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Recovery of analytic functions that is exact on subspaces of entire functions

K. Yu. Osipenko

Abstract. A family of optimal recovery methods is developed for the recovery of analytic functions in a strip and their derivatives from inaccurately specified trace of the Fourier transforms of these functions on the real axis. In addition, the methods must be exact on some subspaces of entire functions.

Bibliography: 12 titles.

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§ 1. Introduction

One popular idea in the development of numerical methods is to look for methods that are exact on some subspace of functions. This is based on the natural observation that if the original function can be approximated sufficiently accurately by elements of this subspace, then the error of the corresponding method (which is usually a linear functional or an operator of the function) is admissible. A typical example here is quadrature formulae, which are constructed to be exact on the algebraic polynomials of some fixed degree: the most spectacular example is Gauss's quadrature formulae (for instance, see [1]).

Another approach to the development of numerical methods, or — in a broader sense — to approximations as such, is connected with Kolmogorov's ideas. In this case one fixes come *a priori* information — a set (class) of functions — for which one develops an optimal (best) method based on the condition that this method must produce the minimum error in this class of functions. A typical example here also is quadrature formulae; in this setting such formulae were constructed for the first time by Nikol'skii (see [2]).

In [3] we proposed to combine these two approaches: the one going back to Gauss and based on developing methods exact on subspaces and the other going back to Kolmogorov and based on finding methods optimal on the class under consideration. In other words, we proposed to look for methods optimal on a class which are at the same time exact on a fixed subspace. In the framework of this approach, in [4] and [5] we solved several recovery problems for solutions of equations of mathematical physics.

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In this paper we consider problems of developing optimal methods for the recovery of analytic functions in a strip and their derivatives from inaccurately prescribed traces of the Fourier transforms of these functions on the real axis. The optimal methods are additionally required to be exact on subspaces of entire functions.

§ 2. Statement of the problem

Let X be a linear space and Y and Z be two normed linear spaces, and let $A\colon X \to Z$ and $I\colon X \to Y$ be linear operators. We consider the problem of the optimal recovery of the values of A on a set $W \subset X$ from the inaccurately prescribed values of I at elements of this set. We assume that for each $x \in W$ we know a value $y \in Y$ such that $\|Ix - y\|_Y \leqslant \delta$, where δ is some positive number characterizing the error of the *a priori* information about elements of W. The problem consists in recovering the value of Ax from y. A recovery method is a map $m\colon Y \to Z$ that assigns to $y \in Y$ an element $m(y) \in Z$, which is set to be the approximate value of Ax.

The *error of the method* m is the quantity

$$e(A, W, I, \delta, m) = \sup_{\substack{x \in W,\ y \in Y \\ \|Ix - y\|_Y \leqslant \delta}} \|Ax - m(y)\|_Z.$$

The *optimal recovery error* is the quantity

$$E(A, W, I, \delta) = \inf_{m\colon Y \to Z} e(A, W, I, \delta, m),$$

while methods delivering the infimum are called *optimal on the set* W. The above problem relates to optimal recovery theory. For more information about this theory and the problems considered it its framework the reader can consult the survey paper [6] and the books [7]–[10].

Let $L \subset X$ be a linear subspace of X. We say that a method $m\colon Y \to Z$ is *exact on* L if $Ax = m(Ix)$ for all $x \in L$. Consider the set $\mathcal{E}_L$ of linear operators $m\colon Y \to Z$ that are exact on L. Set

$$E_L(A, W, I, \delta) = \inf_{m \in \mathcal{E}_L} e(A, W, I, \delta, m).$$

We call methods delivering the infimum in this equality *optimal on* W among the exact methods on L.

By the *sum of two sets* A and B in a linear space we mean the set

$$A + B = \{a + b\colon a \in A,\ b \in B\}.$$

Proposiiton (see [4]). *Let $L \subset X$ be a linear subspace of X, and let $m^*\colon Y \to Z$ be a linear operator presenting an optimal method for the recovery of A on the set $W + L$. Then*

$$E_L(A, W, I, \delta) = E(A, W + L, I, \delta).$$

If $E_L(A, W + L, I, \delta) < \infty$, then m^* is an optimal recovery method on W among the exact methods on L.

Thus, to find a linear method optimal on W among the ones exact on L it is sufficient to find linear methods among the optimal methods on $W+L$.

In this paper we consider the problem of the optimal recovery of analytic functions in a strip

$$S_\beta = \{z \in \mathbb{C}\colon |\operatorname{Im} z| < \beta\}$$

and their derivatives under the assumptions that the recovery methods are exact on the space $\mathcal{B}_{\sigma,2}(\mathbb{R})$ of entire functions, the subspace of $L_2(\mathbb{R})$ formed by the restrictions to $\mathbb{R}$ of entire functions of exponential type σ.

We turn to the precise statement. By the *Hardy space* $\mathcal{H}_2^\beta$ we mean the set of analytic functions f in the strip S_β such that

$$\|f\|_{\mathcal{H}_2^\beta} = \left(\sup_{0\leqslant\eta<\beta} \frac{1}{2}\int_{\mathbb{R}} (|f(t+i\eta)|^2 + |f(t-i\eta)|^2)\,dt\right)^{1/2} < \infty.$$

We let $\mathcal{H}_2^{r,\beta}$ (the Hardy–Sobolev space) denote the set of analytic functions in S_β such that $f^{(r)} \in \mathcal{H}_2^\beta$.

Let $H_2^{r,\beta}$ denote the set of functions $f \in \mathcal{H}_2^{r,\beta} \cap L_2(\mathbb{R})$ satisfying $\|f^{(r)}\|_{\mathcal{H}_2^\beta} \leqslant 1$. If $\sigma > 0$, then $\mathcal{B}_{\sigma,2}(\mathbb{R})$ denotes the subspace of $L_2(\mathbb{R})$ formed by the restrictions to $\mathbb{R}$ of entire functions of exponential type σ. It is well known that $f \in \mathcal{B}_{\sigma,2}(\mathbb{R})$ if and only if the support of the Fourier transform Ff lies on the interval $\Delta_\sigma = [-\sigma,\sigma]$. By definition $\mathcal{B}_{0,2}(\mathbb{R}) = \{0\}$.

Consider the problem of the optimal recovery of the kth derivative of $f \in H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R})$, $k \leqslant r$, from the trace on Δ_{σ_1}, $\sigma_1 > 0$, of its Fourier transform defined with some error in the metric $L_2(\Delta_{\sigma_1})$, that is, we assume that in place of the trace of Ff on Δ_{σ_1} we know a function $y \in L_2(\Delta_{\sigma_1})$ such that

$$\|Ff - y\|_{L_2(\Delta_{\sigma_1})} \leqslant \delta.$$

From y we must recover the function $f^{(k)}$ on $\mathbb{R}$ in the best possible way, that is, the problem consists in the recovery of

$$E(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta) = \inf_{m\colon L_2(\Delta_{\sigma_1})\to L_2(\mathbb{R})} e(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta, m),$$

where $D^k f = f^{(k)}$, $I_{\sigma_1} f = Ff\big|_{\Delta_{\sigma_1}}$ and

$$e(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta, m) = \sup_{\substack{f\in H_2^{r,\beta}+\mathcal{B}_{\sigma,2}(\mathbb{R}),\ y\in L_2(\Delta_{\sigma_1}) \\ \|Ff-y\|_{L_2(\Delta_{\sigma_1})}\leqslant\delta}} \|f^{(k)} - m(y)\|_{L_2(\mathbb{R})}.$$

In other words, we are going to find optimal methods for the recovery of the kth derivative on the class $H_2^{r,\beta}$ among the methods exact on the subspace of entire functions $\mathcal{B}_{\sigma,2}(\mathbb{R})$. Without the assumptions that the method is exact on $\mathcal{B}_{\sigma,2}(\mathbb{R})$ this problem was considered in [11].

§ 3. Main results

Consider the function $y = s(x)$, $x \geqslant 0$, defined parametrically by

$$\begin{cases} x = t^{2r}\cosh 2\beta t, \\ y = t^{2k}, \end{cases} \qquad t \geqslant 0,$$

$k, r \in \mathbb{N}$, $r \geqslant k$, $\beta > 0$. For $t > 0$ its derivative is positive:

$$\frac{dy}{dx} = \frac{kt^{2(k-r)}}{r\cosh 2\beta t + t\beta\sinh 2\beta t} > 0,$$

and it is monotonically decreasing, so that s is an increasing concave function.

The straight line connecting a point $(x(t), y(t))$ with the origin has the form $y = \lambda_2 x$, where

$$\lambda_2 = \frac{y(t)}{x(t)} = \frac{1}{t^{2(r-k)}\cosh 2\beta t}.$$

Since s is concave, there exists t_0 such that the tangent to s at $(x(t_0), y(t_0))$ is parallel to $y = \lambda_2 x$. Thus we can find t_0 from the equation

$$\frac{y'(t_0)}{x'(t_0)} = \lambda_2.$$

This equation can be written as

$$\frac{kt_0^{2(k-r)}}{r\cosh 2\beta t_0 + t_0\beta\sinh 2\beta t_0} = \frac{1}{t^{2(r-k)}\cosh 2\beta t}. \tag{3.1}$$

The tangent line through $(x(t_0), y(t_0))$ has the form $y = \lambda_1 + \lambda_2 x$, where

$$\lambda_1 = t_0^{2k}\biggl(1 - \frac{k}{r + t_0\beta\tanh 2\beta t_0}\biggr). \tag{3.2}$$

Let $h(t)$ denote the point at which $y(h(t)) = \lambda_1$ (Figure 1). Thus,

$$h(t) = t_0\biggl(1 - \frac{k}{r + t_0\beta\tanh 2\beta t_0}\biggr)^{1/(2k)}.$$

As the function on the right-hand side of (3.2) is monotonically increasing in $t_0 \in [0, +\infty)$ from zero to $+\infty$, for each $\lambda_1 > 0$ there exists $t_0 > 0$ such that the tangent to s at $(x(t_0), y(t_0))$ passes through the point $(0, \lambda_1)$. We denote this point t_0 by $h_1(\lambda_1)$.

The function $t^r\sqrt{\cosh 2\beta t}$ is monotonically increasing from 0 to $+\infty$ for $t \in \mathbb{R}_+$. Hence for each $x \in \mathbb{R}_+$ the equation

$$t^r\sqrt{\cosh 2\beta t} = x$$

has a unique solution on the interval $[0, +\infty)$. We denote it by $\mu_{r\beta}(x)$.

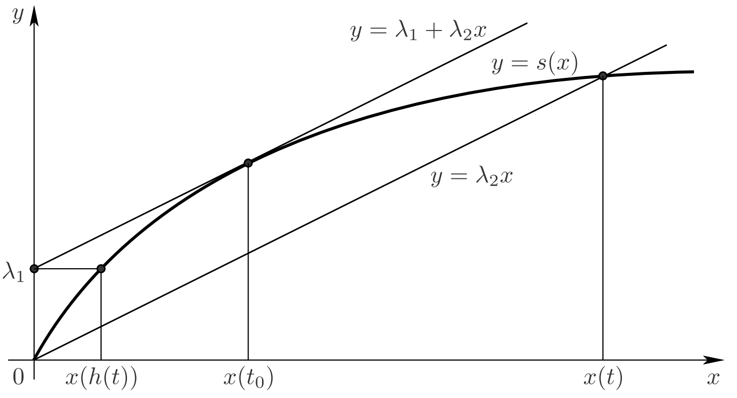

Figure 1

Let $\widehat{\sigma}_1$ denote the value of the parameter t such that $t_0 = \widehat{t}_0 = \mu_{r\beta}(\sqrt{2\pi}/\delta)$, that is, $x(\widehat{t}_0) = 2\pi/\delta^2$. Set $\widehat{\sigma} = h(\widehat{\sigma}_1)$. The tangent line through $(x(\widehat{t}_0), y(\widehat{t}_0))$ has an equation $y = \widehat{\lambda}_1 + \widehat{\lambda}_2 x$, where

$$\widehat{\lambda}_1 = \widehat{t}_0^{2k}\left(1 - \frac{k}{r + \widehat{t}_0\beta \tanh 2\beta\widehat{t}_0}\right) \quad \text{and} \quad \widehat{\lambda}_2 = \frac{k\widehat{t}_0^{2(k-r)}}{r\cosh 2\beta\widehat{t}_0 + \widehat{t}_0\beta\sinh 2\beta\widehat{t}_0}.$$

Thus,

$$\widehat{\sigma} = \widehat{\lambda}_1^{1/(k)} \quad \text{and} \quad \widehat{\sigma}_1 = \mu_{r-k,\beta}\left(\frac{1}{\sqrt{\widehat{\lambda}_2}}\right)$$

(Figure 2).

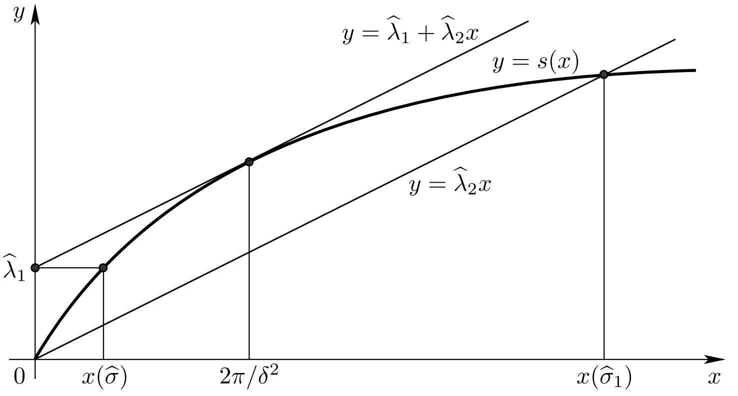

Figure 2

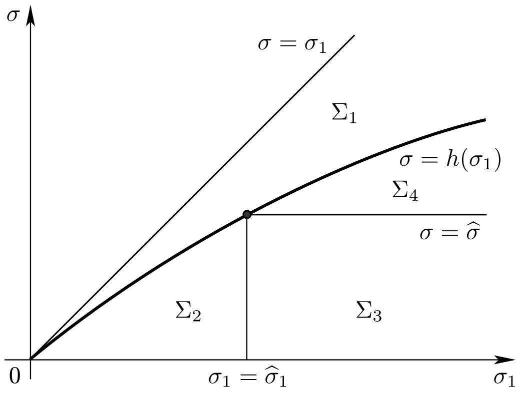

Figure 3

Consider the following four domains in the plane $\mathbb{R}^2$ (Figure 3):

$$\Sigma_1 = \bigl\{(\sigma_1,\sigma)\in\mathbb{R}^2\colon 0<h(\sigma_1)\leqslant\sigma\leqslant\sigma_1\bigr\},$$
$$\Sigma_2 = \bigl\{(\sigma_1,\sigma)\in\mathbb{R}^2\colon 0\leqslant\sigma\leqslant h(\sigma_1),\ 0<\sigma_1\leqslant\widehat{\sigma}_1\bigr\},$$
$$\Sigma_3 = \bigl\{(\sigma_1,\sigma)\in\mathbb{R}^2\colon \sigma_1\geqslant\widehat{\sigma}_1,\ 0\leqslant\sigma\leqslant\widehat{\sigma}\bigr\},$$
$$\Sigma_4 = \bigl\{(\sigma_1,\sigma)\in\mathbb{R}^2\colon \widehat{\sigma}\leqslant\sigma\leqslant h(\sigma_1)\bigr\}.$$

Set

$$(\lambda_1,\lambda_2)=\begin{cases}\left(\sigma^{2k},\dfrac{1}{\sigma_1^{2(r-k)}\cosh 2\beta\sigma_1}\right), & (\sigma_1,\sigma)\in\Sigma_1,\\ \left(h^{2k}(\sigma_1),\dfrac{1}{\sigma_1^{2(r-k)}\cosh 2\beta\sigma_1}\right), & (\sigma_1,\sigma)\in\Sigma_2,\\ \left(\widehat{\sigma}^{2k},\dfrac{1}{\widehat{\sigma}_1^{2(r-k)}\cosh 2\beta\widehat{\sigma}_1}\right), & (\sigma_1,\sigma)\in\Sigma_3,\\ \left(\sigma^{2k},\dfrac{h_1^{2k}(\sigma^{2k})-\sigma^{2k}}{h_1^{2r}(\sigma^{2k})\cosh(2\beta h_1(\sigma^{2k}))}\right), & (\sigma_1,\sigma)\in\Sigma_4.\end{cases} \tag{3.3}$$

We let $\Theta(\sigma,\sigma_1)$ denote the set of measurable functions θ on $[-\sigma_1,-\sigma)\cup(\sigma,-\sigma_1]$ such that $\theta(t)|\leqslant 1$ for almost all $\sigma<|t|\leqslant\sigma_1$.

Theorem. *Let k and r be integers satisfying $0\leqslant k\leqslant r$.*
(1) *If $\sigma>\sigma_1$, then $E(D^k,H_2^{r,\beta}+\mathcal{B}_{\sigma,2}(\mathbb{R}),I_{\sigma_1},\delta)=\infty$.*
(2) *If $k\geqslant 1$, then*

$$E(D^k,H_2^{r,\beta}+\mathcal{B}_{\sigma,2}(\mathbb{R}),I_{\sigma_1},\delta)=\sqrt{\lambda_1\frac{\delta^2}{2\pi}+\lambda_2} \tag{3.4}$$

for all $\sigma_1>0$ and $\sigma\geqslant 0$ such that $\sigma\leqslant\sigma_1$, and for each function $\theta\in\Theta(\sigma,\sigma_1)$ the method

$$\widehat{m}_\theta(y)(x)=\frac{1}{2\pi}\int_{-\sigma}^{\sigma}(it)^k y(t)e^{itx}\,dt+\frac{1}{2\pi}\int_{\sigma<|t|\leqslant\sigma_1}(it)^k a_\theta(t)y(t)e^{itx}\,dt,$$

where

$$a_\theta(t) = \frac{\lambda_1 + \theta(t)|t|^{r-k}\sqrt{\lambda_1\lambda_2\cosh 2\beta t}\,\sqrt{-t^{2k} + \lambda_1 + \lambda_2 t^{2r}\cosh 2\beta t}}{\lambda_1 + \lambda_2 t^{2r}\cosh 2\beta t}, \tag{3.5}$$

is an optimal method.

(3) *If* $k = 0$, *then*

$$E(D^0, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta) = \sqrt{\frac{\delta^2}{2\pi} + \frac{1}{\sigma_1^{2r}\cosh 2\beta\sigma_1}}$$

for all $\sigma_1 > 0$ *and* $\sigma \geqslant 0$ *such that* $\sigma \leqslant \sigma_1$, *and for each* $\theta \in \Theta(\sigma, \sigma_1)$ *the method*

$$\widehat{m}_\theta(y)(x) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} y(t)e^{itx}\,dt + \frac{1}{2\pi}\int_{\sigma<|t|\leqslant\sigma_1} a_\theta(t)y(t)e^{itx}\,dt,$$

where

$$a_\theta(t) = \frac{\sigma_1^{2r}\cosh 2\beta\sigma_1 + \theta(t)t^{2r}\cosh 2\beta t}{\sigma_1^{2r}\cosh 2\beta\sigma_1 + t^{2r}\cosh 2\beta t}, \tag{3.6}$$

is an optimal method.

Proof. By the main theorem on the representation of analytic functions in tube domains (see [12]) we have $f \in \mathcal{H}_2^\beta$ if and only if this function has the form

$$f(z) = \frac{1}{2\pi}\int_{\mathbb{R}} g(t)e^{izt}\,dt, \tag{3.7}$$

where g is a function such that

$$\sup_{|y|<\beta}\int_{\mathbb{R}} |g(t)|^2 e^{-2yt}\,dt < \infty$$

(g is the Fourier transform of $f(x)$, $x \in \mathbb{R}$). By Plancherel's theorem

$$\|f\|^2_{\mathcal{H}_2^\beta} = \frac{1}{2\pi}\sup_{0\leqslant y<\beta}\int_{\mathbb{R}} |Ff(t)|^2\cosh 2yt\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} |Ff(t)|^2\cosh 2\beta t\,dt. \tag{3.8}$$

We show that $f \in \mathcal{H}_2^{r,\beta} \cap L_2(\mathbb{R})$ is in the class $H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R})$ if and only if

$$\frac{1}{2\pi}\int_{|t|>\sigma} t^{2r}|Ff(t)|^2\cosh 2\beta t\,dt \leqslant 1. \tag{3.9}$$

In fact, if $f \in H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R})$, then $f = f_1 + f_2$, where $f_1 \in H_2^{r,\beta}$ and $f_2 \in \mathcal{B}_{\sigma,2}(\mathbb{R})$. Now bearing in mind that Ff_2 has support on Δ_σ, we have

$$\frac{1}{2\pi}\int_{|t|>\sigma} t^{2r}|Ff(t)|^2\cosh 2\beta t\,dt = \frac{1}{2\pi}\int_{|t|>\sigma} t^{2r}|Ff_1(t)|^2\cosh 2\beta t\,dt \leqslant 1.$$

Conversely, let $f \in \mathcal{H}_2^{r,\beta} \cap L_2(\mathbb{R})$ be a function such that (3.9) holds. Let $f_2 \in L_2(\mathbb{R})$ denote the function satisfying $Ff_2 = \chi_\sigma Ff$, where χ_σ is the characteristic function of the interval Δ_σ. Then it is clear that $f_2 \in \mathcal{B}_{\sigma,2}(\mathbb{R})$. Set $f_1 = f - f_2$.

Then it is obvious that $f_1 \in \mathcal{H}_2^{r,\beta} \cap L_2(\mathbb{R})$, and by (3.8) (since $Ff_1 = 0$ on Δ_σ) we have

$$\|f_1^{(r)}\|^2_{\mathcal{H}_2^\beta} = \frac{1}{2\pi}\int_{|t|>\sigma} t^{2r}|Ff_1(t)|^2\cosh 2\beta t\,dt = \frac{1}{2\pi}\int_{|t|>\sigma} t^{2r}|Ff(t)|^2\cosh 2\beta t\,dt \leqslant 1,$$

that is, $f = f_1 + f_2 \in H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R})$.

Let $f \in \mathcal{H}_2^{r,\beta} \cap L_2(\mathbb{R})$ be a function such that $\|Ff\|_{L_2(\Delta_{\sigma_1})} \leqslant \delta$ and inequality (3.9) holds. The for each method $m\colon L_2(\Delta_{\sigma_1}) \to L_2(\mathbb{R})$ we have

$$\begin{aligned}2\|f^{(k)}\|_{L_2(\mathbb{R})} &= \|f^{(k)} - m(0) - (-f^{(k)} - m(0))\|_{L_2(\mathbb{R})}\\ &\leqslant \|f^{(k)} - m(0)\|_{L_2(\mathbb{R})} + \|-f^{(k)} - m(0)\|_{L_2(\mathbb{R})}\\ &\leqslant 2e(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta, m).\end{aligned}$$

Hence

$$\begin{gathered}\sup_{\substack{f\in\mathcal{H}_2^{r,\beta}\cap L_2(\mathbb{R}),\ \|Ff\|_{L_2(\Delta_{\sigma_1})}\leqslant\delta\\ \frac{1}{2\pi}\int_{|t|>\sigma} t^{2r}|Ff(t)|^2\cosh 2\beta t\,dt\leqslant 1}} \|f^{(k)}\|_{L_2(\mathbb{R})}\\ \leqslant e(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta, m) \leqslant E(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta). \qquad (3.10)\end{gathered}$$

Consider the extremal problem on the left-hand side of (3.10). Passing to squares for convenience we can write it as

$$\begin{gathered}\frac{1}{2\pi}\int_{\mathbb{R}} t^{2k}|Ff(t)|^2\,dt \to \max,\\ \int_{|t|\leqslant\sigma_1}|Ff(t)|^2\,dt \leqslant \delta^2, \qquad \frac{1}{2\pi}\int_{|t|>\sigma} t^{2r}|Ff(t)|^2\cosh 2\beta t\,dt \leqslant 1,\\ f \in \mathcal{H}_2^{r,\beta} \cap L_2(\mathbb{R}).\end{gathered} \qquad (3.11)$$

(1) Assume that $\sigma > \sigma_1$. Let f_0 be a function such that

$$Ff_0(t) = \begin{cases} c, & t \in (\sigma_1, \sigma),\\ 0, & t \notin (\sigma_1, \sigma),\end{cases}$$

where $c > 0$. Then f_0 is an admissible function in (3.11) and

$$\|f_0^{(k)}\|^2_{L_2(\mathbb{R})} = \frac{c^2}{2\pi}\int_{\sigma_1}^{\sigma} t^{2k}\,dt.$$

Letting c tend to infinity, from (3.10) we obtain

$$E(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta) = \infty.$$

(2) Let $k \geqslant 1$. We show that in each domain Σ_j, $j = 1, 2, 3, 4$, we have the inequality

$$E(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta) \geqslant \sqrt{\lambda_1\frac{\delta^2}{2\pi} + \lambda_2}. \qquad (3.12)$$

Let $(\sigma_1,\sigma)\in\Sigma_1$. For each $n\in\mathbb{N}$ such that $1/n<\sigma$ consider the function f_n satisfying

$$Ff_n(t)=\begin{cases}\delta\sqrt{n}, & \sigma-\dfrac{1}{n}<t<\sigma,\\ \sqrt{2\pi n}\Big(\sigma_1+\dfrac{1}{n}\Big)^{-r}\cosh^{-1/2}\Big(2\beta\Big(\sigma_1+\dfrac{1}{n}\Big)\Big), & \sigma_1<t<\sigma_1+\dfrac{1}{n},\\ 0 & \text{otherwise.}\end{cases}\tag{3.13}$$

Then we have

$$\|Ff_n\|^2_{L_2(\Delta_{\sigma_1})}=\int_{\sigma-1/n}^{\sigma}\delta^2n\,dt=\delta^2$$

and

$$\begin{aligned}&\frac{1}{2\pi}\int_{|t|>\sigma}t^{2r}|Ff_n(t)|^2\cosh 2\beta t\,dt\\ &\qquad=\frac{n}{(\sigma_1+1/n)^{2r}\cosh(2\beta(\sigma_1+1/n))}\int_{\sigma_1}^{\sigma_1+1/n}t^{2r}\cosh 2\beta t\,dt\leqslant 1.\end{aligned}$$

Hence the functions f_n are admissible in problem (3.11). From (3.10) we obtain

$$\begin{aligned}&E^2(D^k,H_2^{r,\beta}+\mathcal{B}_{\sigma,2}(\mathbb{R}),I_{\sigma_1},\delta)\\ &\quad\geqslant\frac{1}{2\pi}\int_{\mathbb{R}}t^{2k}|Ff_n(t)|^2\,dt\\ &\quad=\frac{1}{2\pi}\int_{\sigma-1/n}^{\sigma}t^{2k}\delta^2n\,dt+\frac{n}{(\sigma_1+1/n)^{2r}\cosh(2\beta(\sigma_1+1/n))}\int_{\sigma_1}^{\sigma_1+1/n}t^{2k}\,dt\\ &\quad=\frac{\delta^2n(\sigma^{2k+1}-(\sigma-1/n)^{2k+1})}{2\pi(2k+1)}\\ &\qquad+\frac{n}{(\sigma_1+1/n)^{2r}\cosh(2\beta(\sigma_1+1/n))}\,\frac{(\sigma_1+1/n)^{2k+1}-\sigma_1^{2k+1}}{2k+1}.\end{aligned}$$

Taking the limit as $n\to\infty$ yields

$$E^2(D^k,H_2^{r,\beta}+\mathcal{B}_{\sigma,2}(\mathbb{R}),I_{\sigma_1},\delta)\geqslant\frac{\delta^2\sigma^{2k}}{2\pi}+\frac{1}{\sigma_1^{2(r-k)}\cosh 2\beta\sigma_1}=\lambda_1\frac{\delta^2}{2\pi}+\lambda_2.$$

Now let $(\sigma_1,\sigma)\in\Sigma_2$. The straight line connecting $(x(\sigma_1),y(\sigma_1))$ with the origin has the form $y=\lambda_2x$, where

$$\lambda_2=\frac{y(\sigma_1)}{x(\sigma_1)}=\frac{1}{\sigma_1^{2(r-k)}\cosh 2\beta\sigma_1}.$$

As mentioned above, since s is concave, there exists a point t_0 such that the tangent to s at $(x(t_0),y(t_0))$ is parallel to the line $y=\lambda_2x$. The tangent through $(x(t_0),y(t_0))$ itself has the form $y=\lambda_1+\lambda_2x$, where

$$\lambda_1=t_0^{2k}-\lambda_2t_0^{2r}\cosh 2\beta t_0=t_0^{2k}\bigg(1-\frac{t_0^{2(r-k)}\cosh 2\beta t_0}{\sigma_1^{2(r-k)}\cosh 2\beta\sigma_1}\bigg)=h^{2k}(\sigma_1).$$

Since $\sigma_1 \leqslant \widehat{\sigma}_1$, it follows that $t_0 \leqslant \widehat{t}_0$. Therefore, $t_0^{2r} \cosh 2\beta t_0 \leqslant 2\pi/\delta^2$. For each $n \in \mathbb{N}$ such that $h(\sigma_1) < t_0 - 1/n$ consider the function f_n such that

$$Ff_n(t) = \begin{cases} \delta\sqrt{n}, & t_0 - \dfrac{1}{n} < t < t_0, \\ \dfrac{\sqrt{n(2\pi - \delta^2 t_0^{2r} \cosh 2\beta t_0)}}{(\sigma_1 + 1/n)^r \sqrt{\cosh(2\beta(\sigma_1 + 1/n))}}, & \sigma_1 < t < \sigma_1 + \dfrac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\|Ff_n\|^2_{L_2(\Delta_{\sigma_1})} = \int_{t_0 - 1/n}^{t_0} \delta^2 n \, dt = \delta^2$$

and

$$\begin{aligned} &\frac{1}{2\pi} \int_{|t|>\sigma} t^{2r} |Ff_n(t)|^2 \cosh 2\beta t \, dt \\ &\quad = \frac{\delta^2 n}{2\pi} \int_{t_0 - 1/n}^{t_0} t^{2r} \cosh 2\beta t \, dt \\ &\qquad + \frac{n(2\pi - \delta^2 t_0^{2r} \cosh 2\beta t_0)}{2\pi(\sigma_1 + 1/n)^{2r} \cosh(2\beta(\sigma_1 + 1/n))} \int_{\sigma_1}^{\sigma_1 + 1/n} t^{2r} \cosh 2\beta t \, dt \\ &\quad \leqslant \frac{\delta^2}{2\pi} t_0^{2r} \cosh 2\beta t_0 + 1 - \frac{\delta^2}{2\pi} t_0^{2r} \cosh 2\beta t_0 = 1. \end{aligned}$$

Hence the function f_n are admissible in problem (3.11). From (3.10) we obtain

$$\begin{aligned} &E^2(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta) \\ &\quad \geqslant \frac{1}{2\pi} \int_{\mathbb{R}} t^{2k} |Ff_n(t)|^2 \, dt \\ &\quad = \frac{1}{2\pi} \int_{t_0 - 1/n}^{t_0} t^{2k} \delta^2 n \, dt + \frac{n(2\pi - \delta^2 t_0^{2r} \cosh 2\beta t_0)}{2\pi(\sigma_1 + 1/n)^{2r} \cosh(2\beta(\sigma_1 + 1/n))} \int_{\sigma_1}^{\sigma_1 + 1/n} t^{2k} \, dt \\ &\quad = \frac{\delta^2 n (t_0^{2k+1} - (t_0 - 1/n)^{2k+1})}{2\pi(2k+1)} \\ &\qquad + \frac{n(2\pi - \delta^2 t_0^{2r} \cosh 2\beta t_0)((\sigma_1 + 1/n)^{2k+1} - \sigma_1^{2k+1})}{2\pi(2k+1)(\sigma_1 + 1/n)^{2r} \cosh(2\beta(\sigma_1 + 1/n))}. \end{aligned}$$

Taking the limit as $n \to \infty$ yields

$$\begin{aligned} &E^2(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta) \\ &\quad \geqslant \frac{\delta^2 t_0^{2k}}{2\pi} + \frac{2\pi - \delta^2 t_0^{2r} \cosh 2\beta t_0}{2\pi \sigma_1^{2(r-k)} \cosh 2\beta\sigma_1} \\ &\quad = \frac{\delta^2 t_0^{2k}}{2\pi} \left(1 - \frac{t_0^{2(r-k)} \cosh 2\beta t_0}{\sigma_1^{2(r-k)} \cosh 2\beta\sigma_1}\right) + \frac{1}{\sigma_1^{2(r-k)} \cosh 2\beta\sigma_1} \\ &\quad = \lambda_1 \frac{\delta^2}{2\pi} + \lambda_2. \end{aligned}$$

Let $(\sigma_1, \sigma) \in \Sigma_3$. For each $n \in \mathbb{N}$ such that $\sigma < \widehat{t}_0 - 1/n$ consider the function f_n such that

$$Ff_n(t) = \begin{cases} \delta\sqrt{n}, & \widehat{t}_0 - \dfrac{1}{n} < t < \widehat{t}_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\|Ff_n\|^2_{L_2(\Delta_{\sigma_1})} = \int_{\widehat{t}_0 - 1/n}^{\widehat{t}_0} \delta^2 n\, dt = \delta^2$$

and

$$\begin{aligned} \frac{1}{2\pi}\int_{|t|>\sigma} t^{2r}|Ff_n(t)|^2 \cosh 2\beta t\, dt &= \frac{\delta^2 n}{2\pi}\int_{\widehat{t}_0 - 1/n}^{\widehat{t}_0} t^{2r}\cosh 2\beta t\, dt \\ &\leqslant \frac{\delta^2}{2\pi}\widehat{t}_0^{\,2r}\cosh 2\beta\widehat{t}_0 = 1. \end{aligned}$$

Thus, the f_n are admissible functions in (3.11). From (3.10) we obtain

$$\begin{aligned} &E^2(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta) \\ &\quad\geqslant \frac{1}{2\pi}\int_{\mathbb{R}} t^{2k}|Ff_n(t)|^2\, dt = \frac{1}{2\pi}\int_{\widehat{t}_0 - 1/n}^{\widehat{t}_0} t^{2k}\delta^2 n\, dt \\ &\quad= \frac{\delta^2 n(\widehat{t}_0^{\,2k+1} - (\widehat{t}_0 - 1/n)^{2k+1})}{2\pi(2k+1)}. \end{aligned}$$

Taking the limit as $n \to \infty$ yields

$$E^2(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta) \geqslant \frac{\delta^2\widehat{t}_0^{\,2k}}{2\pi} = \lambda_1\frac{\delta^2}{2\pi} + \lambda_2.$$

Let $(\sigma_1, \sigma) \in \Sigma_4$, and let t_0 be the point defined as for $(\sigma_1, \sigma) \in \Sigma_2$. Set $\xi = h_1(\sigma^{2k})$. Since $\sigma \leqslant h(\sigma_1)$, we obtain $\xi \leqslant t_0 < \sigma_1$. Moreover, as $\sigma \geqslant \widehat{\sigma}$, it follows that $\xi \geqslant \widehat{t}_0$. Hence $\xi^{2r}\cosh 2\beta\xi \geqslant 2\pi/\delta^2$. For each $n \in \mathbb{N}$ satisfying $1/n < \sigma$ and $\xi - 1/n > \sigma$ consider the function f_n such that

$$Ff_n(t) = \begin{cases} \sqrt{n}\sqrt{\delta^2 - \dfrac{2\pi}{\xi^{2r}\cosh 2\beta\xi}}, & \sigma - \dfrac{1}{n} < t < \sigma, \\ \dfrac{\sqrt{2\pi n}}{\xi^r\sqrt{\cosh 2\beta\xi}}, & \xi - \dfrac{1}{n} < t < \xi, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\|Ff_n\|^2_{L_2(\Delta_{\sigma_1})} = \int_{\sigma - 1/n}^{\sigma} n\left(\delta^2 - \frac{2\pi}{\xi^{2r}\cosh 2\beta\xi}\right) dt + \int_{\xi - 1/n}^{\xi} \frac{2\pi n}{\xi^{2r}\cosh 2\beta\xi}\, dt = \delta^2,$$

$$\frac{1}{2\pi}\int_{|t|>\sigma} t^{2r}|Ff_n(t)|^2\cosh 2\beta t\, dt = \frac{n}{\xi^{2r}\cosh 2\beta\xi}\int_{\xi - 1/n}^{\xi} t^{2r}|Ff_n(t)|^2\cosh 2\beta t\, dt \leqslant 1.$$

Thus the functions f_n are admissible in problem (3.11). By (3.10) we have

$$
\begin{aligned}
&E^2(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta) \\
&\quad\geqslant \frac{1}{2\pi}\int_{\mathbb{R}} t^{2k}|Ff_n(t)|^2\,dt \\
&\quad= \frac{n}{2\pi}\left(\delta^2 - \frac{2\pi}{\xi^{2r}\cosh 2\beta\xi}\right)\int_{\sigma-1/n}^{\sigma} t^{2k}\,dt + \frac{n}{\xi^{2r}\cosh 2\beta\xi}\int_{\xi-1/n}^{\xi} t^{2k}\,dt \\
&\quad= \frac{n}{2\pi}\left(\delta^2 - \frac{2\pi}{\xi^{2r}\cosh 2\beta\xi}\right)\frac{\sigma^{2k+1} - (\sigma-1/n)^{2k+1}}{2k+1} \\
&\qquad+ \frac{n}{\xi^{2r}\cosh 2\beta\xi}\frac{\xi^{2k+1} - (\xi-1/n)^{2k+1}}{2k+1}.
\end{aligned}
$$

Taking the limit as $n \to \infty$ yields

$$
\begin{aligned}
E^2(D^k, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta) &\geqslant \frac{\sigma^{2k}}{2\pi}\left(\delta^2 - \frac{2\pi}{\xi^{2r}\cosh 2\beta\xi}\right) + \frac{\xi^{2k}}{\xi^{2r}\cosh 2\beta\xi} \\
&= \lambda_1\frac{\delta^2}{2\pi} + \lambda_2.
\end{aligned}
$$

We look for optimal recovery methods $m_a\colon L_2(\Delta_{\sigma_1}) \to L_2(\mathbb{R})$ in the class of maps with the following representation in terms of Fourier transforms:

$$
Fm_a(y)(t) = \begin{cases} (it)^k a(t)y(t), & t \in \Delta_{\sigma_1}, \\ 0, & t \notin \Delta_{\sigma_1}. \end{cases} \tag{3.14}
$$

For an estimate of the error of such a method we must estimate the value of the extremal problem

$$
\begin{gathered}
\|f^{(k)} - m_a(y)\|_{L_2(\mathbb{R})} \to \max, \\
\|Ff - y\|_{L_2(\Delta_{\sigma_1})} \leqslant \delta, \qquad f \in H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}).
\end{gathered} \tag{3.15}
$$

Considering Fourier images in the functional to be maximized, from Plancherel's theorem we obtain that the square of the value of problem (3.15) is the value of the following problem:

$$
\begin{gathered}
\frac{1}{2\pi}\int_{-\sigma_1}^{\sigma_1} t^{2k}|Ff(t) - a(t)y(t)|^2\,dt + \frac{1}{2\pi}\int_{|t|>\sigma_1} t^{2k}|Ff(t)|^2\,dt \to \max, \\
\int_{-\sigma_1}^{\sigma_1}|Ff(t) - y(t)|^2\,dt \leqslant \delta^2, \qquad \frac{1}{2\pi}\int_{|t|\geqslant\sigma} t^{2r}|Ff(t)|^2\cosh 2\beta t\,dt \leqslant 1.
\end{gathered} \tag{3.16}
$$

Note that on pairs (f, y) admissible for this problem, where $f \in \mathcal{B}_{\sigma,2}(\mathbb{R})$ and $y = Ff$, the functional to be maximized takes the form

$$
\frac{1}{2\pi}\int_{-\sigma}^{\sigma} t^{2k}|Ff(t)|^2|1 - a(t)|^2\,dt.
$$

Hence, if the function a is not almost everywhere equal to one on Δ_σ, then, as $\mathcal{B}_{\sigma,2}(\mathbb{R})$ is a linear space, the value of problem (3.16) (and therefore of (3.15)) is infinite, that is, the method with this a has an infinitely large error.

Let $a \equiv 1$ on Δ_σ. We estimate the functional maximized in (3.16) from above by representing it as a sum of three terms,

$$I_1 = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} t^{2k}|Ff(t) - y(t)|^2\,dt,$$

$$I_2 = \frac{1}{2\pi}\int_{\sigma<|t|\leqslant\sigma_1} t^{2k}|Ff(t) - a(t)y(t)|^2\,dt$$

and

$$I_3 = \frac{1}{2\pi}\int_{|t|>\sigma_1} t^{2k}|Ff(t)|^2\,dt.$$

We show that

$$I_1 \leqslant \frac{\lambda_1}{2\pi}\int_{-\sigma}^{\sigma} |Ff(t) - y(t)|^2\,dt \tag{3.17}$$

in all domains Σ_j, $j = 1, 2, 3, 4$. In fact, the inequality

$$I_1 \leqslant \frac{\sigma^{2k}}{2\pi}\int_{-\sigma}^{\sigma} |Ff(t) - y(t)|^2\,dt$$

is obvious. Since $\sigma^{2k} = \lambda_1$ in Σ_1 and Σ_4, (3.17) holds for these domains. If $(\sigma_1, \sigma) \in \Sigma_2$, then

$$\lambda_1 = h^{2k}(\sigma_1) \geqslant \sigma^{2k},$$

while if $(\sigma_1, \sigma) \in \Sigma_3$, then

$$\lambda_1 = \widehat{\sigma}^{2k} \geqslant \sigma^{2k},$$

so that (3.17) holds for all domains.

Next we estimate I_2. Using the Cauchy–Schwarz–Bunyakovsky inequality we obtain

$$\begin{aligned} t^{2k}&|Ff(t) - a(t)y(t)|^2 \\ &= t^{2k}|(1 - a(t))Ff(t) + a(t)(Ff(t) - y(t))|^2 \\ &\leqslant t^{2k}\left(\frac{|1 - a(t)|^2}{\lambda_2 t^{2r}\cosh 2\beta t} + \frac{|a(t)|^2}{\lambda_1}\right)(\lambda_2 t^{2r}|Ff(t)|^2\cosh 2\beta t + \lambda_1|Ff(t) - y(t)|^2). \end{aligned} \tag{3.18}$$

Set

$$S_a = \operatorname*{ess\,max}_{\sigma<|t|\leqslant\sigma_1} t^{2k}\left(\frac{|1 - a(t)|^2}{\lambda_2 t^{2r}\cosh 2\beta t} + \frac{|a(t)|^2}{\lambda_1}\right).$$

Then integrating (3.18) we arrive at the following bound for I_2:

$$I_2 \leqslant \frac{S_a}{2\pi}\int_{\sigma<|t|\leqslant\sigma_1} (\lambda_2 t^{2r}|Ff(t)|^2\cosh 2\beta t + \lambda_1|Ff(t) - y(t)|^2)\,dt. \tag{3.19}$$

Now we show that

$$I_3 \leqslant \frac{\lambda_2}{2\pi}\int_{|t|>\sigma_1} t^{2r}|Ff(t)|^2\cosh 2\beta t\,dt \tag{3.20}$$

in all domains Σ_j, $j=1,2,3,4$. We have

$$I_3=\frac{1}{2\pi}\int_{|t|>\sigma_1}t^{2(k-r)}t^{2r}|Ff(t)|^2\,dt\leqslant\frac{\sigma_1^{2(k-r)}}{2\pi\cosh 2\beta\sigma_1}\int_{|t|>\sigma_1}t^{2r}|Ff(t)|^2\cosh 2\beta t\,dt.$$

Since in Σ_1 and Σ_2 we have

$$\lambda_2=\frac{\sigma_1^{2(k-r)}}{\cosh 2\beta\sigma_1},$$

inequality (3.20) holds in these domains. If $(\sigma_1,\sigma)\in\Sigma_3$, then $\sigma_1\geqslant\widehat{\sigma}_1$. Therefore,

$$\lambda_2=\frac{1}{\widehat{\sigma}_1^{2(r-k)}\cosh 2\beta\widehat{\sigma}_1}\geqslant\frac{\sigma_1^{2(k-r)}}{\cosh 2\beta\sigma_1}.$$

Let $(\sigma_1,\sigma)\in\Sigma_4$. Then λ_2 is the slope of the tangent to s at $(x(\xi),y(\xi))$, and $\sigma_1^{2(k-r)}\cosh^{-1}2\beta\sigma_1$ is the slope of the tangent to s at $(x(t_0),y(t_0))$ (we defined t_0 when we considered the lower bound in the case $(\sigma_1,\sigma)\in\Sigma_2$). Since $\xi\leqslant t_0$ and s is a concave function, it follows that

$$\lambda_2\geqslant\frac{\sigma_1^{2(k-r)}}{\cosh 2\beta\sigma_1}.$$

Thus, (3.20) holds in all domains.

Assuming that a is a function such that $S_a\leqslant 1$, adding (3.17), (3.19) and (3.20) we obtain the following estimate for the functional in (3.16):

$$\begin{aligned}&\frac{\lambda_1}{2\pi}\int_{-\sigma}^{\sigma}|Ff(t)-y(t)|^2\,dt\\&\qquad+\frac{1}{2\pi}\int_{\sigma<|t|\leqslant\sigma_1}(\lambda_2t^{2r}|Ff(t)|^2\cosh 2\beta t+\lambda_1|Ff(t)-y(t)|^2)\,dt\\&\qquad+\frac{\lambda_2}{2\pi}\int_{|t|>\sigma_1}t^{2r}|Ff(t)|^2\cosh 2\beta t\,dt\\&=\frac{\lambda_1}{2\pi}\int_{-\sigma_1}^{\sigma_1}|Ff(t)-y(t)|^2\,dt+\frac{\lambda_2}{2\pi}\int_{|t|>\sigma}t^{2r}|Ff(t)|^2\cosh 2\beta t\,dt\\&\leqslant\lambda_1\frac{\delta^2}{2\pi}+\lambda_2.\end{aligned}$$

Hence

$$e(D^k,H_2^{r,\beta}+\mathcal{B}_{\sigma,2}(\mathbb{R}),I_{\sigma_1},\delta,m_a)\leqslant\sqrt{\lambda_1\frac{\delta^2}{2\pi}+\lambda_2}.$$

Taking (3.12) into account we obtain

$$E(D^k,H_2^{r,\beta}+\mathcal{B}_{\sigma,2}(\mathbb{R}),I_{\sigma_1},\delta)=\sqrt{\lambda_1\frac{\delta^2}{2\pi}+\lambda_2},$$

and the methods m_a are optimal.

We show that there exist functions a such that $S_a \leqslant 1$. Note (by extracting a 'full square') that the condition $S_a \leqslant 1$ is equivalent to the following one: for almost all $\sigma < |t| \leqslant \sigma_1$ we have

$$\left|a(t) - \frac{\lambda_1}{\lambda_1 + \lambda_2 t^{2r}\cosh 2\beta t}\right|^2 \leqslant \frac{\lambda_1\lambda_2 t^{2(r-k)}\cosh 2\beta t(-t^{2k} + \lambda_1 + \lambda_2 t^{2r}\cosh 2\beta t)}{(\lambda_1 + \lambda_2 t^{2r}\cosh 2\beta t)^2}.$$

If

$$-t^{2k} + \lambda_1 + \lambda_2 t^{2r}\cosh 2\beta t \geqslant 0 \tag{3.21}$$

for $\sigma < |t| \leqslant \sigma_1$, then it is obvious that such functions a exist and can be described by equality (3.5).

If $(\sigma_1, \sigma) \in \Sigma_1$, then the straight line $y = \lambda_1 + \lambda_2 x$ is parallel to the tangent to s at the point $(x(t_0), y(t_0))$, where t_0 is defined by the equality

$$\frac{k t_0^{2(k-r)}}{r\cosh 2\beta t_0 + t_0\beta\sinh 2\beta t_0} = \frac{1}{\sigma_1^{2(r-k)}\cosh 2\beta\sigma_1}$$

(see (3.1)), and since $\sigma \geqslant h(\sigma_1)$, it does not lie below the tangent. Hence, as s is concave, for all $x \geqslant 0$ we have the inequality $\lambda_1 + \lambda_2 x \geqslant s(x)$. This yields condition (3.21). In the other three cases the lines $y = \lambda_1 + \lambda_2 x$ are tangent to s, and condition (3.21) holds for the same reasons.

(3) Let $k = 0$, $\sigma_1 > 0$, $\sigma \geqslant 0$ and $\sigma \leqslant \sigma_1$. As shown above, the functions f_n defined by (3.13) are admissible in problem (3.11). Hence

$$\begin{aligned}
&E^2(D^0, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta)\\
&\quad\geqslant \frac{1}{2\pi}\int_{\mathbb{R}} |Ff_n(t)|^2\,dt\\
&\quad= \frac{1}{2\pi}\int_{\sigma-1/n}^{\sigma} \delta^2 n\,dt + \frac{n}{(\sigma_1 + 1/n)^{2r}\cosh(2\beta(\sigma_1 + 1/n))}\int_{\sigma_1}^{\sigma_1+1/n} dt\\
&\quad= \frac{\delta^2}{2\pi} + \frac{1}{(\sigma_1 + 1/n)^{2r}\cosh 2\beta(\sigma_1 + 1/n)}.
\end{aligned}$$

Taking the limit as $n \to \infty$ we obtain

$$E^2(D^0, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta) \geqslant \frac{\delta^2}{2\pi} + \widetilde{\lambda}, \qquad \widetilde{\lambda} = \frac{1}{\sigma_1^{2r}\cosh 2\beta\sigma_1}. \tag{3.22}$$

We look for optimal recovery methods $m_a\colon L_2(\Delta_{\sigma_1}) \to L_2(\mathbb{R})$ among the maps with representation (3.14) for $k = 0$ in terms of Fourier transforms. Following the above scheme we assume that $a \equiv 1$ on Δ_σ and estimate the functional maximized in (3.16) (for $k = 0$) by representing it as a sum of three terms:

$$\begin{aligned}
I_1 &= \frac{1}{2\pi}\int_{-\sigma}^{\sigma} |Ff(t) - y(t)|^2\,dt,\\
I_2 &= \frac{1}{2\pi}\int_{\sigma<|t|\leqslant\sigma_1} |Ff(t) - a(t)y(t)|^2\,dt,\\
I_3 &= \frac{1}{2\pi}\int_{|t|>\sigma_1} |Ff(t)|^2\,dt.
\end{aligned}$$

We estimate I_2. Using the Cauchy–Schwarz–Bunyakovsky inequality we obtain

$$\begin{aligned} &|Ff(t) - a(t)y(t)|^2 \\ &\quad = |(1-a(t))Ff(t) + a(t)(Ff(t) - y(t))|^2 \\ &\quad \leqslant \left(\frac{|1-a(t)|^2}{\widetilde{\lambda} t^{2r}\cosh 2\beta t} + |a(t)|^2\right)\bigl(\widetilde{\lambda} t^{2r}|Ff(t)|^2\cosh 2\beta t + |Ff(t)-y(t)|^2\bigr). \quad (3.23) \end{aligned}$$

Set

$$\widetilde{S}_a = \operatorname*{ess\,max}_{\sigma<|t|\leqslant\sigma_1}\left(\frac{|1-a(t)|^2}{\widetilde{\lambda} t^{2r}\cosh 2\beta t} + |a(t)|^2\right).$$

Then integrating (3.23) we arrive at the following estimate for I_2:

$$I_2 \leqslant \frac{\widetilde{S}_a}{2\pi}\int_{\sigma<|t|\leqslant\sigma_1}(\widetilde{\lambda} t^{2r}|Ff(t)|^2\cosh 2\beta t + |Ff(t)-y(t)|^2)\,dt.$$

For I_3 we have

$$I_3 \leqslant \frac{\widetilde{\lambda}}{2\pi}\int_{|t|>\sigma_1} t^{2r}|Ff(t)|^2\cosh 2\beta t\,dt.$$

Assume that for the function a we have $\widetilde{S}_a \leqslant 1$. Then taking the estimates for I_2 and I_3 into account we obtain the following estimate for the functional in (3.16) (for $k=0$):

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\sigma}^{\sigma}|Ff(t)-y(t)|^2\,dt + \frac{1}{2\pi}\int_{\sigma<|t|\leqslant\sigma_1}(\widetilde{\lambda} t^{2r}|Ff(t)|^2\cosh 2\beta t + |Ff(t)-y(t)|^2)\,dt \\ &\qquad + \frac{\widetilde{\lambda}}{2\pi}\int_{|t|>\sigma_1} t^{2r}|Ff(t)|^2\cosh 2\beta t\,dt \\ &\quad = \frac{1}{2\pi}\int_{-\sigma_1}^{\sigma_1}|Ff(t)-y(t)|^2\,dt + \frac{\widetilde{\lambda}}{2\pi}\int_{|t|>\sigma} t^{2r}|Ff(t)|^2\cosh 2\beta t\,dt \\ &\quad \leqslant \frac{\delta^2}{2\pi} + \widetilde{\lambda}. \end{aligned}$$

Hence

$$e(D^0, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta, m_a) \leqslant \sqrt{\frac{\delta^2}{2\pi} + \widetilde{\lambda}}.$$

Taking (3.22) into account we obtain

$$E(D^0, H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R}), I_{\sigma_1}, \delta) = \sqrt{\frac{\delta^2}{2\pi} + \widetilde{\lambda}},$$

and the methods m_a are optimal.

The condition $\widetilde{S}_a \leqslant 1$ is equivalent to the following one: for almost all $\sigma < |t| \leqslant \sigma_1$ we have the inequality

$$\left|a(t) - \frac{1}{1+\widetilde{\lambda} t^{2r}\cosh 2\beta t}\right| \leqslant \frac{\widetilde{\lambda} t^{2r}\cosh 2\beta t}{1+\widetilde{\lambda} t^{2r}\cosh 2\beta t}.$$

It is obvious that such a exist and are described by (3.6).

The proof is complete.

§ 4. Discussion of optimal methods

When we recover $f^{(k)}$ on the class $H_2^{r,\beta} + \mathcal{B}_{\sigma,2}(\mathbb{R})$ from an inaccurately given Fourier transform of the function f on the interval $[-\sigma_1, \sigma_1]$, the following two questions arise:

- can we reduce the interval $[-\sigma_1, \sigma_1]$ on which the *a priori* information about f is set without increasing the optimal recovery error?
- can we also extend the subspace $\mathcal{B}_{\sigma,2}(\mathbb{R})$ on which the optimal method is exact without increasing the optimal recovery error?

In other words, the question is whether part of the information on the function f that we obtain is excessive and among the family of optimal methods we can find one that is exact on a wider subspace and does not increase the optimal recovery error. We look at the case $k \geqslant 1$. The answers to the above questions depend on the domain Σ_j, $j = 1, 2, 3, 4$, in which the point (σ_1, σ) occurs.

When $(\sigma_1, \sigma) \in \Sigma_1$, it is clear from (3.4) that, as σ_1 decreases or σ increases the optimal recovery error grows. Thus, the answers to the questions are in the negative in this case.

If $(\sigma_1, \sigma) \in \Sigma_2$, then the optimal recovery error does not change for the point $(\sigma_1, h(\sigma_1)$. This means that we can extend the original subspace $\mathcal{B}_{\sigma,2}(\mathbb{R})$ to $\mathcal{B}_{h(\sigma_1),2}(\mathbb{R})$ without increasing the optimal recovery error.

For $(\sigma_1, \sigma) \in \Sigma_4$ we can reduce the interval on which the information about f is set to the interval $[-\sigma_1', \sigma_1']$, where σ_1' is such that $h(\sigma_1') = \sigma$.

Finally, if $(\sigma_1, \sigma) \in \Sigma_3$, then we can both reduce the interval on which the information on f is prescribed to $[-\widehat{\sigma}_1, \widehat{\sigma}_1]$ and extend the subspace to $\mathcal{B}_{\widehat{\sigma},2}(\mathbb{R})$.

We show these transitions schematically in Figure 4.

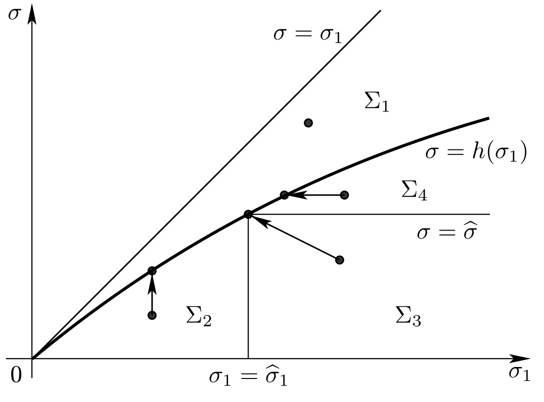

Figure 4

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Konstantin Yu. Osipenko
Faculty of Mechanics and Mathematics,
Lomonosov Moscow State University,
Moscow, Russia;
Institute for Information Transmission Problems
of the Russian Academy of Sciences
(Kharkevich Institute),
Moscow, Russia
E-mail: kosipenko@yahoo.com

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