Optimal recovery of linear operators
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Let $X$ be a linear space, $Y_j, j = 1, \ldots, n$ be linear spaces with semi-inner products, $I_j : X \to Y_j, j = 1, \ldots, n$ be linear operators, and $Z$ be a normed linear space. We consider the problem of optimal recovery of a linear operator $T : X \to Z$ on a set

$$W = \{ x \in X : \| I_j x \|_{Y_j} \leq \delta_j, \quad j = 1, \ldots, k, \quad k < n \}$$

by inaccurate information about values of the operators $I_{k+1}, \ldots, I_n$. More precisely, we are interested in the value

$$E(T, W, I, \delta) = \inf_{\varphi : Y_{k+1} \times \ldots \times Y_n \to Z} \sup_{x \in W} \| Tx - \varphi(y) \|_Z,$$

and in a method $\hat{\varphi}$ for which this infimum is attained (we call it an optimal method of recovery).

Consider the following extremal problem

$$(1) \quad \| Tx \|_Z^2 \to \max, \quad \| I_j x \|_{Y_j} \leq \delta_j, \quad j = 1, \ldots, n, \quad x \in X. \quad \text{Denote by}$$

$$L(x, \lambda) = -\| Tx \|_Z^2 + \sum_{j=1}^n \lambda_j \| I_j x \|_{Y_j}^2$$

the Lagrange function of this problem.

**Theorem 1.** Suppose that there exist nonnegative $\hat{\lambda}_j, \quad j = 1, \ldots, n$ such that $L(x, \hat{\lambda}) \geq 0$ for all $x \in X$. Let $\{x_m\}$ be a sequence of admissible elements in (1) such that

$$(a) \quad \lim_{m \to \infty} L(x_m, \hat{\lambda}) = 0,$$

$$(b) \quad \lim_{m \to \infty} \sum_{j=1}^n \hat{\lambda}_j (\| I_j x_m \|_{Y_j}^2 - \delta_j^2) = 0.$$ 

If for all $y = (y_{k+1}, \ldots, y_n) \in Y_{k+1} \times \ldots \times Y_n$ there exists an element $x_y$ which is a solution of the problem

$$\sum_{j=1}^k \hat{\lambda}_j \| I_j x \|_{Y_j}^2 + \sum_{j=k+1}^n \hat{\lambda}_j \| I_j x - y_j \|_{Y_j}^2 \to \min, \quad x \in X,$$

then the method

$$\hat{\varphi}(y) = Tx_y$$

is optimal and

$$E(T, W, I, \delta) = \sqrt{\sum_{j=1}^n \hat{\lambda}_j \delta_j^2}.$$ 

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