int_{\Pi_{d-1}} J(\omega)\, d\omega = \frac{2\pi^{d/2}}{\Gamma(d/2)} = I_0.$$

Now it suffices to employ Theorem 3.

For $p = \infty$, the conclusion of the corollary was proved in [24].

The expression for $E_{22}((-\Delta)^{k/2}, (-\Delta)^{n/2})$ and the corresponding optimal method were obtained in [23].

Note that the optimal method (4.6) employs information on the noisy Fourier transform of the function $x(\,\cdot\,)$ which is only measured in the ball

$$|\xi| < \beta^{-1/(2(n-k))}.$$

Moreover, the larger the error δ in the original information, the smaller the ball containing the 'useful' information.

Consider another example. Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}_+^d$. We define the operator D^α (the derivative of order α) as follows:

$$D^\alpha x(\,\cdot\,) = F^{-1}((i\xi)^\alpha Fx(\xi))(\,\cdot\,),$$

where $(i\xi)^\alpha = (i\xi_1)^{\alpha_1} \cdots (i\xi_d)^{\alpha_d}$. The function $|(i\xi)^\alpha|$ is a homogeneous function of order $k = \alpha_1 + \dots + \alpha_d$. Consider problem (4.1) with $\Lambda = D^\alpha$ and $D = (-\Delta)^{n/2}$.

Corollary 3. *Let $n > k$, $2 < p \leqslant \infty$. Then*

$$E_{p2}(D^\alpha, (-\Delta)^{n/2}) = \frac{1}{(2\pi)^{d\widetilde{\gamma}/2}} C_p(n,k) I^{1/\widetilde{q}} \delta^{\widetilde{\gamma}},$$

where

$$I = 2\frac{\Gamma((\alpha_1\widetilde{q}+1)/2)\cdots\Gamma((\alpha_d\widetilde{q}+1)/2)}{\Gamma((k\widetilde{q}+d)/2)}.$$

The method

$$\widehat{m}(y)(\,\cdot\,) = F^{-1}\left(\left(1-\beta\frac{|\xi|^{2n}}{|\xi^{2\alpha}|}\right)_+ (i\xi)^\alpha y(\xi)\right)(\,\cdot\,), \tag{4.7}$$

where

$$\beta = \frac{k+d(1/2-1/p)}{n+d(1/2-1/p)} C_p^2(n,k)\left(\frac{\delta I^{1/2-1/p}}{(2\pi)^{d/2}}\right)^{\frac{2(n-k)}{n+d(1/2-1/p)}},$$

is optimal.

Proof. From Dirichlet's well-known formula we have

$$\int_{\substack{\xi_1\geqslant 0,\dots,\xi_d\geqslant 0\\ \xi_1^2+\dots+\xi_d^2\leqslant 1}} \xi_1^{p_1-1}\cdots\xi_d^{p_d-1}\,d\xi_1\cdots d\xi_d = \frac{\Gamma(p_1/2)\cdots\Gamma(p_d/2)}{2^d\Gamma(p_1/2+\dots+p_d/2+1)},$$

$p_1,\dots,p_d > 0$. Therefore,

$$\int_{\xi_1^2+\dots+\xi_d^2\leqslant 1} |\xi_1|^{p_1-1}\cdots|\xi_d|^{p_d-1}\,d\xi_1\cdots d\xi_d = \frac{\Gamma(p_1/2)\cdots\Gamma(p_d/2)}{\Gamma(p_1/2+\dots+p_d/2+1)}.$$

Changing to the spherical coordinates we obtain

$$\int_{\Pi_{d-1}} \Phi(\omega,p_1,\dots,p_d)J(\omega)\,d\omega \int_0^1 \rho^{p_1+\dots+p_d-1}\,d\rho = \frac{\Gamma(p_1/2)\cdots\Gamma(p_d/2)}{\Gamma(p_1/2+\dots+p_d/2+1)},$$

where

$$\Phi(\omega,p_1,\dots,p_d) = |\cos\omega_1|^{p_1-1}\cdots|\sin\omega_1\sin\omega_2\cdots\sin\omega_{d-2}\sin\omega_{d-1}|^{p_d-1}.$$

As a result,

$$\int_{\Pi_{d-1}} \Phi(\omega,p_1,\dots,p_d)J(\omega)\,d\omega = 2\frac{\Gamma(p_1/2)\cdots\Gamma(p_d/2)}{\Gamma(p_1/2+\dots+p_d/2)}.$$

So, for the quantity I in Theorem 3 we have

$$\begin{aligned} I &= \int_{\Pi_{d-1}} |\cos\omega_1|^{\alpha_1\widetilde{q}}\cdots|\sin\omega_1\sin\omega_2\cdots\sin\omega_{d-2}\sin\omega_{d-1}|^{\alpha_d\widetilde{q}} J(\omega)\,d\omega \\ &= \int_{\Pi_{d-1}} \Phi(\omega,\alpha_1\widetilde{q}+1,\dots,\alpha_d\widetilde{q}+1)J(\omega)\,d\omega \\ &= 2\frac{\Gamma((\alpha_1\widetilde{q}+1)/2)\cdots\Gamma((\alpha_d\widetilde{q}+1)/2)}{\Gamma((k\widetilde{q}+d)/2)}. \end{aligned} \tag{4.8}$$

Now the conclusion of the corollary follows from Theorem 3.

Consider the case $p=2$. The situation here is fairly close to that considered in [27] and [28], even though here the class on which the operator D^α is recovered is different.

Theorem 4. *Let $n>k>0$. Then*

$$E_{22}(D^\alpha,(-\Delta)^{n/2})=\frac{\alpha^{\alpha/2}}{k^{k/2}}\biggl(\frac{\delta}{(2\pi)^{d/2}}\biggr)^{1-k/n}, \tag{4.9}$$

and all the methods

$$\widehat{m}(y)(\,\cdot\,)=F^{-1}(a(\xi)(i\xi)^\alpha y(\xi))(\,\cdot\,), \tag{4.10}$$

where $a(\,\cdot\,)$ are measurable functions satisfying the condition

$$|\xi^{2\alpha}|\biggl(\frac{|1-a(\xi)|^2}{\lambda_2|\xi|^{2n}}+\frac{|a(\xi)|^2}{(2\pi)^d\lambda_1}\biggr)\leqslant 1 \tag{4.11}$$

in which

$$\lambda_1=\frac{\alpha^\alpha(n-k)}{(2\pi)^dk^kn}\biggl(\frac{\delta^2}{(2\pi)^d}\biggr)^{-k/n} \quad \text{and} \quad \lambda_2=\lambda_1\frac{k}{n-k}\delta^2,$$

are optimal.

Proof. Given $\varepsilon>0$, we set

$$\widehat{\xi}=\frac{1}{\sqrt{k}}\biggl(\frac{(2\pi)^d}{\delta^2}\biggr)^{1/(2n)}(\sqrt{\alpha_1},\dots,\sqrt{\alpha_d}), \qquad \widehat{\xi}_\varepsilon=\widehat{\xi}\biggl(1-\frac{\varepsilon}{|\widehat{\xi}|}\biggr),$$
$$B_\varepsilon=\{\xi\in\mathbb{R}^d\colon |\xi-\widehat{\xi}_\varepsilon|<\varepsilon\}.$$

Consider a function $x_\varepsilon(\,\cdot\,)$ such that

$$Fx_\varepsilon(\xi)=\begin{cases}\dfrac{\delta}{\sqrt{\operatorname{mes}B_\varepsilon}}, & \xi\in B_\varepsilon,\\ 0, & \xi\notin B_\varepsilon.\end{cases}$$

We have $\|Fx_\varepsilon(\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}=\delta^2$ and

$$\|(-\Delta)^{n/2}x_\varepsilon(\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}=\frac{\delta^2}{(2\pi)^d\operatorname{mes}B_\varepsilon}\int_{B_\varepsilon}|\xi|^{2n}\,d\xi\leqslant\frac{\delta^2}{(2\pi)^d}|\widehat{\xi}|^{2n}=1.$$

From an estimate similar to (4.3) we have

$$\begin{aligned}E^2_{22}(D^\alpha,(-\Delta)^{n/2})&\geqslant\sup_{\substack{\|(-\Delta)^{n/2}x(\,\cdot\,)\|_{L_2(\mathbb{R}^d)}\leqslant1\\ \|Fx(\,\cdot\,)\|_{L_2(\mathbb{R}^d)}\leqslant\delta}}\|D^\alpha x(\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}\\ &\geqslant\|D^\alpha x_\varepsilon(\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}=\frac{\delta^2}{(2\pi)^d\operatorname{mes}B_\varepsilon}\int_{B_\varepsilon}|\xi^{2\alpha}|\,d\xi\\ &=\frac{\delta^2}{(2\pi)^d}|\xi_0^{2\alpha}|,\end{aligned}$$

where ξ_0 is some point in B_ε. Letting $\varepsilon \to 0$ we obtain the estimate

$$E_{22}^2(D^\alpha, (-\Delta)^{n/2}) \geqslant \frac{\delta^2}{(2\pi)^d}|\widehat{\xi^{2\alpha}}| = \frac{\alpha^\alpha}{k^k}\left(\frac{\delta^2}{(2\pi)^d}\right)^{1-k/n}. \tag{4.12}$$

The optimal methods will be sought among the methods of the form (4.10). Passing to the Fourier transform we have

$$\|D^\alpha x(\,\cdot\,) - \widehat{m}(y)(\,\cdot\,)\|_{L_2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi^{2\alpha}|\,\big|Fx(\xi) - a(\xi)y(\xi)\big|^2\,d\xi.$$

We set $z(\,\cdot\,) = Fx(\,\cdot\,) - y(\,\cdot\,)$ and note that

$$\int_{\mathbb{R}^d} |z(\xi)|^2\,d\xi \leqslant \delta^2 \quad \text{and} \qquad \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^{2n}\,|Fx(\xi)|^2\,d\xi \leqslant 1.$$

Hence

$$\|D^\alpha x(\,\cdot\,) - \widehat{m}(y)(\,\cdot\,)\|_{L_2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi^{2\alpha}|\,\big|\big(1-a(\xi)\big)Fx(\xi) + a(\xi)z(\xi)\big|^2\,d\xi.$$

We write the integrand as

$$|\xi^{2\alpha}|\left|\frac{(1-a(\xi))\sqrt{\lambda_2}|\xi|^n Fx(\xi)}{\sqrt{\lambda_2}|\xi|^n} + \frac{a(\xi)}{(2\pi)^{d/2}\sqrt{\lambda_1}}(2\pi)^{d/2}\sqrt{\lambda_1}z(\xi)\right|^2.$$

Applying the Cauchy-Bunyakovskii-Schwarz inequality we obtain the estimate

$$\begin{aligned}&\|D^\alpha x(\,\cdot\,) - \widehat{m}(y)(\,\cdot\,)\|_{L_2(\mathbb{R}^d)}^2\\ &\qquad \leqslant \operatorname*{vraisup}_{\xi\in\mathbb{R}^d} S(\xi)\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\big(\lambda_2|\xi|^{2n}\,|Fx(\xi)|^2 + (2\pi)^d\lambda_1|z(\xi)|^2\big)\,d\xi,\end{aligned}$$

where

$$S(\xi) = |\xi^{2\alpha}|\left(\frac{|1-a(\xi)|^2}{\lambda_2|\xi|^{2n}} + \frac{|a(\xi)|^2}{(2\pi)^d\lambda_1}\right).$$

If we assume that $S(\xi) \leqslant 1$ for almost all ξ, then by (4.12),

$$\begin{aligned}&e_{22}^2(D^\alpha, (-\Delta)^{n/2}, \widehat{m})\\ &\qquad \leqslant \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\big(\lambda_2|\xi|^{2n}\,|Fx(\xi)|^2 + (2\pi)^d\lambda_1|z(\xi)|^2\big)\,d\xi\\ &\qquad \leqslant \lambda_2 + \lambda_1\delta^2 = \frac{\alpha^\alpha}{k^k}\left(\frac{\delta^2}{(2\pi)^d}\right)^{1-k/n} \leqslant E_{22}^2(D^\alpha, (-\Delta)^{n/2}).\end{aligned} \tag{4.13}$$

This proves (4.9) and shows that the methods under consideration are optimal.

It remains to verify that the set of functions $a(\,\cdot\,)$ satisfying (4.11) is nonempty. Condition (4.11) can be rewritten in an equivalent form:

$$\begin{aligned}&\left|a(\xi) - \frac{(2\pi)^d\lambda_1}{(2\pi)^d\lambda_1 + \lambda_2|\xi|^{2n}}\right|^2\\ &\qquad \leqslant \frac{(2\pi)^d\lambda_1\lambda_2|\xi|^{2n}}{|\xi^{2\alpha}|((2\pi)^d\lambda_1 + \lambda_2|\xi|^{2n})^2}(-|\xi^{2\alpha}| + (2\pi)^d\lambda_1 + \lambda_2|\xi|^{2n}).\end{aligned}$$

Hence it suffices to show that, for all $\xi \in \mathbb{R}^d$,

$$-|\xi^{2\alpha}| + (2\pi)^d\lambda_1 + \lambda_2|\xi|^{2n} \geqslant 0. \tag{4.14}$$

From the theorem of the arithmetic and geometric means (see [29]) it follows that

$$|\xi^{2\alpha}| \leqslant \frac{\alpha^\alpha}{k^k}|\xi|^{2k}.$$

Consider the function $y(s) = s^{k/n}$, $s \geqslant 0$. The tangent to this function at any point $s_0 > 0$ has the form

$$y = \frac{k}{n}s_0^{k/n-1}s + \frac{n-k}{n}s_0^{k/n}.$$

The function $y(\,\cdot\,)$ is concave, hence for all $s \geqslant 0$,

$$s^{k/n} \leqslant \frac{k}{n}s_0^{k/n-1}s + \frac{n-k}{n}s_0^{k/n}.$$

Setting $s_0 = |\widehat{\xi}|^{2n}$ and $s = |\xi|^{2n}$ we find that

$$|\xi^{2\alpha}| \leqslant \frac{\alpha^\alpha}{k^k}|\xi|^{2k} \leqslant \frac{\alpha^\alpha}{k^k}\left(\frac{k}{n}|\widehat{\xi}|^{2(k-n)}|\xi|^{2n} + \frac{n-k}{n}|\widehat{\xi}|^{2k}\right).$$

It is easily checked that

$$\lambda_1 = \frac{\alpha^\alpha(n-k)}{(2\pi)^d k^k n}|\widehat{\xi}|^{2k} \quad \text{and} \quad \lambda_2 = \frac{\alpha^\alpha}{k^{k-1}n}|\widehat{\xi}|^{2(k-n)}.$$

As a result, we obtain

$$|\xi^{2\alpha}| \leqslant (2\pi)^d\lambda_1 + \lambda_2|\xi|^{2n},$$

which is equivalent to (4.14).

4.2. Recovery in the metric $L_\infty(\mathbb{R}^d)$. We set

$$\begin{gathered}\gamma_1 = \frac{n-k-d/2}{n+d(1/2-1/p)}, \qquad q_1 = \frac{1}{1/2+\gamma_1(1/2-1/p)}, \\ \widetilde{C}_p(n,k) = \gamma_1^{-\gamma_1/p}(1-\gamma_1)^{-(1-\gamma_1)/2}\left(\frac{B\bigl(q_1\gamma_1/p+1, q_1(1-\gamma_1)/2\bigr)}{2(n-k-d/2)}\right)^{1/q_1}.\end{gathered} \tag{4.15}$$

For $1 < p < \infty$ let the function $\kappa_1(\,\cdot\,)$ be defined by

$$\frac{\kappa_1(t)}{(1-\kappa_1(t))^{p-1}} = \frac{|\psi(t)|^{p-2}}{|\varphi(t)|^{2(p-1)}},$$

for $p = 1$ by

$$\kappa_1(t) = \min\{1, |\psi(t)|^{-1}\},$$

and, for $p = \infty$, by

$$\kappa_1(t) = \left(1 - \frac{|\varphi(t)|^2}{|\psi(t)|}\right)_+.$$

Theorem 5. *Let $k \geqslant 0$, $n > k + d/2$, $1 \leqslant p \leqslant \infty$, $k + p > 1$,*

$$I = \int_{\Pi_{d-1}} \frac{\widetilde{\psi}^{q_1}(\omega)}{\widetilde{\varphi}^{q_1(1-\gamma_1)}(\omega)} J(\omega)\, d\omega < \infty \quad \textit{and} \quad \Pi_{d-1} = [0,\pi]^{d-2} \times [0, 2\pi].$$

Then

$$E_{p\infty}(\Lambda, D) = \frac{1}{(2\pi)^{d(1+\gamma_1)/2}} \widetilde{C}_p(n,k) I^{1/q_1} \delta^{\gamma_1}.$$

Then the method

$$\widehat{m}(y)(\,\cdot\,) = F^{-1}\big(\kappa_1\big(\xi_1^{\overline{\frac{1}{n+d(1/2-1/p)}}}\xi\big)\psi(\xi)y(\xi)\big)(\,\cdot\,),$$

where

$$\xi_1 = \delta\bigg(\frac{(1-\gamma_1)^{p-1}}{\gamma_1}\bigg)^{q_1/(2p)} \bigg(\frac{\widetilde{C}_p(n,k) I^{1/q_1}}{(2\pi)^{d(1+\gamma_1)/2}}\bigg)^{q_1(1/2-1/p)},$$

is optimal.

Proof. Using an estimate similar to (4.3) we have

$$E_{p\infty}(\Lambda, D) \geqslant \sup_{\substack{x(\,\cdot\,)\in W_p \\ \|Fx(\,\cdot\,)\|_{L_p(\mathbb{R}^d)} \leqslant \delta}} \|\Lambda x(\,\cdot\,)\|_{L_\infty(\mathbb{R}^d)}.$$

Assume that $x(\,\cdot\,) \in W_p$ and $\|Fx(\,\cdot\,)\|_{L_p(\mathbb{R}^d)} \leqslant \delta$. If $\widehat{x}(\,\cdot\,)$ is such that

$$F\widehat{x}(\xi) = \varepsilon(\xi) e^{-i\langle t,\xi\rangle} Fx(\xi),$$

where

$$\varepsilon(\xi) = \begin{cases} \dfrac{\overline{\psi(\xi)Fx(\xi)}}{|\psi(\xi)Fx(\xi)|}, & \psi(\xi)Fx(\xi) \neq 0, \\ 0, & \psi(\xi)Fx(\xi) = 0, \end{cases}$$

then we obtain $\widehat{x}(\,\cdot\,) \in W_p$, $\|F\widehat{x}(\,\cdot\,)\|_{L_p(\mathbb{R}^d)} \leqslant \delta$ and

$$\left|\int_{\mathbb{R}^d} \psi(\xi) F\widehat{x}(\xi) e^{i\langle t,\xi\rangle}\, d\xi\right| = \int_{\mathbb{R}^d} |\psi(\xi)Fx(\xi)|\, d\xi.$$

Hence

$$E_{p\infty}(\Lambda, D) \geqslant \frac{1}{(2\pi)^d} \sup_{\substack{x(\,\cdot\,)\in W_p \\ \|Fx(\,\cdot\,)\|_{L_p(\mathbb{R}^d)} \leqslant \delta}} \int_{\mathbb{R}^d} |\psi(\xi)Fx(\xi)|\, d\xi. \tag{4.16}$$

Let $1 \leqslant p < \infty$. It follows from (3.4) that

$$E_{p\infty}(\Lambda, D) \geqslant E_{p12},$$

where, in the problem of the evaluation of E_{p12}, the function $\varphi(\,\cdot\,)$ should be replaced by the function $(2\pi)^{-d/2}\varphi(\,\cdot\,)$, and the function $\psi(\,\cdot\,)$ by $(2\pi)^{-d}\psi(\,\cdot\,)$. From Theorem 2 we obtain

$$E_{p\infty}(\Lambda, D) \geqslant E_{p12} = \frac{1}{(2\pi)^{d(1+\gamma_1)/2}} \widetilde{C}_p(n,k) I^{1/q_1} \delta^{\gamma_1}.$$

Another appeal to Theorem 2 shows that

$$\int_{\mathbb{R}^d}\left|\frac{1}{(2\pi)^d}\psi(\xi)Fx(\xi)-m(y)(\xi)\right|d\xi\leqslant E_{p12},$$

where

$$m(y)(\xi)=\frac{1}{(2\pi)^d}\kappa_1\big(\xi_1^{\overline{\frac{1}{n+d(1/2-1/p)}}}\xi\big)\psi(\xi)y(\xi).$$

Therefore,

$$\begin{aligned}\left|\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\psi(\xi)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi-\int_{\mathbb{R}^d}m(y)(\xi)e^{i\langle t,\xi\rangle}\,d\xi\right|\\ \leqslant\int_{\mathbb{R}^d}\left|\frac{1}{(2\pi)^d}\psi(\xi)Fx(\xi)-m(y)(\xi)\right|d\xi\leqslant E_{p12}\leqslant E_{p\infty}(\Lambda,D).\end{aligned}$$

It follows that the method $\widehat{m}(y)(\,\cdot\,)$ is optimal, and the error of optimal recovery coincides with E_{p12}.

Now consider the case when $p=\infty$. We set

$$s(\xi)=\begin{cases}\dfrac{\psi(\xi)}{|\psi(\xi)|}, & \psi(\xi)\neq 0,\\ 1, & \psi(\xi)=0.\end{cases}$$

Let $\widehat{x}(\,\cdot\,)$ be a function such that

$$F\widehat{x}(\xi)=\begin{cases}\delta\overline{s(\xi)}, & |\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2,\\ \dfrac{\delta}{\lambda}\,\dfrac{\overline{\psi(\xi)}}{|\varphi(\xi)|^2}, & |\psi(\xi)|<\lambda|\varphi(\xi)|^2.\end{cases}$$

We choose $\lambda>0$ such that $\|D\widehat{x}(\,\cdot\,)\|_{L_2(\mathbb{R}^d)}=1$. Now, to find λ we have the equation

$$\frac{\delta^2}{(2\pi)^d}\int_{|\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2}|\varphi(\xi)|^2\,d\xi+\frac{\delta^2\lambda^{-2}}{(2\pi)^d}\int_{|\psi(\xi)|<\lambda|\varphi(\xi)|^2}\frac{|\psi(\xi)|^2}{|\varphi(\xi)|^2}\,d\xi=1.$$

Changing to the spherical coordinates we obtain

$$\begin{aligned}\frac{\delta^2}{(2\pi)^d}\int_{\Pi_{d-1}}\widetilde{\varphi}^2(\omega)J(\omega)\,d\omega\int_0^{\Phi_2(\omega)}\rho^{2n+d-1}\,d\rho\\ +\frac{\delta^2\lambda^{-2}}{(2\pi)^d}\int_{\Pi_{d-1}}\frac{\widetilde{\psi}^2(\omega)}{\widetilde{\varphi}^2(\omega)}J(\omega)\,d\omega\int_{\Phi_2(\omega)}^{+\infty}\rho^{-2n+2k+d-1}\,d\rho=1,\end{aligned}$$

where

$$\Phi_2(\omega)=\left(\frac{\widetilde{\psi}(\omega)}{\lambda\widetilde{\varphi}^2(\omega)}\right)^{1/(2n-k)}.$$

This gives us the equation

$$\frac{\delta^2}{(2\pi)^d}\lambda^{-(2n+d)/(2n-k)}\frac{4n-2k}{(2n+d)(2n-2k-d)}I=1.$$

As a result,

$$\lambda = \left(\frac{\delta^2(4n-2k)}{(2\pi)^d(2n+d)(2n-2k-d)}I\right)^{(2n-k)/(2n+d)}.$$

From (4.16) we find that

$$\begin{aligned}
E_{\infty\infty}(\Lambda, D) &\geqslant \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\psi(\xi)||F\widehat{x}(\xi)|\,d\xi \\
&= \frac{\delta}{(2\pi)^d}\int_{|\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2}|\psi(\xi)|\,d\xi + \frac{\delta}{\lambda(2\pi)^d}\int_{|\psi(\xi)|<\lambda|\varphi(\xi)|^2}\frac{|\psi(\xi)|^2}{|\varphi(\xi)|^2}\,d\xi \\
&= \frac{\delta}{(2\pi)^d}\int_{\Pi_{d-1}}\widetilde{\psi}(\omega)J(\omega)\,d\omega\int_0^{\Phi_2(\omega)}\rho^{k+d-1}\,d\rho \\
&\quad + \frac{\delta}{\lambda(2\pi)^d}\int_{\Pi_{d-1}}\frac{\widetilde{\psi}^2(\omega)}{\widetilde{\varphi}^2(\omega)}J(\omega)\,d\omega\int_{\Phi_2(\omega)}^{+\infty}\rho^{-2n+2k+d-1}\,d\rho \\
&= \frac{\delta\lambda^{-(k+d)/(2n-k)}}{(2\pi)^d(k+d)}I + \frac{\delta}{\lambda(2\pi)^d(2n-2k-d)}\lambda^{(2n-2k-d)/(2n-k)}I \\
&= \frac{\delta(2n-k)\lambda^{-(k+d)/(2n-k)}I}{(2\pi)^d(k+d)(2n-2k-d)} = \nu, \qquad (4.17)
\end{aligned}$$

where

$$\nu = \frac{(n+d/2)^{(k+d)/(2n+d)}}{k+d}\left(\frac{(2n-k)I}{(2\pi)^d(2n-2k-d)}\right)^{(2n-k)/(2n+d)}\delta^{(2n-2k-d)/(2n+d)}.$$

Let us prove that, for all $x(\,\cdot\,)\in X_\infty$,

$$\begin{aligned}
\Lambda x(t) &= \frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2}\big(\psi(\xi)-\lambda s(\xi)|\varphi(\xi)|^2\big)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi \\
&\quad + \frac{\lambda}{\delta(2\pi)^d}\int_{\mathbb{R}^d}|\varphi(\xi)|^2Fx(\xi)\overline{F\widehat{x}(\xi)}e^{i\langle t,\xi\rangle}\,d\xi. \qquad (4.18)
\end{aligned}$$

Indeed,

$$\begin{aligned}
&\frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2}\big(\psi(\xi)-\lambda s(\xi)|\varphi(\xi)|^2\big)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi \\
&\qquad + \frac{\lambda}{\delta(2\pi)^d}\int_{\mathbb{R}^d}|\varphi(\xi)|^2Fx(\xi)\overline{F\widehat{x}(\xi)}e^{i\langle t,\xi\rangle}\,d\xi \\
&= \frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2}\big(\psi(\xi)-\lambda s(\xi)|\varphi(\xi)|^2\big)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi \\
&\qquad + \frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2}\lambda s(\xi)|\varphi(\xi)|^2Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi \\
&\qquad + \frac{1}{(2\pi)^d}\int_{|\psi(\xi)|<\lambda|\varphi(\xi)|^2}\psi(\xi)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\psi(\xi)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi = \Lambda x(t).
\end{aligned}$$

We estimate the error of the method

$$m(y)(t)=\frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2}\bigl(\psi(\xi)-\lambda s(\xi)|\varphi(\xi)|^2\bigr)y(\xi)e^{i\langle t,\xi\rangle}\,d\xi.$$

We have

$$\begin{aligned}
&|\Lambda x(t)-m(y)(t)|\\
&\quad=\biggl|\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\psi(\xi)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi\\
&\qquad-\frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2}\bigl(\psi(\xi)-\lambda s(\xi)|\varphi(\xi)|^2\bigr)y(\xi)e^{i\langle t,\xi\rangle}\,d\xi\biggr|\\
&\quad\leqslant\biggl|\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\psi(\xi)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi\\
&\qquad-\frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2}\bigl(\psi(\xi)-\lambda s(\xi)|\varphi(\xi)|^2\bigr)Fx(\xi)e^{i\langle t,\xi\rangle}\,d\xi\biggr|\\
&\qquad+\frac{1}{(2\pi)^d}\int_{|\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2}|\psi(\xi)-\lambda s(\xi)|\varphi(\xi)|^2|\,|Fx(\xi)-y(\xi)|\,d\xi.
\end{aligned}$$

For $x(\,\cdot\,)$ satisfying

$$\|Fx(\,\cdot\,)-y(\,\cdot\,)\|_{L_\infty(\mathbb{R}^d)}\leqslant\delta,\qquad \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\varphi(\xi)|^2\,|Fx(\xi)|^2\,d\xi\leqslant 1,$$

in view of (4.18) we have

$$|\Lambda x(t)-m(y)(t)|\leqslant\frac{\lambda}{\delta(2\pi)^d}\int_{\mathbb{R}^d}|\varphi(\xi)|^2|Fx(\xi)||F\widehat{x}(\xi)|\,d\xi+\mu\leqslant\frac{\lambda}{\delta}+\mu,$$

where

$$\mu=\frac{\delta}{(2\pi)^d}\int_{|\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2}(|\psi(\xi)|-\lambda|\varphi(\xi)|^2)\,d\xi.$$

It was found above that

$$\frac{\delta}{(2\pi)^d}\int_{|\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2}|\psi(\xi)|\,d\xi=\frac{\delta\lambda^{-(k+d)/(2n-k)}}{(2\pi)^d(k+d)}I$$

(see the first term in (4.17)). Next,

$$\begin{aligned}
&\frac{\delta\lambda}{(2\pi)^d}\int_{|\psi(\xi)|\geqslant\lambda|\varphi(\xi)|^2}|\varphi(\xi)|^2\,d\xi\\
&\qquad=\frac{\delta\lambda}{(2\pi)^d}\int_{\Pi_{d-1}}\widetilde{\varphi}^2(\omega)J(\omega)\,d\omega\int_0^{\Phi_2(\omega)}\rho^{2n+d-1}\,d\rho\\
&\qquad=\frac{\delta\lambda^{-(k+d)/(2n-k)}}{(2\pi)^d(2n+d)}I,
\end{aligned}$$

and so

$$\mu = \frac{\delta\lambda^{-(k+d)/(2n-k)}(2n-k)}{(2\pi)^d(k+d)(2n+d)} I.$$

It is easily checked that $\lambda/\delta + \mu = \nu$, and therefore

$$e_{\infty\infty}(\Lambda, D, m) \leqslant \nu \leqslant E_{\infty\infty}(\Lambda, D).$$

It follows that $m(y)(\,\cdot\,)$ is an optimal method, and the error of optimal recovery is ν. It is straightforward that, for $p = \infty$,

$$\frac{1}{(2\pi)^{d(1+\gamma_1)/2}} \widetilde{C}_\infty(n, k) I^{1/q_1} \delta^{\gamma_1} = \nu.$$

We evaluate ξ_1 for $p = \infty$. We have

$$\xi_1 = \delta(1-\gamma_1)^{q_1/2} \left(\frac{\widetilde{C}_\infty(n, k) I^{1/q_1}}{(2\pi)^{d(1+\gamma_1)/2}} \right)^{q_1/2} = \lambda^{(n+d/2)/(2n-k)}. \tag{4.19}$$

The method $m(y)(\,\cdot\,)$ can be written as

$$m(y)(\,\cdot\,) = F^{-1}\left(\left(1 - \lambda \frac{|\varphi(\xi)|^2}{|\psi(\xi)|} \right)_+ \psi(\xi) y(\xi) \right)(\,\cdot\,).$$

In view of (4.19) we have

$$m(y)(\,\cdot\,) = F^{-1}\bigl(\kappa_1(\xi_1^{1/(n+d/2)}\xi)\psi(\xi)y(\xi)\bigr)(\,\cdot\,) = \widehat{m}(y)(\,\cdot\,).$$

Corollary 4. *Let $k \geqslant 0$, $n > k$, $1 \leqslant p \leqslant \infty$ and $k + p > 1$. Then*

$$E_{p\infty}\left(\frac{d^k}{dt^k}, \frac{d^n}{dt^n} \right) = \frac{1}{(2\pi)^{(1+\gamma_1)/2}} \widetilde{C}_p(n, k) 2^{1/q_1} \delta^{\gamma_1},$$

where γ_1, q_1 and $\widetilde{C}_p(n, k)$ are defined by (4.15) for $d = 1$. The method

$$\widehat{m}(y)(\,\cdot\,) = F^{-1}\bigl(\kappa_1\bigl(\xi_1^{\frac{1}{n+1/2-1/p}}\xi\bigr)(i\xi)^k y(\xi)\bigr)(\,\cdot\,),$$

where

$$\xi_1 = \delta \left(\frac{(1-\gamma_1)^{p-1}}{\gamma_1} \right)^{q_1/(2p)} \left(\frac{\widetilde{C}_p(n, k) 2^{1/q_1}}{(2\pi)^{(1+\gamma_1)/2}} \right)^{q_1(1/2-1/p)},$$

is optimal.

The result of Corollary 4 for $p = 1, 2, \infty$ was obtained in [30] where the case when $p = 1$ and $k = 0$ was also examined.

Corollary 5. *Let $k \geqslant 0$, $n > k + d/2$, $1 \leqslant p \leqslant \infty$ and $k + p > 1$. Then*

$$E_{p\infty}((-\Delta)^{k/2}, (-\Delta)^{n/2}) = \frac{1}{(2\pi)^{d(1+\gamma_1)/2}} \widetilde{C}_p(n, k) I_0^{1/q_1} \delta^{\gamma_1},$$

where I_0 *is defined by* (4.5). *Then the method*

$$\widehat{m}(y)(\,\cdot\,) = F^{-1}\big(\kappa_1\big(\xi_1^{\frac{1}{n+d(1/2-1/p)}}\xi\big)|\xi|^k y(\xi)\big)(\,\cdot\,),$$

where

$$\xi_1 = \delta\biggl(\frac{(1-\gamma_1)^{p-1}}{\gamma_1}\biggr)^{q_1/(2p)}\biggl(\frac{\widetilde{C}_p(n,k)I_0^{1/q_1}}{(2\pi)^{d(1+\gamma_1)/2}}\biggr)^{q_1(1/2-1/p)},$$

is optimal.

The result of Corollary 5 for $p=\infty$ was obtained in [25].

Let us now apply Theorem 5 to the operators $\Lambda = D^\alpha$ and $D = (-\Delta)^{n/2}$.

Corollary 6. *Let* $k=\alpha_1+\dots+\alpha_d>0$, $n>k+d/2$, $1\leqslant p\leqslant\infty$. *Then*

$$E_{p\infty}(D^\alpha,(-\Delta)^{n/2}) = \frac{1}{(2\pi)^{d(1+\gamma_1)/2}}\widetilde{C}_p(n,k)I^{1/q_1}\delta^{\gamma_1},$$

where

$$I = 2\frac{\Gamma((\alpha_1q_1+1)/2)\cdots\Gamma((\alpha_dq_1+1)/2)}{\Gamma((kq_1+d)/2)}. \tag{4.20}$$

The method

$$\widehat{m}(y)(\,\cdot\,) = F^{-1}\big(\kappa_1\big(\xi_1^{\frac{1}{n+d(1/2-1/p)}}\xi\big)(i\xi)^\alpha y(\xi)\big)(\,\cdot\,),$$

where

$$\xi_1 = \delta\gamma_1^{-q_1/(2p)}(1-\gamma_1)^{\frac{q_1}{2}(1-1/p)}\biggl(\frac{\widetilde{C}_p(n,k)I^{1/q_1}}{(2\pi)^{d(1+\gamma_1)/2}}\biggr)^{q_1(1/2-1/p)},$$

is optimal.

Proof. In the case under consideration the quantity I in Theorem 5 has the form

$$I = \int_{\Pi_{d-1}}|\cos\omega_1|^{\alpha_1q_1}\cdots|\sin\omega_1\sin\omega_2\cdots\sin\omega_{d-2}\sin\omega_{d-1}|^{\alpha_dq_1}J(\omega)\,d\omega.$$

Taking (4.8) into account we arrive at (4.20). Now the conclusion of the corollary follows directly from Theorem 5.

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