SCHWARZ LEMMA AND OPTIMAL RECOVERY OF FUNCTIONS IN $H^2$

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Let $D \subset C^k$ be a domain, $\nu$ be a probability measure on $D$ and $X$ be a closed subspace of $L^2(\nu)$. Consider $D_0, \ldots, D_n \subset D$ and probability measures $\mu_0, \ldots, \mu_n$ on $D_0, \ldots, D_n$ respectively. We suppose that $X \subset L^2(\mu_j), j = 0, 1, \ldots, n$. We allow one of $D_j$ to coincide with $D$. In this case we assume that $\mu_j$ coincides with $\nu$.

Write $D = (D_0, \ldots, D_n), \mu = (\mu_0, \ldots, \mu_n), \mu = (\mu_1, \ldots, \mu_n), y = (y_1, \ldots, y_n)$.

1. Optimal recovery problem

Given $y_1, \ldots, y_n$ defined on $D_1, \ldots, D_n$ such that

$$\|f_j - y_j\|_{L^2(\mu_j)} \leq \delta_j, \quad j = 1, \ldots, n,$$

we are to reconstruct $f$. Here $f_j$ is the restriction of $f$ to $D_j$ and $\delta_j \geq 0, \quad j = 1, \ldots, n$ are accuracy levels. In particular, $\delta_j = 0$ means that $f$ is known precisely on $D_j$.

A recovery algorithm (method, procedure, etc.) is an operator

$$A: L^2(\mu_1) \times \cdots \times L^2(\mu_n) \rightarrow L^2(\mu_0).$$

We consider $A(y), y = (y_1, \ldots, y_n)$, to be the recovered value of $f$ on $D_0$. At this point we impose no conditions on $A$.

The maximal possible error of a method $A$ is

$$e(X, D, \mu, \delta, A) = \sup_{f \in X, y \in L^2(\mu_1) \times \cdots \times L^2(\mu_n)} \|f_0 - A(y)\|_{L^2(\mu_0)} \quad \|f_j - y_j\|_{L^2(\mu_j)} \leq \delta_j, j = 1, \ldots, n.$$  

The optimal recovery error is

$$E(X, D, \mu, \delta) = \inf_{A: L^2(\mu_1) \times \cdots \times L^2(\mu_n) \rightarrow L^2(\mu_0)} e(X, D, \mu, \delta, A).$$
A method \( \hat{A} \) such that
\[
E(X, D, \mu, \delta) = e(X, D, \mu, \delta, \hat{A})
\]
is called an \textit{optimal recovery method}.

The problem of finding an optimal recovery method (and sometimes an extremal function at which the optimal recovery error is attained) is usually referred to as \textit{optimal recovery problem}.

2. **Extremal problem**

The optimal recovery problem is closely related to the following extremal problem. Find

(1) \[
\|f_0\|_{L^2(\mu_0)} \rightarrow \max, \quad \|f_j\|^2_{L^2(\mu_j)} \leq \delta_j^2, \quad j = 1, \ldots, n, \quad f \in X.
\]

A special case of this extremal problem is when \( D \) is the unit disk \( \mathbb{D} \), \( \mu_0 \) and \( \mu_1 \) are point masses and \( \mu_2 \) is the normalized Lebesgue measure on the unit circle. Here the problem turns into
\[
\max\{\|f(a_0)\| : \|f(a_1)\| \leq \delta_1, \|f\|_{H^2} \leq \delta_2\},
\]
which might be viewed as a version of the classical Schwarz lemma. Here we consider another variant of Schwarz Lemma. Let \( a \in \mathbb{D} \) and \( \Gamma \) be a circle inside of the unit disk, \( \mu \) be the normalized Lebesgue measure on \( \Gamma \), and \( \mu > 0 \). Find

(2) \[
\sup \left\{ \int_{\Gamma} |f|^2 d\mu : f \in H^2, \|f\|_{H^2}^2 \leq 1, \|f(a)\| \leq \delta \right\}.
\]

We will consider the case when the circle \( \Gamma \) passes through the origin and its center lies on the real axis, so that
\[
\Gamma = \{z \in \mathbb{C} : |z - \rho| = \rho\}, \quad 0 < \rho < 1/2.
\]

The corresponding optimal recovery problem is: \textit{Reconstruct a Hardy function \( f \) on the circle \( \Gamma \) from its value at a given with some tolerance}.

There are several papers where similar problems were considered for Hardy and Bergman spaces in connection with optimal recovery in both one and several dimensional cases (see, for example, [4]–[6]).

3. **Euler equation for the general problem**

Let \( K(z, w) \) be the reproducing kernel of \( X \). Write
\[
\tilde{\mu} = -\mu_0 + \sum_{j=1}^{n} \lambda_j \mu_j.
\]
Then $\hat{\mu}$ is a regular measure on $D$ and every function from $X$ is square-integrable with respect to $\hat{\mu}$. For $w \in D$ we introduce
\[ d\hat{\mu}_w(z) = K(z, w)d\hat{\mu}(z). \]
Obviously every function from $X$ is $\hat{\mu}_w$-integrable.

We further define
\[ \tau^\lambda_w(z) = \int_D K(z\tau)d\hat{\mu}_w(\tau). \]

**Theorem 1.** If $\hat{f} \in X$ is a solution of the general extremal problem above, then there exists a non-negative vector $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_n)$ such that
\[ \hat{f} = (\text{span}\{\tau^\lambda_w, w \in D\})^\perp, \]
and
\[ \hat{\lambda}_j(\|f\|_{L^2(\mu_j)} - \delta_j) = 0, \quad j = 1, \ldots, n. \]

We say that a non-negative vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ belongs to the spectrum of the problem, if there exists an admissible for this problem function $f \in X$ such that
1. $\lambda_j(\|f\|_{L^2(\mu_j)} - \mu_j) = 0$.
2. $f \in (\text{span}\{\tau^\lambda_w : w \in D\})^\perp$.

In this case we call $f$ a spectral function.

**Theorem 2.** Let $\Lambda$ be the spectrum of the problem. Then
\[ (3) \quad \sup_{\|f\|_{L^2(\mu_j)} \leq \delta_j, \ j = 1, \ldots, n} \frac{\lambda_j \delta_j^2}{\|f\|_{L^2(\mu_j)}} = \sup_{\lambda \in \Lambda} \sum_{j=1}^n \lambda_j \delta_j^2. \]

We call a spectral point $(\hat{\lambda}_1, \ldots, \hat{\lambda}_n)$ extremal, if the maximum of the right-hand side of (3) is attained at $(\hat{\lambda}_1, \ldots, \hat{\lambda}_n)$.

4. **Spectrum of the Schwarz Lemma**

Here we have.
\[ \tau^\lambda_w = -\frac{1}{\pi} \int_\Gamma \frac{1}{1 - z\rho} \cdot \frac{1}{1 - \tau w} \cdot \frac{|d\tau|}{|\tau - \rho|} + \lambda_1 \frac{1}{1 - z\rho} \cdot \frac{1}{1 - \tau w} + \lambda_2 \frac{1}{2\pi} \int_{|\tau| = 1} \frac{1}{1 - z\rho} \cdot \frac{1}{1 - \tau w} |d\tau| = \]
\[ -\frac{1}{1 - z\rho - \rho w} + \lambda_1 \frac{1}{1 - z\rho} \cdot \frac{1}{1 - \tau w} + \lambda_2 \frac{1}{1 - z\rho - \rho w}. \]
By Theorem 1 every extremal function satisfies the following equation
\[
\frac{1}{1 - \rho w} f \left( \frac{\rho}{1 - \rho w} \right) = \lambda_1 \frac{f(a)}{1 - \overline{a}w}
\]
for some \( \lambda_1, \lambda_2 \geq 0 \) and all \( w \in \mathbb{D} \). Let
\[
b = \frac{1 - \sqrt{1 - 4\rho^2}}{2\rho}.
\]
Then \( b \) is the Denjoy-Wolff point of the following self-mapping of \( \mathbb{D} \)
\[
z \rightarrow \frac{\rho}{1 - \rho z},
\]
and the disk bounded by the circle \( \Gamma \) is a hyperbolic neighborhood of \( b \).

Consider the following functions
\[
\varphi_j(z) = \frac{\sqrt{1 - b^2}}{1 - bz} \left( \frac{b - z}{1 - bz} \right)^j, \quad j = 0, 1, \ldots.
\]
These functions form an orthonormal basis of \( H^2 \), and they are eigenfunctions of the operator
\[
T f(z) = \frac{1}{1 - \rho z} f \left( \frac{\rho}{1 - \rho z} \right),
\]
and the corresponding eigenvalues are
\[
\alpha_j = \frac{b^{2j}}{1 - \rho b}.
\]

**Theorem 3.** Let \( a \neq b \).

1. If
\[
\left| a - \frac{\rho}{1 - \rho^2} \right| \geq \frac{\rho^2}{1 - \rho^2},
\]
or
\[
\delta > \frac{\sqrt{|a|^2 \rho^2 - |\rho - a|^2}}{a \rho + \overline{a} \rho - |a|^2},
\]
then the spectrum of Schwarz Lemma extremal problem consists of two parts \( \Lambda = \Lambda_1 \cup \Lambda_2 \), where
\[
\Lambda_1 = \{(0, \alpha_j) : |\varphi_j(a)| \leq \delta\},
\]
\[
\Lambda_2 = \{(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 > 0, \ F(\lambda_2) = \delta^{-2}, \ \lambda_1 = h(\lambda_2)\},
\]
where
\[
F(\lambda) = \sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{(a_j - \lambda)^2} h^2(\lambda), \quad h(\lambda) = \left( \sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{a_j - \lambda} \right)^{-1}.
\]
2. If
\[ |a - \frac{\rho}{1 - \rho^2}| < \frac{\rho^2}{1 - \rho^2}, \]
and
\[ \delta \leq \sqrt{\frac{|a|^2 \rho^2 - |\rho - a|^2}{a \rho + \overline{a} \rho - |a|^2}}, \]
then the spectrum includes in addition the point
\[ \Lambda_3 = \left\{ \left( \frac{a \rho + \overline{a} \rho - |a|^2}{\rho^2}, 0 \right) \right\}. \]

Theorem 4. Let \( a = b \),
\[ \Lambda_1 = \{ (0, \alpha_j) : j = 1, 2, \ldots, \}, \]
\[ \Lambda_2 = \{ ((1 - b^2)(\alpha_0 - \alpha_j), \alpha_j) : j = 1, 2, \ldots, \}. \]
Then the spectrum of problem is \( \Lambda = \Lambda_1 \cup \Lambda_2 \), if \( \delta < \frac{1}{\sqrt{1 - b^2}} \), and \( \Lambda = \Lambda_1 \cup \Lambda_2 \cup \{ (0, \alpha_0) \} \), if \( \delta \geq \frac{1}{\sqrt{1 - b^2}} \).

It turns out that \( \Lambda_2 \) is the most important part of the spectrum.

Proposition 1. If \( a \) lies outside \( \Gamma \), then \( F(\lambda) \to \infty \) as \( \lambda \to 0 \).

This Proposition implies that if \( a \) lies outside \( \Gamma \), then \( \Lambda_2 \) contains only finite number of points.

Now we will use Theorem 2 to describe the extremal points of the spectrum.

Proposition 2. If \( \delta \geq |\varphi_0(a)| \), then \( (0, \alpha_0) \) is the extremal point of the spectrum.

Proposition 3. If \( a = b \) and \( \delta < 1/\sqrt{1 - b^2} \), then the extremal spectral point is
\[ (\hat{\lambda}_1, \hat{\lambda}_2) = ( (1 - b^2)(\alpha_0 - \alpha_1), \alpha_1 ). \]

Proposition 4. If \( \delta < |\varphi_0(a)| \), then \( \Lambda_1 \) does not contain extremal spectral points.

Note that the function
\[ g(\lambda) = \sum_{j=0}^{\infty} \frac{|\varphi_j(a)|^2}{\alpha_j - \lambda} \]
is monotone and increases from \( -\infty \) to \( +\infty \) when \( \lambda \in (\alpha_{j+1}, \alpha_j) \). Let \( \zeta_j \) be the only zero of \( g \) on the interval \( (\alpha_{j+1}, \alpha_j) \).
Proposition 5. Let \( a \neq b \). If \( \delta \leq |\varphi_1(a)| \), then the extremal spectral point \((\hat{\lambda}_1, \hat{\lambda}_2)\) is unique, belongs to \( \Lambda_2 \) and is determined by the condition \( \zeta_0 < \hat{\lambda}_2 < \alpha_0 \).

Proposition 6. Assume that \( |\varphi_1(a)| < \delta < |\varphi_0(a)| \) and

\[
\gamma = \left| \frac{b - a}{1 - ab} \right| \geq b^{2/3},
\]

then the conclusion of Proposition 5 is valid, that is, the extremal spectral point \((\hat{\lambda}_1, \hat{\lambda}_2)\) is unique, belongs to \( \Lambda_2 \) and is determined by the condition \( \zeta_0 < \hat{\lambda}_2 < \alpha_0 \).

5. Optimal Recovery Method

To construct optimal recovery methods we need the following result (several results of this type may be found in [2], [1], [3]).

Theorem 5. Assume that there exist \( \hat{\lambda}_j \geq 0, j = 1, \ldots, n \), such that the value of the extremal problem

\[
\| f_0 \|_{L_2(\mu_0)}^2 \rightarrow \max, \quad \sum_{j=1}^{\infty} \hat{\lambda}_j \| f_j \|_{L_2(0, \mu_j)}^2 \leq \sum_{j=1}^{\infty} \hat{\lambda}_j \delta_j^2, \quad f \in X,
\]

is the same as in (1). Moreover, assume that for every \( \tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n) \in Y_1 \times \cdots \times Y_n \), where \( Y_j \) are dense in \( L^2(\mu_j) \), there exists \( f_{\tilde{y}} \) which is a solution of the extremal problem

\[
\sum_{j=1}^{\infty} \hat{\lambda}_j \| f_j - \tilde{y}_j \|_{L_2(0, \mu_j)}^2 \rightarrow \min, \quad f \in X.
\]

Moreover, let \( \hat{A} : L^2(\mu_1) \times \cdots \times L^2(\mu_n) \rightarrow L^2(\mu_0) \) be a linear continuous operator, where the norm in \( L^2(\mu_1) \times \cdots \times L^2(\mu_n) \) is defined as

\[
\| y \| = \left( \sum_{j=1}^{n} \| y_j \|_{L_2(\mu_j)}^2 \right)^{1/2},
\]

such that for all \( \tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n) \in Y_1 \times \cdots \times Y_n \),

\[
\hat{A}(\tilde{y}) = (f_{\tilde{y}})_0.
\]

Then

\[
E(X, D, \mu, \delta) = \sup_{f \in X : \| f \|_{L_2(\mu_0)} \leq \delta} \| f_0 \|_{L_2(\mu_0)}
\]

and the method \( \hat{A}(y) \) is optimal.
We will apply Theorem 5 to the construction of optimal recovery method for the Schwarz Lemma type problem considered above.

Consider the extremal problem

\[(4) \int_{\Gamma} |f|^2 d\mu \to \max, \quad \hat{\lambda}_1 |f(a)|^2 + \hat{\lambda}_2 \|f\|^2_{H^2} \leq \hat{\lambda}_1 \delta^2 + \hat{\lambda}_2, \quad f \in H^2,\]

where as before \(\mu\) is the normalized Lebesgue measure on \(\Gamma\) and \((\hat{\lambda}_1, \hat{\lambda}_2)\) is an extremal spectral point for problem (2).

**Proposition 7.** Suppose that either

1. \(a \neq b\) and \(\delta \leq |\varphi_1(a)|\), or \(|\varphi_1(a)| < \delta < |\varphi_0(a)|\) and \(\gamma = \left|\frac{b - a}{1 - ab}\right| \geq b^{2/3}\),

or

2. \(a = b\) and \(\delta < \varphi(b) = 1/\sqrt{1 - b^2}\).

Then the values of extremal problems (2) and (4) are the same.

**Theorem 6.** Suppose that one of the following conditions is satisfied

1. \(\delta \geq |\varphi_0(a)|\),
2. \(\delta \leq |\varphi_1(a)|\),
3. \(|\varphi_1(a)| < \delta < |\varphi_0(a)|\), \(\gamma \geq b^{2/3}\),
4. \(a = b\),

and \((\hat{\lambda}_1, \hat{\lambda}_2)\) is the corresponding extremal spectral point. Then the error of optimal recovery is given by

\[\sqrt{\hat{\lambda}_1 \delta^2 + \hat{\lambda}_2}\]

and the method

\[(5) \quad \hat{A}(y)(z) = \frac{\hat{\lambda}_1 y}{\hat{\lambda}_1 + \hat{\lambda}_2 (1 - |a|^2)} \cdot \frac{1 - |a|^2}{1 - az}\]

is optimal.

Note that for \(a = b\) the optimal method of recovery (5) does not depend on \(\delta\) and has the form

\[\hat{A}(y)(z) = \frac{1 - |b|^2}{1 - bz}.\]

6. **Open problems**

1. It would be desirable to identify the extremal spectral point in all possible cases. We have shown that in a number of cases the extremal spectral point is the only point in \(\Lambda_2\) such that \(\zeta_0 < \hat{\lambda}_2 < \alpha_0\). Our attempts to find a nontrivial-case when this point is not extremal failed.
Thus, we are tempted to conjecture that the point of $\Lambda_2$ with the biggest $\lambda_2$ is always extremal.

**Conjecture.** If $a \neq b$ and $\delta < |\varphi_0(a)|$, the point in $\Lambda_2$ such that $\zeta_0 < \tilde{\lambda}_2 < \alpha_0$ is always the spectral extremal point for problem (2).

2. It is natural to ask which choice of $a$ minimizes the value of problem (2) (of course, this choice of $a$ leads to the least optimal recovery error). It follows from above discussion that the point $b$ plays a special role.

**Problem.** Does the choice $a = b$ always lead to the least mean square optimal recovery error?

3. Finally, if in problem (2) we replace the constraint $|\varphi(a)| \leq \delta$ with
\[
\frac{1}{2\pi r} \int_{|z-a|=r} |f(z)|^2 |d(z-a)| \leq \delta, \quad 0 < r < 1 - |a|,
\]
then the problem becomes even more difficult. The reason is that in the right hand side of Euler’s equation the term $\lambda_1 \frac{f(a)}{1 - \overline{a}z}$ is replaced with
\[
\lambda_1 f \left( a - \frac{r^2 z}{1 - \overline{a}z} \right)
\]
and the equation turns into
\[
\frac{1}{1 - \rho w} f \left( \frac{\rho}{1 - \rho w} \right) = \lambda_1 \frac{f(a)}{1 - \overline{a}z} + \lambda_2 f(w).
\]
Thus, finding the spectrum in this case is reduced to finding eigenvalues of an operator which is a linear combination of two compact non-commuting operators. It would be very interesting to find the eigenbasis which corresponds to this problem and to find the solution.

**References**


