2016 年 12 月 应用数学与计算数学学报 第 30 卷 第 4 期
Dec. 2016 Communication on Applied Mathematics and Computation Vol. 30 No. 4

DOI 10.3969/j.issn.1006-6330.2016.04.001

Review

Optimal recovery of linear functionals and operators

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(Communicated by NI Mingkang)

Abstract The paper is concerned with recovery problems of linear functionals and operators from precise and noisy information. We present several basic results from the theory of optimal recovery. Special attention is paid to the construction of optimal recovery methods. As an example of application, the problem of recovery of functions and their derivatives from inaccurate Fourier coefficients is considered.

Key words optimal recovery; linear functional; linear operator; Fourier coefficients

2010 Mathematics Subject Classification 41A46; 49N05; 93C05

Chinese Library Classification O177

线性算子和线性泛函的最优反演理论

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(倪明康推荐)

摘要 研究了几类在给出全部和部分信息时线性泛函和线性算子的反演问题. 介绍了最优反

Received 2016-07-20; **Revised** 2016-08-23

Project supported by the financial support of the Russian Foundation for Basic Research (14-01-00456, 14-01-00744)

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演理论的基本结果，特别是最优反演方法的构造．作为例子，讨论了在给出部分傅里叶系数时反演函数和函数的导数问题.

关键词 最优反演；线性泛函；线性算子；傅里叶系数
2010 数学分类号 41A46; 49N05; 93C05
中图分类号 O177 **文献标志码** A **文章编号** 1006-6330(2016)04-0459-23

0 Introduction

In this paper, we give a short history of optimal recovery problems and some general results. There are several surveys and monographs devoted to the theory of optimal recovery[1-6]. Here, we try to pay special attention to the construction of optimal recovery methods.

One of the first example of optimal recovery problems is the problem of the best quadrature. Let W be some class of functions integrable on the interval $[a, b]$. The problem is to find

$$Lf = \int_a^b f(x)\mathrm{d}x,$$

knowing the information about function values at the system of knots $a \leqslant x_1 < \cdots < x_n \leqslant b$. Thus, using the vector

$$If = (f(x_1), \cdots, f(x_n)),$$

we have to give an approximate value of Lf. Any linear method of approximation

$$\varphi(If) = \sum_{j=1}^{n} p_j f(x_j)$$

is called the quadrature formula.

The quadrature formula

$$\widehat{\varphi}(If) = \sum_{j=1}^{n} \widehat{p}_j f(x_j)$$

is called the best quadrature formula if

$$\sup_{f\in W} |Lf - \widehat{\varphi}(If)| = \inf_{p_1,\cdots,p_n\in\mathbb{R}} \sup_{f\in W} \Big|Lf - \sum_{j=1}^{n} p_j f(x_j)\Big|. \tag{1}$$

The first setting of such problems were given by Sard[7] and Nikol'skii[8]. The development of these problems may be found in [9].

Smolyak[10] considered the following generalization of (1). Let X be a linear space and L be a linear functional on X. Put

$$Ix = (l_1x, \cdots, l_nx), \quad x \in X,$$

where l_j, $j = 1, \cdots, n$, are linear functionals on X. For $W \subset X$, we consider the problem of the optimal recovery of L on W by the information operator I. Any method of recovery is a mapping $\varphi\colon \mathbb{R}^n \to \mathbb{R}$. For a given method φ, we define the error of this method by

$$e(L, W, I, \varphi) = \sup_{x \in W} |Lx - \varphi(Ix)|.$$

We want to find the optimal error of recovery

$$E(L, W, I) = \inf_{\varphi\colon \mathbb{R}^n \to \mathbb{R}} e(L, W, I, \varphi),$$

and an optimal method $\widehat{\varphi}$ for which

$$e(L, W, I, \widehat{\varphi}) = E(L, W, I).$$

Theorem 1[10] If W is a convex and centrally-symmetric set, then among all optimal methods, there exists a linear optimal method and

$$E(L, W, I) = \sup_{\substack{x \in W \\ Ix = 0}} |LX|. \tag{2}$$

Thus, if W is a convex and centrally-symmetric set, then there exist $\widehat{p}_1, \cdots, \widehat{p}_n$ such that the method

$$\widehat{\varphi}(Ix) = \sum_{j=1}^{n} \widehat{p}_j l_j x$$

is an optimal method of recovery.

Any element $x_0 \in W$ for which $Ix_0 = 0$ and

$$|Lx_0| = \sup_{\substack{x \in W \\ Ix = 0}} |Lx|,$$

we call extremal. The problem of finding an extremal element often turns out more simple than the problem of finding an optimal recovery method.

Let us consider a simple example. Let $\mathcal{H}_\infty^{\mathbb{R}}$ be the space of functions analytic in the unit disk

$$D := \{z \in \mathbb{C} : |z| < 1\},$$

bounded, and real in the interval $(-1, 1)$. As the set W, we consider $H_\infty^{\mathbb{R}}$ which is the set of functions from $\mathcal{H}_\infty^{\mathbb{R}}$ satisfying the condition

$$\sup_{z \in D} |f(z)| \leqslant 1.$$

For the problem of optimal recovery of functions from $H_\infty^{\mathbb{R}}$ at the point $\tau \in (-1,1)$ by their values at zero, the dual problem (2) may be solved immediately using the Schwarz lemma,

$$\sup_{\substack{f\in H_\infty^{\mathbb{R}}\\ f(0)=0}} |f(\tau)| = |\tau|.$$

Thus, the function $f_0(z) = z$ is extremal for the considered problem. However, the problem of finding an optimal method of recovery is not so evident.

1 Method of parametrization

In [11], we offer an approach allowing to obtain an optimal method of recovery using some parametrization of extremal element.

Theorem 2[11] Let X be a real linear space, W a convex centrally symmetric set from X, and x_0 an extremal element in the problem of optimal recovery of a linear functional L on the set W by the values of linear functionals $l_1x, \cdots, l_nx$. Assume that for all $M = (t_1, \cdots, t_n) \in \mathbb{R}^n$ from some neighborhood of $M_0 \in \mathbb{R}^n$, there exists $x(M) \in W$ such that $x(M_0) = x_0$. Then, if the functions $\omega(M) = Lx(M)$, $\omega_j(M) = l_jx(M)$, $j = 1, \cdots, n$, have continuous partial derivatives with respect to all variables in a neighborhood of M_0 and the determinant of the matrix

$$J(M) = \begin{pmatrix} \dfrac{\partial\omega_1}{\partial t_1} & \cdots & \dfrac{\partial\omega_n}{\partial t_1} \\ \vdots & & \vdots \\ \dfrac{\partial\omega_1}{\partial t_n} & \cdots & \dfrac{\partial\omega_n}{\partial t_n} \end{pmatrix}$$

does not vanish at M_0, then the method

$$\widehat{\varphi}(Ix) = \sum_{j=1}^{n} C_j l_j x, \tag{3}$$

where $C_1, \cdots, C_n$ are solutions of the system

$$J(M_0)\begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} \dfrac{\partial\omega}{\partial t_1}(M_0) \\ \vdots \\ \dfrac{\partial\omega}{\partial t_n}(M_0) \end{pmatrix},$$

is the unique linear optimal method of recovery.

It is sometimes convenient to use another form of the optimal method of recovery.

Corollary 1 Let the conditions of Theorem 2 are fulfilled. Then, the unique linear optimal method of recovery is

$$\widehat{\varphi}(Ix) = \sum_{j=1}^{n} y_j \frac{\partial \omega}{\partial t_j}(M_0),$$

where $y_1, \cdots, y_n$ are the solutions of the system

$$\sum_{j=1}^{n} y_j \frac{\partial \omega_k}{\partial t_j}(M_0) = l_k x, \quad k = 1, \cdots, n. \tag{4}$$

Proof For

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

put

$$(a, b) = \sum_{j=1}^{n} a_j b_j.$$

Set

$$\widehat{\omega} = \begin{pmatrix} \dfrac{\partial \omega}{\partial t_1}(M_0) \\ \vdots \\ \dfrac{\partial \omega}{\partial t_n}(M_0) \end{pmatrix}, \quad z = \begin{pmatrix} l_1 x \\ \vdots \\ l_n x \end{pmatrix}.$$

Then, the optimal method (3) has the form

$$\widehat{\varphi}(Ix) = (J^{-1}(M_0)\widehat{\omega}, z) = (\widehat{\omega}, (J^{\mathrm{T}}(M_0))^{-1} z).$$

Put $y = (J^{\mathrm{T}}(M_0))^{-1} z$. Then, the coordinates of $y = (y_1, \cdots, y_n)^{\mathrm{T}}$ satisfy the system (4).

Now, let us construct the optimal recovery method of functions from $H_\infty^{\mathbb{R}}$ at the point $\tau \in (-1, 1)$ by their values at zero. Put

$$f_1(z, t) = \frac{z + t}{1 + tz}.$$

It is easy to see that $f_1(z, t) \in H_\infty^{\mathbb{R}}$ for all $t \in (-1, 1)$. Moreover, $f_1(z, 0) = f_0(z) = z$ and $f_1(0, t) = t$. Thus, here $M = t \in \mathbb{R}$, $M_0 = 0$, $x(t) = f_1(z, t)$, $\omega(t) = f_1(\tau, t)$, $\omega_1(t) = f_1(0, t) = t$.

From Corollary 1, we obtain that the unique linear optimal method of recovery has the form $\widehat{\varphi}(f(0)) = y_1 \varphi'(0)$, where y_1 satisfies the equality $y_1 \omega_1'(0) = f(0)$. Consequently, it has the form

$$\widehat{\varphi}(f(0)) = \Big(\frac{\partial f_1}{\partial t}(0, 0)\Big)^{-1} \frac{\partial f_1}{\partial t}(\tau, 0) f(0) = (1 - \tau^2) f(0).$$

More general results concerning the considered problem may be found in [12] and [13] (they also can be obtained by the proposed method).

2 Optimal interpolation of smooth functions

Denote by $W^r_\infty[-1,1]$, $r \in \mathbb{N}$, the Sobolev class of functions $x(t)$, $t \in [-1,1]$, for which $x^{(r-1)}$ is absolutely continuous on $[-1,1]$ and

$$\operatorname*{ess\,sup}_{t\in[-1,1]} |x^{(r)}(t)| \leqslant 1.$$

Let

$$-1 \leqslant t_1 < \cdots < t_n \leqslant 1, \quad \nu_j \in \mathbb{N}, \quad 1 \leqslant \nu_j \leqslant r, \quad j = 1, \cdots, n, \quad m = \nu_1 + \cdots + \nu_n \geqslant r. \tag{5}$$

Assume that for any $x \in W^r_\infty([-1,1])$, we know

$$Fx = (x(t_1), \cdots, x^{(\nu_1-1)}(t_1), \cdots, x(t_n), \cdots, x^{(\nu_n-1)}(t_n)). \tag{6}$$

Consider the problem of optimal recovery of $x(\tau)$, $\tau \in [-1,1]$, $x \in W^r_\infty([-1,1])$, by the information Fx. In other words, we would like to interpolate a function $x \in W^r_\infty([-1,1])$ at the point τ using values of x and its derivatives at some system of points $t_1, \cdots, t_n$. In this case, we put

$$E(\tau, W^r_\infty([-1,1]), F) = \inf_{\varphi:\, \mathbb{R}^N \to \mathbb{R}} \sup_{x \in W^r_\infty([-1,1])} |x(\tau) - \varphi(Fx)|.$$

To obtain the solution of this optimal recovery problem, we recall some definitions and results about splines.

A perfect spline of degree $r \in \mathbb{N}$ with knots $s_1 < \cdots < s_N$ is a function of the form

$$s(t) = p_{r-1}(t) + \frac{\alpha}{r!}\Big(t^r + 2\sum_{j=1}^{N}(-1)^j (t - s_j)^r_+\Big),$$

where p_{r-1} is a polynomial of degree $r-1$, $\alpha = -1$ or $\alpha = 1$, and

$$t_+ = \begin{cases} t, & t \geqslant 0, \\ 0, & t < 0. \end{cases}$$

A polynomial spline of degree $r-1$, $r \in \mathbb{N}$, with N knots $s_1 < \cdots < s_N$ is a function of the form

$$S(t) = \sum_{j=0}^{r-1} a_j t^j + \sum_{j=1}^{N} b_j (t - s_j)^{r-1}_+.$$

Suppose that conditions (5) are fulfilled. Then, it is known (see, for example, [14]) that there exists a perfect spline s of degree r with $m - r$ knots

$$-1 < s_1 < \cdots < s_{m-r} < 1 \tag{7}$$

such that

$$s^{(\nu)}(t_j)=0,\quad \nu=0,1,\cdots,\nu_j-1,\quad j=1,\cdots,n.$$

Moreover, for any $x_{j\nu}$, $\nu=0,1,\cdots,\nu_j-1$, $j=1,\cdots,n$, there exists the unique polynomial spline S of order $r-1$ with knots (7) for which

$$S^{(\nu)}(t_j)=x_{j\nu},\quad \nu=0,1,\cdots,\nu_j-1,\quad j=1,\cdots,n.$$

Theorem 3[15-16] Assume that conditions (5) are fulfilled and $s_1<\cdots<s_{m-r}$ are the knots of a perfect spline s such that

$$s^{(\nu)}(t_j)=0,\quad \nu=0,1,\cdots,\nu_j-1,\quad j=1,\cdots,n.$$

Then, for any $\tau\in[-1,1]$,

$$E(\tau,W_\infty^r([-1,1]),F)=|s(\tau)|, \tag{8}$$

and the unique linear optimal recovery method is the polynomial spline S of order $r-1$ with knots $s_1,\cdots,s_{m-r}$ satisfying conditions

$$S^{(\nu)}(t_j)=x^{(\nu)}(t_j),\quad \nu=0,1,\cdots,\nu_j-1,\quad j=1,\cdots,n. \tag{9}$$

We give a simple proof of this theorem using the method of parametrization which was described in the previous section. Moreover, using this method, we can prove the uniqueness of the linear optimal method (which was not done in [15] and [16]).

Proof It follows from (2) that

$$E(\tau,W_\infty^r([-1,1]),F)=\sup_{\substack{x\in W_\infty^r([-1,1])\\ Fx=0}}|x(\tau)|.$$

Assume that there exists $x\in W_\infty^r([-1,1])$ such that $F\widehat{x}=0$ and $|\widehat{x}(\tau)|>|s(\tau)|$. Put

$$y=s-\rho\widehat{x},\quad \rho=\frac{s(\tau)}{\widehat{x}(\tau)}.$$

Then, y has $m+1$ zeros with multiplicities, and consequently, $y^{(r)}$ has at least $m-r+1$ sign changes. On the other hand, taking into account that $|\rho|<1$ on every interval $(-1,s_1),(s_1,s_2),\cdots,(s_{m-r},1)$, we can get that the function $y^{(r)}$ has the same sign as $s^{(r)}(\cdot)$. Thus, $y^{(r)}$ has exactly $m-r$ sign changes. The obtained contradiction proves (8).

Assume that the perfect spline s has the form

$$s(t)=\sum_{j=0}^{r-1}a_jt^j+\frac{\alpha}{r!}\Big(t^r+2\sum_{j=1}^{m-r}(-1)^j(t-s_j)_+^r\Big).$$

For points $M = (b_0, \cdots, b_{r-1}, u_1, \cdots, u_{m-r}) \in \mathbb{R}^m$ sufficiently close to the point $M_0 = (a_0, \cdots, a_{r-1}, s_1, \cdots, s_{m-r}) \in \mathbb{R}^m$ consider functions

$$s_M(t) = \sum_{j=0}^{r-1} b_j t^j + \frac{\alpha}{r!}\Big(t^r + 2\sum_{j=1}^{m-r}(-1)^j (t-u_j)_+^r\Big).$$

It is clear that $s_M \in W_\infty^r([-1,1])$ for all M from sufficiently small neighborhood of M_0. Moreover, $s_{M_0} = s$. We have

$$\frac{\partial s_M(t)}{\partial b_j}\Big|_{M_0} = t^j, \quad j = 0, \cdots, r-1,$$
$$\frac{\partial s_M(t)}{\partial u_j}\Big|_{M_0} = \frac{2\alpha(-1)^{j+1}}{(r-1)!}(t-s_j)_+^{r-1}, \quad j = 1, \cdots, m-r.$$

Putting

$$S(t) = \sum_{j=0}^{r-1} y_j t^j + \sum_{j=1}^{m-r} y_j (t-s_j)_+^{r-1},$$

we obtain that the system (4) has the same form as (9). Thus, by Corollary 1, the value of the interpolation spline S at the point τ is the unique linear optimal method of recovery.

3 Optimal recovery of linear functionals from inaccurate information

Let X be a linear space, L be a linear functional on X, $Ix = (l_1x, \cdots, l_nx)$, $x \in X$, where l_j, $j = 1, \cdots, n$, are linear functionals on X, and $W \subset X$. Now, assume that for all $x \in W$ instead of exact values of Ix, we know approximate values $y \in \mathbb{R}^n$ such that $\|Ix - y\| \leqslant \delta$, where $\|\cdot\|$ is any norm in $\mathbb{R}^n$, and $\delta \geqslant 0$ is the error of approximate values. In this case, the error of a recovery method φ is defined as follows:

$$e(L, W, I, \delta, \varphi) = \sup_{\substack{x\in W, y\in\mathbb{R}^n \\ \|Ix-y\|\leqslant\delta}} |Lx - \varphi(y)|.$$

Again, we are interested in the optimal error of recovery

$$E(L, W, I, \delta) = \inf_{\varphi\colon \mathbb{R}^n\to\mathbb{R}} e(L, W, I, \delta, \varphi)$$

and in an optimal method $\widehat{\varphi}$ for which

$$e(L, W, I, \delta, \widehat{\varphi}) = E(L, W, I, \delta).$$

It was proved in [17] an analog of Smolyak's result.

Theorem 4[17] If W is a convex and centrally-symmetric set, then among all optimal methods, there exists a linear optimal method, and

$$E(L,W,I,\delta)=\sup_{\substack{x\in W\\ \|Ix\|\leqslant\delta}}|Lx|.$$

We consider a more general problem of optimal recovery. Let X and Y be linear spaces, L be a linear functional on X, and $W\subset X$. Let $F\colon W\to Y$ be a multivalued mapping. It means that for any $x\in W$, $F(x)$ is a subset of Y. The problem is to recover Lx, $x\in W$ by the information $F(x)$. The multivalued mapping F is modeling inaccurate information. Usually, F has the form

$$F(x)=\{y\in Y:\|Ix-y\|\leqslant\delta\},\tag{10}$$

where $I:X\to Y$ is a linear operator, Y is a normed linear space, and $\delta\geqslant 0$. In this case, we speak about optimal recovery of L on W by inaccurate values of the operator I.

For any recovery method $\varphi:Y\to\mathbb{R}$, we define the error of the method φ by

$$e(L,F,\varphi)=\sup_{(x,y)\in\operatorname{gr}F}|Lx-\varphi(y)|,\tag{11}$$

where

$$\operatorname{gr}F=\{(x,y):x\in W,y\in F(x)\}.$$

The optimal error of recovery is defined as follows:

$$E(L,F)=\inf_{\varphi:Y\to\mathbb{R}}e(L,F,\varphi).\tag{12}$$

Let $A\subset X$. Denote by $\operatorname{bco}A$ the convex centrally-symmetric hull of A

$$\operatorname{bco}A=\Big\{x:x=\sum_{j=1}^n\lambda_jx_j,x_j\in A,\sum_{j=1}^n|\lambda_j|\leqslant 1,n\in\mathbb{N}\Big\}.$$

For any multivalued mapping $F\colon W\to Y$, we define the convex centrally-symmetric multivalued mapping $\operatorname{bco}F:\operatorname{bco}W\to Y$ by

$$\operatorname{bco}F(x)=\{y\in Y:(x,y)\in\operatorname{bco}\operatorname{gr}F\}.$$

Let $y\in F(W)$. The value

$$r(L,F,y)=\inf_{c\in\mathbb{R}}\sup_{x\in F^{-1}(y)}|Lx-c|$$

is called the Chebyshev radius of the set $L(F^{-1}(y))$. The value

$$R(L,F)=\sup_{y\in F(W)}r(L,F,y)$$

is called the radius of information in problem (12).

Theorem 5[18] For the existence of the linear optimal recovery method in (12), it is necessary and sufficient that

$$R(L,F)=R(L,\mathrm{bco}\ F).$$

Moreover, in this case,

$$E(L,F)=\sup_{x\in(\mathrm{bco}\ F)^{-1}(0)}|Lx|.$$

For F defined by (10), we put

$$e(L,F,\varphi)=e(L,W,I,\delta,\varphi),\quad E(L,F)=E(L,W,I,\delta).$$

If W is a convex and centrally-symmetric set, then bco $F=F$. Consequently, from Theorem 5, we immediately obtain that Theorem 4 holds in this general multi-dimensional case.

4 Optimal recovery methods for inaccurate information

Consider the problem (12) for F defined by (10).

Theorem 6 Let W be a convex and centrally-symmetric set and Y be a normed linear space. Assume that there exist such linear continuous functionals $\widehat{\varphi}$ and $\widehat{x}\in W$ that

(i) $\sup\limits_{x\in W}|Lx-\widehat{\varphi}(Ix)|=L\widehat{x}-\widehat{\varphi}(I\widehat{x})$,

(ii) $\widehat{\varphi}(I\widehat{x})=\delta\|\widehat{\varphi}\|$,

(iii) $\|I\widehat{x}\|\leqslant\delta$.

Then, $\widehat{\varphi}$ is an optimal method of recovery and

$$E(L,W,I,\delta)=L\widehat{x}.\tag{13}$$

Proof It follows from generalization of Theorem 4 that

$$E(L,W,I,\delta)=\sup_{\substack{x\in W\\ \|Ix\|\leqslant\delta}}|Lx|\geqslant|L\widehat{x}|=L\widehat{x}.$$

On the other hand, using the conditions (i)∼(iii), for all $x\in W$ and $y\in Y$ such that $\|Ix-y\|\leqslant\delta$, we have

$$\begin{aligned}|Lx-\widehat{\varphi}(y)|&=|Lx-\widehat{\varphi}(Ix)+\widehat{\varphi}(Ix-y)|\leqslant|Lx-\widehat{\varphi}(Ix)|+|\widehat{\varphi}(Ix-y)|\\&\leqslant L\widehat{x}-\widehat{\varphi}(I\widehat{x})+\|\widehat{\varphi}\|\delta=L\widehat{x}\leqslant E(L,W,I,\delta).\end{aligned}$$

Thus,

$$e(L,W,I,\delta,\widehat{\varphi})\leqslant L\widehat{x}\leqslant E(L,W,I,\delta)\leqslant e(L,W,I,\delta,\widehat{\varphi}).$$

Consequently, $\widehat{\varphi}$ is an optimal method of recovery and (13) holds.

We apply this result to optimal recovery of function values from their Fourier coefficients. Let $L_2(\mathbb{T})$ be the space of 2π periodic functions defined on the interval $\mathbb{T}=[-\pi,\pi]$ with identified endpoints with the norm

$$\|x\|=\Big(\frac{1}{\pi}\int_{\mathbb{T}}|x(t)|^2\mathrm{d}t\Big)^{1/2}.$$

Denote by $W_2^r(\mathbb{T})$ the Sobolev class of 2π periodic functions defined on $\mathbb{T}$ with absolutely continuous $x^{(r-1)}$ and $\|x^{(r)}\|\leqslant 1$. For any $x\in W_2^r(\mathbb{T})$ and all $t\in\mathbb{T}$, we have

$$x(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kt+b_k\sin kt).$$

We consider the problem of optimal recovery of $x(\tau)$, $\tau\in\mathbb{T}$, on the class $W_2^r(\mathbb{T})$ from the information about inaccurate values of Fourier coefficients a_k, $k\in A$, and b_k, $k\in B$, where A and B are some finite subsets of $\mathbb{Z}_+=\{0,1,\cdots\}$. More precisely, instead of a_k, $k\in A$, and b_k, $k\in B$, we know $\widetilde{a}_k,\widetilde{b}_k$, such that

$$|a_k-\widetilde{a}_k|\leqslant\delta,\quad k\in A,\quad |b_k-\widetilde{b}_k|\leqslant\delta,\quad k\in B.$$

Set $N=\operatorname{card}A+\operatorname{card}B$ and

$$F_{A,B}x=(\{a_k\}_{k\in A},\{b_k\}_{k\in B}).$$

Denote by l_∞^N the space of vectors $y=(y_1,\cdots,y_N)$ with the norm

$$\|y\|_{l_\infty^N}=\max_{1\leqslant k\leqslant N}|y_k|.$$

Thus, for every $x\in W_2^r(\mathbb{T})$, we know the vector

$$y=(\{\widetilde{a}_k\}_{k\in A},\{\widetilde{b}_k\}_{k\in B})$$

such that

$$\|F_{A,B}x-y\|_{l_\infty^N}\leqslant\delta.$$

In accordance with (11) and (12), we put

$$e(W_2^r(\mathbb{T}),F_{A,B},\delta,\varphi)=\sup_{\substack{x\in W_2^r(\mathbb{T}),y\in l_\infty^N\\ \|F_{A,B}x-y\|_{l_\infty^N}\leqslant\delta}}|x(\tau)-\varphi(y)|,$$
$$E(W_2^r(\mathbb{T}),F_{A,B},\delta)=\inf_{\varphi:l_\infty^N\to\mathbb{R}}e(W_2^r(\mathbb{T}),F_{A,B},\delta,\varphi).$$

We say that $\widehat{\varphi}$ is the optimal method of recovery if

$$E(W_2^r(\mathbb{T}),F_{A,B},\delta)=e(W_2^r(\mathbb{T}),F_{A,B},\delta,\widehat{\varphi}).$$

It is easy to show that if $0 \notin A$, then $E(W_2^r(\mathbb{T}), F_{A,B}, \delta) = \infty$ (for the proof, it suffices to consider only constant functions from $W_2^r(\mathbb{T})$). Therefore, in what follows, we assume that $0 \in A$. Set $\widetilde{A} = A \backslash \{0\}$ and consider the vector

$$\Big(\Big\{\frac{\cos k\tau}{k^{2r}}\Big\}_{k\in\widetilde{A}}, \Big\{\frac{\sin k\tau}{k^{2r}}\Big\}_{k\in B}\Big).$$

Let

$$|\gamma_2| \geqslant \cdots \geqslant |\gamma_N| \tag{14}$$

be the modules of the elements of this vector, sorted in a descending order. If $\gamma_s = k_s^{-2r} \cos k_s\tau$, then the corresponding index will be denoted by $k_s(A)$, and if $\gamma_s = k_s^{-2r} \sin k_s\tau$, then the corresponding index will be denoted by $k_s(B)$. For every $2 \leqslant s \leqslant N$, we denote by A_s and B_s the subsets of indexes $k_2(C), \cdots, k_s(C)$ for $C = A$ and $C = B$, respectively. For convenience, we put $A_1 = B_1 = \varnothing$. We also assume that the sum over the empty set equals 0.

Put

$$\begin{aligned} p = p(\delta) \ &= \max \Big\{ s : \gamma_s^2 \Big(1 - \delta^2 \sum_{k\in A_s \cup B_s} k^{2r}\Big) \\ &> \delta^2 \sum_{k\in\mathbb{N}\backslash A_s} \frac{\cos^2 k\tau}{k^{2r}} + \delta^2 \sum_{k\in\mathbb{N}\backslash B_s} \frac{\sin^2 k\tau}{k^{2r}}, 2 \leqslant s \leqslant N \Big\} \end{aligned}$$

(we assume that $p = 1$, if the set of such s is empty),

$$\lambda = \left(\frac{\displaystyle\sum_{k\in\mathbb{N}\backslash A_p} \frac{\cos^2 k\tau}{k^{2r}} + \sum_{k\in\mathbb{N}\backslash B_p} \frac{\sin^2 k\tau}{k^{2r}}}{\displaystyle 1 - \delta^2 \sum_{k\in A_p\cup B_p} k^{2r}} \right)^{1/2},$$

and

$$\lambda_s = k_s^{2r}(C)(|\gamma_s| - \lambda\delta), \quad \widetilde{c}_s = \begin{cases} \text{sign } \gamma_s \widetilde{a}_{k_s(C)}, & C = A, \\ \text{sign } \gamma_s \widetilde{b}_{k_s(C)}, & C = B. \end{cases}$$

Theorem 7 For all $\delta > 0$,

$$E(W_2^r(\mathbb{T}), F_{A,B}, \delta) = \frac{\delta}{2} + \delta \sum_{s=2}^{p} \lambda_s + \lambda,$$

and the method

$$\widehat{\varphi}(y) = \frac{\widetilde{a}_0}{2} + \sum_{s=2}^{p} \lambda_s \widetilde{c}_s \tag{15}$$

is the optimal method of recovery.

Proof We define the sequences $\widetilde{a}_k$ and $\widetilde{b}_k$ as follows:

$$\widehat{a}_k = \begin{cases} \delta \ \operatorname{sign} \cos k\tau, & k \in A_p \cup 0; \\ \dfrac{\cos k\tau}{\lambda k^{2r}}, & k \notin A_p \cup 0. \end{cases} \qquad \widehat{b}_k = \begin{cases} \delta \ \operatorname{sign} \sin k\tau, & k \in B_p; \\ \dfrac{\sin k\tau}{\lambda k^{2r}}, & k \notin B_p. \end{cases}$$

It is easy to check that the following equality:

$$\sum_{k=1}^{\infty} k^{2r} (\widehat{a}_k^2 + \widehat{b}_k^2) = 1 \tag{16}$$

holds.

Put

$$\widehat{x}(t) = \frac{\widehat{a}_0}{2} + \sum_{k=1}^{\infty} (\widehat{a}_k \cos kt + \widehat{b}_k \sin kt).$$

It follows from (16) that $\|\widehat{x}^{(r)}\| = 1$. Thus, $\widehat{x} \in W_2^r(\mathbb{T})$.

We will apply Theorem 6. It suffices to check conditions (i)∼(iii). We begin with the condition (iii). Let us show that $\|F_{A,B}\widehat{x}\|_{l_\infty^N} \leqslant \delta$. In other words, we should show that $|\widehat{a}_k| \leqslant \delta$ for all $k \in A$ and $|\widehat{b}_k| \leqslant \delta$ for all $k \in B$. If $p = N$, then it is obvious. Let $p < N$. If for some $k > 0$ and $k \in A \backslash A_p$, the inequality $|\widehat{a}_k| > \delta$ holds or for some $k \in B \backslash B_p$, the inequality $|\widehat{b}_k| > \delta$ holds, then there exists γ_s, $p < s \leqslant N$, for which

$$\gamma_s^2 > \delta^2 \lambda^2.$$

In view of (14), it implies that

$$\gamma_{p+1}^2 > \delta^2 \lambda^2. \tag{17}$$

Assume that

$$\gamma_{p+1}^2 = \frac{\cos^2 k_{p+1}(A)\tau}{k_{p+1}^{4r}(A)}.$$

Then, (17) may be written in the form

$$\frac{\cos^2 k_{p+1}(A)\tau}{k_{p+1}^{4r}(A)} \Big(1 - \delta^2 \sum_{k \in A_p \cup B_p} k^{2r}\Big)$$
$$> \ \delta^2 \sum_{k \in \mathbb{N} \backslash A_p} \frac{\cos^2 k\tau}{k^{2r}} + \delta^2 \sum_{k \in \mathbb{N} \backslash B_p} \frac{\sin^2 k\tau}{k^{2r}}.$$

Since $A_{p+1} = A_p \cup \{k_{p+1}\}$, and $B_{p+1} = B_p$, the last inequality may be rewritten in the form

$$\frac{\cos^2 k_{p+1}(A)\tau}{k_{p+1}^{4r}(A)} \Big(1 - \delta^2 \sum_{k \in A_{p+1} \cup B_{p+1}} k^{2r}\Big)$$
$$> \ \delta^2 \sum_{k \in \mathbb{N} \backslash A_{p+1}} \frac{\cos^2 k\tau}{k^{2r}} + \delta^2 \sum_{k \in \mathbb{N} \backslash B_{p+1}} \frac{\sin^2 k\tau}{k^{2r}}.$$

However, this contradicts the definition of p. The case when

$$\gamma_{p+1}^2 = \frac{\sin^2 k_{p+1}(B)\tau}{k_{p+1}^{4r}(B)},$$

may be considered analogously.

Let us prove that for all sequences $\{a_k\}$, $k = 0, 1, \cdots$, and $\{b_k\}$, $k = 1, 2, \cdots$, such that

$$\sum_{k=1}^{\infty} k^{2r}(a_k^2 + b_k^2) < \infty,$$

the equality

$$\sum_{k=1}^{\infty}(a_k \cos k\tau + b_k \sin k\tau) = \sum_{s=2}^{p} \lambda_s c_s + \lambda \sum_{k=1}^{\infty} k^{2r}(\widehat{a}_k a_k + \widehat{b}_k b_k) \tag{18}$$

holds, where

$$c_s = \begin{cases} \text{sign } \gamma_s a_{k_s(C)}, & C = A, \\ \text{sign } \gamma_s b_{k_s(C)}, & C = B. \end{cases}$$

Indeed, we have

$$\begin{aligned} \sum_{s=2}^{p} \lambda_s c_s + \lambda \sum_{k=1}^{\infty} k^{2r}(\widehat{a}_k a_k + \widehat{b}_k b_k) &= \sum_{s=2}^{p} k_s^{2r} |\gamma_{k_s}| c_s - \lambda\delta \sum_{s=2}^{p} k_s^{2r} c_s \\ &\quad + \lambda\delta \sum_{s=2}^{p} k_s^{2r} c_s + \lambda \sum_{k \in \mathbb{N} \setminus A_p} k^{2r} \frac{\cos k\tau}{\lambda k^{2r}} a_k + \lambda \sum_{k \in \mathbb{N} \setminus B_p} k^{2r} \frac{\sin k\tau}{\lambda k^{2r}} b_k \\ &= \sum_{k=1}^{\infty}(a_k \cos k\tau + b_k \sin k\tau). \end{aligned}$$

It follows from (18) that for any $x \in W_2^r(\mathbb{T})$,

$$x(\tau) - \widehat{\varphi}(F_{A,B} x) = \lambda \sum_{k=1}^{\infty} k^{2r}(\widehat{a}_k a_k + \widehat{b}_k b_k).$$

Using the Cauchy-Schwarz inequality, we obtain

$$|x(\tau) - \widehat{\varphi}(F_{A,B} x)| \leqslant \lambda \Big(\sum_{k=1}^{\infty} k^{2r}(\widehat{a}_k^2 + \widehat{b}_k^2) \Big)^{1/2} \Big(\sum_{k=1}^{\infty} k^{2r}(a_k^2 + b_k^2) \Big)^{1/2} \leqslant \lambda.$$

On the other hand,

$$|\widehat{x}(\tau) - \widehat{\varphi}(F_{A,B} \widehat{x})| = \lambda \sum_{k=1}^{\infty} k^{2r}(\widehat{a}_k^2 + \widehat{b}_k^2) = \lambda.$$

Consequently, the condition (i) holds.

It follows from the definition of p that $\lambda_p > 0$. In view of (14), we obtain that $\lambda_s > 0$ for all $s = 2, \cdots, p-1$. We have

$$\widehat{\varphi}(F_{A,B}\widehat{x}) = \delta\Big(\frac{1}{2} + \sum_{s=2}^{p} \lambda_s\Big) = \delta\|\widehat{\varphi}\|.$$

It means that the condition (ii) is fulfilled. Now, the assertion of the theorem follows from Theorem 6.

The case when the Fourier coefficients are known with different errors, that is,

$$|a_k - \widetilde{a}_k| \leqslant \delta_k, \quad k \in A, \quad |b_k - \widetilde{b}_k| \leqslant \delta_k, \quad k \in B$$

may be considered in a similar way (see [19]).

5 Optimal recovery of linear operators

Let $Y_0, Y_1, \cdots, Y_n$ be normed linear spaces and $I_j : X \to Y_j$, $j = 0, 1, \cdots, n$, be linear operators. We consider the problem of optimal recovery of the operator I_0 on the set

$$W = \{x \in X : \|I_j x\|_{Y_j} \leqslant \delta_j, \delta_j \geqslant 0, j = 1, \cdots, k\},$$

where $0 \leqslant k < n$, from inaccurate values of $I_{k+1}, \cdots, I_n$ (if $k = 0$, we set $W = X$). More precisely, we assume that for every $x \in W$, we know a vector $y = (y_{k+1}, \cdots, y_n) \in Y_{k+1} \times \cdots \times Y_n$ such that $\|I_j x - y_j\|_{Y_j} \leqslant \delta_j$, $\delta_j \geqslant 0$, $j = k+1, \cdots, n$.

By the analogy with the previous setting, we define the error of a recovery method $\varphi : Y_{k+1} \times \cdots \times Y_n \to Y_0$ as follows:

$$e(I, \delta, \varphi) = \sup_{\substack{x \in W, y \in Y_{k+1} \times \cdots \times Y_n \\ \|I_j x - y_j\|_{Y_j} \leqslant \delta_j, j = k+1, \cdots, n}} \|I_0 x - \varphi(y)\|_{Y_0}.$$

The value

$$E(I, \delta) = \inf_{\varphi : Y_{k+1} \times \cdots \times Y_n \to Y_0} e(I, \delta, \varphi) \tag{19}$$

is called the optimal error of recovery (here $I = (I_0, I_1 \cdots, I_n)$, $\delta = (\delta_1, \cdots, \delta_n)$). Methods for which the lower bound in (19) is attained, we call optimal methods of recovery.

For the problem of optimal recovery of linear operators, there are no such general results similar to Theorem 4 or Theorem 5. Moreover, sometimes there is no linear optimal method even for the problem of optimal recovery from exact information and with Hilbert spaces $Y_0, Y_1, \cdots, Y_n$. Let us consider the corresponding example.

Let $X = \mathbb{R}^3$, $Y_0 = l_2^2$ (l_2^n is the space $\mathbb{R}^n$ with the usual Euclidean metric), $Y_1 = Y_2 = Y_3 = Y_4 = l_2^1$. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we set

$$I_0 x = (x_1, x_2), \quad I_1 x = x_1 + 2x_2, \quad I_2 x = x_1 - 2x_2,$$
$$I_3 x = x_3, \quad I_4 x = x_1 + x_3.$$

Let $k = 3$, $\delta_1 = \delta_2 = 1$, $\delta_3 = 2/15$, $\delta_4 = 0$. Thus, we consider the problem of optimal recovery of I_0 on the set

$$W = \left\{x \in \mathbb{R}^3 : |I_1x| \leqslant 1, |I_2x| \leqslant 1, |I_3x| \leqslant \frac{2}{15}\right\}$$

from exact values of the functional I_4. It is easy to see that the set W is the parallelepiped

$$W = \left\{x \in \mathbb{R}^3 : |x_1| + 2|x_2| \leqslant 1, |x_3| \leqslant \frac{2}{15}\right\}.$$

Consider the method

$$\varphi_0(y) = \begin{cases} 0, & |y| \leqslant \dfrac{4}{15}, \\ (y, 0), & |y| > \dfrac{4}{15}. \end{cases}$$

If $|x_1 + x_3| \leqslant 4/15$, then

$$\sup_{\substack{(x_1,x_2,x_3)\in W \\ |x_1+x_3|\leqslant 4/15}} ||(x_1, x_2) - \varphi_0(x_1 + x_3)||_{l_2^2} = \sup_{\substack{(x_1,x_2,x_3)\in W \\ |x_1+x_3|\leqslant 4/15}} ||(x_1, x_2)||_{l_2^2}.$$

Since

$$|x_1| = |x_1 + x_3 - x_3| \leqslant |x_1 + x_3| + |x_3| \leqslant \frac{2}{5},$$

we have

$$\sup_{\substack{(x_1,x_2,x_3)\in W \\ |x_1+x_3|\leqslant 4/15}} ||(x_1, x_2) - \varphi_0(x_1 + x_3)||_{l_2^2} \leqslant \sup_{\substack{(x_1,x_2,x_3)\in W \\ |x_1|\leqslant 2/5}} ||(x_1, x_2)||_{l_2^2} = \frac{1}{2}.$$

If $(x_1, x_2, x_3) \in W$ and $|x_1 + x_3| > 4/15$, then $|x_1| \geqslant |x_1 + x_3| - |x_3| > 2/15$. Consequently, $|x_2| < 13/30$. Therefore,

$$\begin{aligned} \sup_{\substack{(x_1,x_2,x_3)\in W \\ |x_1+x_3|>4/15}} ||(x_1, x_2) - \varphi_0(x_1 + x_3)||_{l_2^2} &\leqslant \sup_{\substack{(x_1,x_2,x_3)\in W \\ |x_2|<13/30}} ||(-x_3, x_2)||_{l_2^2} \\ &< \sqrt{\frac{4}{225} + \frac{169}{900}} < \frac{1}{2}. \end{aligned}$$

Thus,

$$E(I, \delta) \leqslant e(I, \delta, \varphi_0) \leqslant \frac{1}{2}.$$

On the other hand, for any linear method $\varphi(y) = (c_1y, c_2y)$, $c_1, c_2 \in \mathbb{R}$, we have

$$e(I, \delta, \varphi) = \sup_{(x_1,x_2,x_3)\in W} \sqrt{(x_1 - c_1(x_1 + x_3))^2 + (x_2 - c_2(x_1 + x_3))^2}.$$

If $c_1 \leqslant 0$, considering the point $(1, 0, 0) \in W$, we obtain

$$e(I, \delta, \varphi) \geqslant \sqrt{(1 - c_1)^2 + c_2^2} \geqslant 1.$$

If $c_1 > 0$, considering the point $(0, 1/2, 2/15\text{sign}\, c_2) \in W$, we obtain

$$e(I,\delta,\varphi) \geqslant \sqrt{c_1^2\frac{4}{15} + \Big(\frac{1}{2} + |c_2|\frac{2}{15}\Big)^2} > \frac{1}{2}.$$

Consequently, for any linear method φ,

$$e(I,\delta,\varphi) > \frac{1}{2} \geqslant E(I,\delta).$$

Nevertheless, we prove a result which sometimes helps to construct a family of linear optimal methods.

Theorem 8 Assume that there exist such $\lambda_j \geqslant 0$, $j = 1, \cdots, n$, and an element $\widehat{x} \in W$, for which $\|I_j\widehat{x}\|_{Y_j} \leqslant \delta_j$, $j = 1, \cdots, n$, and

$$\|I_0\widehat{x}\|_{Y_0} \geqslant \Big(\sum_{j=1}^n \lambda_j\delta_j^2\Big)^{1/2}.$$

Moreover, assume that the linear operators $S_j\colon Y_j \to Y_0$ satisfy the conditions

(a) $I_0 = \sum\limits_{j=1}^n S_jI_j$,

(b) $\Big\|\sum\limits_{j=1}^n S_jz_j\Big\|_{Y_0}^2 \leqslant \sum\limits_{j=1}^n \lambda_j\|z_j\|_{Y_j}^2$, for all $z_j \in Y_j, j = 1, \cdots, n$.

Then, for any such operators, the method

$$\widehat{\varphi}(y) = S_{k+1}y_{k+1} + \cdots + S_ny_n, \quad y \in Y_{k+1} \times \cdots \times Y_k$$

is optimal, and

$$E(I,\delta) = \Big(\sum_{j=1}^n \lambda_j\delta_j^2\Big)^{1/2}.$$

Proof Let $\varphi : Y_{k+1} \times \cdots \times Y_n \to Y_0$ be an arbitrary method of recovery. Then,

$$\begin{aligned} 2\|I_0\widehat{x}\|_{Y_0} &= \|I_0\widehat{x} - \varphi(0) - (I_0(-\widehat{x}) - \varphi(0))\|_{Y_0} \\ &\leqslant \|I_0\widehat{x} - \varphi(0)\|_{Y_0} + \|I_0(-\widehat{x}) - \varphi(0)\|_{Y_0} \leqslant 2e(I,\delta,\varphi). \end{aligned}$$

In view of the arbitrariness φ, we have

$$E(I,\delta) \geqslant \|I_0\widehat{x}\|_{Y_0} \geqslant \Big(\sum_{j=1}^n \lambda_j\delta_j^2\Big)^{1/2}. \tag{20}$$

To estimate the error of the method $\widehat{\varphi}$, consider the following extremal problem:

$$\begin{aligned} \Big\|I_0x - \sum_{j=k+1}^n S_jy_j\Big\|_{Y_0}^2 \to \max, \quad \|I_jx\|_{Y_j}^2 \leqslant \delta_j^2, \quad j = 1, \cdots, k, \\ \|I_jx - y_j\|_{Y_j}^2 \leqslant \delta_j^2, \quad j = k+1, \cdots, n, \quad x \in X. \end{aligned}$$

Set $z_j = I_j x$, $j = 1, \cdots, k$, $z_j = I_j x - y_j$, $j = k+1, \cdots, n$. Then, taking into account (a), this problem may be rewritten in the form

$$\Big\| \sum_{j=1}^{n} S_j z_j \Big\|_{Y_0}^2 \to \max, \quad \|z_j\|_{Y_j}^2 \leqslant \delta_j^2, \quad j = 1, \cdots, n, \quad x \in X. \tag{21}$$

In view of (b), we obtain

$$\Big\| \sum_{j=1}^{n} S_j z_j \Big\|_{Y_0}^2 \leqslant \sum_{j=1}^{n} \lambda_j \|z_j\|_{Y_j}^2 \leqslant \sum_{j=1}^{n} \lambda_j \delta_j^2.$$

Thus,

$$E(I, \delta) \leqslant e(I, \delta, \widehat{\varphi}) \leqslant \Big(\sum_{j=1}^{n} \lambda_j \delta_j^2 \Big)^{1/2}.$$

These inequalities together with (20) prove the theorem.

We apply this theorem to construct a family of optimal recovery methods of the k-th derivative, $1 \leqslant k < r$, for functions from the Sobolev class $W_2^r(\mathbb{T})$ knowing a finite number of their Fourier coefficients given inaccurately. To simplify calculations, we will consider the complex case.

Assume that we have the Fourier series for some 2π-periodic function x,

$$x(t) = \sum_{j=-\infty}^{+\infty} x_j \mathrm{e}^{ijt}.$$

Suppose that we know only a finite number of the Fourier coefficients which are given with an error. That is, we know $\widetilde{x} = (\widetilde{x}_{-N}, \cdots, \widetilde{x}_N)$ such that

$$\sum_{|j| \leqslant N} |x_j - \widetilde{x}_j|^2 \leqslant \delta^2. \tag{22}$$

Using the information $\widetilde{x}$, we want to recover the k-th derivative of x.

One of the simplest methods of recovery is as follows:

$$x^{(k)}(t) \approx \sum_{|j| \leqslant N} (ij)^k \widetilde{x}_j \mathrm{e}^{ijt}.$$

However, it is not very good because the error of terms $(ij)^k \widetilde{x}_j$ in $|j|^k$ times larger than the error of $\widetilde{x}_j$.

In practice, this effect is known very well. Those who deal with such problems simply cut the terms with high frequencies and smooth other terms by some filter. Such filter was constructed in a similar problem in Theorem 7.

The problem which we would like to pose is: what is a best method of recovery? In other words, what is a best possible filter? Now, we give the exact setting of the problem.

Define $L_2(\mathbb{T})$ as the space of square integrable real-valued or complex-valued functions x on $\mathbb{T}$ with the norm

$$\|x\| = \Big(\frac{1}{2\pi}\int_{\mathbb{T}} |x(t)|^2\,\mathrm{d}t\Big)^{1/2}.$$

The Sobolev space $\mathcal{W}_2^r(\mathbb{T})$ is the set of all 2π-periodic real-valued or complex-valued functions x for which the $(r-1)$-st derivative is absolutely continuous and $\|x^{(r)}\| < \infty$. The Sobolev class $W_2^r(\mathbb{T})$ is the set of all functions $x \in \mathcal{W}_2^r(\mathbb{T})$ for which $\|x^{(r)}\| \leqslant 1$.

Denote by l_2^{2N+1}, $N \in \mathbb{Z}_+$, the space of vectors $y = (y_{-N}, \cdots, y_N)$ with the norm

$$\|y\|_{l_2^{2N+1}} = \Big(\sum_{j=-N}^{N} |y_j|^2\Big)^{1/2}.$$

We consider the problem (19) for $X = \mathcal{W}_2^r(\mathbb{T})$, $Y_0 = Y_1 = L_2(\mathbb{T})$, $Y_2 = l_2^{2N+1}$, $I_0 x = x^{(k)}$, $I_1 x = x^{(r)}$, $I_2 x = \{x_j\}_{j=-N}^N$,

$$x_j = \frac{1}{2\pi}\int_{\mathbb{T}} x(t)\mathrm{e}^{-ijt}\mathrm{d}t,$$

$\delta_1 = 1$, and $\delta_2 = \delta > 0$. The appropriate error of optimal recovery, we denote by $E(D^k, W_2^r(\mathbb{T}), \delta)$.

Set

$$s_0 = \min\left\{ s \in \mathbb{Z}_+ : \frac{(s+1)^{2k} - s^{2k}}{(s+1)^{2r} - s^{2r}} \leqslant \frac{1}{(N+1)^{2(r-k)}} \right\}. \tag{23}$$

It is easy to prove that $s_0 \leqslant N$. We will consider three cases:

i) $s_0 > 1, \dfrac{1}{(s+1)^r} \leqslant \delta < \dfrac{1}{s^r}, 1 \leqslant s \leqslant s_0 - 1$;

ii) $s_0 > 0, 0 < \delta < \dfrac{1}{s_0^r}$;

iii) $s_0 = 0$ or $\delta \geqslant 1$.

In Case i), we put

$$\lambda_1 = \frac{(s+1)^{2k} - s^{2k}}{(s+1)^{2r} - s^{2r}}, \quad \lambda_2 = \frac{(s+1)^{2r}s^{2k} - s^{2r}(s+1)^{2k}}{(s+1)^{2r} - s^{2r}};$$

in Case ii), we put

$$\lambda_1 = \frac{1}{(N+1)^{2(r-k)}}, \quad \lambda_2 = s_0^{2k} - \frac{s_0^{2r}}{(N+1)^{2(r-k)}},$$

and in Case iii), we put $\lambda_1 = 1$, $\lambda_2 = 0$.

Theorem 9 For all $\delta > 0$,

$$E(D^k, W_2^r(\mathbb{T}), \delta) = \sqrt{\lambda_1 + \lambda_2\delta^2}.$$

If $s_0 > 0$ and $\delta < 1$, then for all θ_j, $|\theta_j| \leqslant 1$, $0 < |j| \leqslant N$, the methods

$$\widehat{\varphi}(\widetilde{x})(t) = \sum_{0<|j|\leqslant N} (ij)^k \alpha_j \widetilde{x}_j \mathrm{e}^{ijt}, \tag{24}$$

where

$$\alpha_j = \frac{\lambda_2 + \theta_j j^{r-k}\sqrt{\lambda_1\lambda_2(\lambda_2 + \lambda_1 j^{2r} - j^{2k})}}{\lambda_2 + \lambda_1 j^{2r}}, \tag{25}$$

are optimal.

If $s_0 = 0$ or $\delta \geqslant 1$, then the method $\widehat{\varphi}(\widetilde{x})(t) = 0$ is optimal.

Proof In Case i), put

$$\widehat{x}_s = \Big(\frac{\delta^2(s+1)^{2r} - 1}{(s+1)^{2r} - s^{2r}}\Big)^{1/2}, \quad \widehat{x}_{s+1} = \Big(\frac{1 - \delta^2 s^{2r}}{(s+1)^{2r} - s^{2r}}\Big)^{1/2},$$

$$\widehat{x}(t) = \widehat{x}_s \mathrm{e}^{ist} + \widehat{x}_{s+1}\mathrm{e}^{i(s+1)t}.$$

We have

$$\|\widehat{x}^{(r)}\|^2 = s^{2r}|\widehat{x}_s|^2 + (s+1)^{2r}|\widehat{x}_{s+1}|^2 = 1,$$
$$\|I_2\widehat{x}\|^2_{l_2^{2N+1}} = |\widehat{x}_s|^2 + |\widehat{x}_{s+1}|^2 = \delta^2.$$

Moreover,

$$\|I_0\widehat{x}\|^2 = \|\widehat{x}^{(k)}\|^2 = s^{2k}|\widehat{x}_s|^2 + (s+1)^{2k}|\widehat{x}_{s+1}|^2 = \lambda_1 + \lambda_2\delta^2.$$

In Case ii), we put

$$\widehat{x}_{s_0} = \delta, \quad \widehat{x}_{N+1} = \frac{\sqrt{1 - \delta^2 s_0^{2r}}}{(N+1)^r}, \quad \widehat{x}(t) = \widehat{x}_{s_0}\mathrm{e}^{is_0 t} + \widehat{x}_{N+1}\mathrm{e}^{i(N+1)t}.$$

We have

$$\|\widehat{x}^{(r)}\|^2 = s_0^{2r}|\widehat{x}_{s_0}|^2 + (N+1)^{2r}|\widehat{x}_{N+1}|^2 = 1, \quad \|I_2\widehat{x}\|^2_{l_2^{2N+1}} = |\widehat{x}_{s_0}|^2 = \delta^2,$$

and

$$\|I_0\widehat{x}\|^2 = \|\widehat{x}^{(k)}\|^2 = s_0^{2k}|\widehat{x}_{s_0}|^2 + (N+1)^{2k}|\widehat{x}_{N+1}|^2 = \lambda_1 + \lambda_2\delta^2.$$

In Case iii), we consider $\widehat{x}(t) = \mathrm{e}^{it}$. Then,

$$\|\widehat{x}^{(r)}\| = 1, \quad \|I_2\widehat{x}\|_{L^2_{2N+1}} = \begin{cases} 1, & N > 0, \\ 0, & N = 0, \end{cases}$$

and

$$\|I_0\widehat{x}\|^2 = \|\widehat{x}^{(k)}\|^2 = 1 = \lambda_1 + \lambda_2\delta^2.$$

Now, to apply Theorem 8, we will construct the operators S_1 and S_2. Let $u \in L_2(\mathbb{T})$,

$$u(t) = \sum_{j=-\infty}^{+\infty} u_j e^{ijt},$$

and $v = (v_{-N}, \cdots, v_N) \in L_{2N+1}^2$. We will search the operators S_1 and S_2 in the forms

$$S_1 u = \sum_{\substack{j=-\infty \\ j\neq 0}}^{+\infty} \beta_j u_j e^{ijt}, \quad S_2 v = \sum_{|j|\leqslant N} (ij)^k \alpha_j v_j e^{ijt}.$$

From condition (a) of Theorem 8, we obtain

$$\beta_j = \begin{cases} (ij)^{k-r}(1-\alpha_j), & 0 < |j| \leqslant N, \\ (ij)^{k-r}, & |j| > N. \end{cases}$$

First, we consider Case iii) ($\lambda_1 = 1$, $\lambda_2 = 0$). Put $\alpha_j = 0$ for all $j = -N, \cdots, N$. Then, by virtue of the Parseval equality, we have

$$\begin{aligned} \|S_1 u + S_2 v\|^2 = \|S_1 u\|^2 = \sum_{\substack{j=-\infty \\ j\neq 0}}^{+\infty} j^{2(k-r)} |u_j|^2 \leqslant \sum_{\substack{j=-\infty \\ j\neq 0}}^{+\infty} |u_j|^2 \\ \leqslant \|u\|^2 = \lambda_1 \|u\|^2 + \lambda_2 \|v\|^2_{l_2^{2N+1}}. \end{aligned}$$

Now, consider Cases i) and ii), we have

$$\|S_1 u + S_2 v\|^2 = \sum_{0<|j|\leqslant N} |\beta_j u_j + (ij)^k \alpha_j v_j|^2 + \sum_{|j|>N} |\beta_j|^2 |u_j|^2. \tag{26}$$

Using the Cauchy-Schwarz inequality, we obtain

$$|\beta_j u_j + (ij)^k \alpha_j v_j|^2 \leqslant A_j (\lambda_1 |u_j|^2 + \lambda_2 |v_j|^2), \tag{27}$$

where

$$A_j = \frac{|\beta_j|^2}{\lambda_1} + \frac{j^{2k}|\alpha_j|^2}{\lambda_2} = \frac{|1-\alpha_j|^2}{j^{2(r-k)}\lambda_1} + \frac{j^{2k}|\alpha_j|^2}{\lambda_2}.$$

Assume that we find α_j such that $A_j \leqslant 1$ for all $0 < |j| \leqslant N$. Then, from (26), (27), taking into account that $\lambda_1 \geqslant (N+1)^{-2(r-k)}$, we obtain

$$\begin{aligned} \|S_1 u + S_2 v\|^2 &\leqslant \lambda_1 \sum_{0<|j|\leqslant N} |u_j|^2 + \sum_{|j|>N} j^{2(k-r)} |u_j|^2 + \lambda_2 \sum_{|j|\leqslant N} |v_j|^2 \\ &\leqslant \lambda_1 \sum_{j=-\infty}^{+\infty} |u_j|^2 + \lambda_2 \sum_{|j|\leqslant N} |v_j|^2 = \lambda_1 \|u\|^2 + \lambda_2 \|v\|^2_{l_2^{2N+1}}. \end{aligned}$$

It remains to show that there exist α_j such that

$$\frac{|1-\alpha_j|^2}{j^{2(r-k)}\lambda_1} + \frac{j^{2k}|\alpha_j|^2}{\lambda_2} \leqslant 1 \tag{28}$$

for all $0 < |j| \leqslant N$. This inequality may be rewritten in the form

$$\left|\alpha_j - \frac{\lambda_2}{\lambda_2 + \lambda_1 j^{2r}}\right|^2 \leqslant \frac{\lambda_1 \lambda_2 j^{2(r-k)}(\lambda_2 + \lambda_1 j^{2r} - j^{2k})}{(\lambda_2 + \lambda_1 j^{2r})^2}. \tag{29}$$

It suffices to prove that

$$\lambda_2 + \lambda_1 j^{2r} - j^{2k} \geqslant 0 \tag{30}$$

for all $j = 1, \cdots, N$. Consider the set of points on the plane $\mathbb{R}^2$,

$$\begin{cases} x_j = j^{2r}, \\ y_j = j^{2k}, \end{cases} \quad j = 0, 1, \cdots. \tag{31}$$

If we plot the function

$$\begin{cases} x = t^{2r}, \\ y = t^{2k}, \end{cases} \quad t \in [0, +\infty), \tag{32}$$

then the points (31) belong to the plot of this function. The function defined by (32) can be written in the form

$$y = x^{k/r}, \quad 0 < \frac{k}{r} < 1.$$

It is a concave function. In Case i), the line $y = \lambda_2 + \lambda_1 x$ passes through the points (s^{2r}, s^{2k}) and $((s+1)^{2r}, (s+1)^{2k}))$. In view of concavity, the inequality (30) holds for all $j \geqslant 0$.

In Case ii), the line $y = \lambda_2 + \lambda_1 x$ passes through the point (s_0^{2r}, s_0^{2k}) and

$$\frac{(s_0+1)^{2k} - s_0^{2k}}{(s_0+1)^{2r} - s_0^{2r}} \leqslant \lambda_1.$$

It means that the inequality (30) holds for all $j \geqslant s_0$. On the other hand, in view of definition of s_0,

$$\frac{s_0^{2k} - (s_0-1)^{2k}}{s_0^{2r} - (s_0-1)^{2r}} > \lambda_1.$$

Consequently, the inequality (30) holds for all $0 \leqslant j \leqslant s_0$.

Now, it remains to note that the set of all α_j satisfying (29) may be written in the form (25) with $|\theta_j| \leqslant 1$.

Among the family of optimal methods (24), we find the ones that use minimal information about the input data. If in (24) $\alpha_j = 0$ for some j, then the information about $\widetilde{x}_j$ is not used. Thus, we would like to find all such j. It follows from (28) that if $\alpha_j = 0$, then $|j| \geqslant \lambda_1^{\frac{1}{2(k-r)}}$. It is interesting to find also those j for which $\alpha_j = 1$ (that is, for such j, we do not smooth the information). From (28), we see that we may take $\alpha_j = 1$ for $|j| \leqslant \lambda_2^{\frac{1}{2k}}$. Thus, we obtain the following result.

Corollary 2 If $s_0 > 0$ and $\delta < 1$, then for all θ_j, $|\theta_j| \leqslant 1$, $0 < |j| \leqslant N$, the methods

$$\widehat{\varphi}(\widetilde{x})(t) = \sum_{0<|j|\leqslant\lambda_2^{\frac{1}{2k}}} (ij)^k \widetilde{x}_j \mathrm{e}^{ijt} + \sum_{\lambda_2^{\frac{1}{2k}}<|j|<\lambda_1^{\frac{1}{2(k-r)}}} (ij)^k \alpha_j \widetilde{x}_j \mathrm{e}^{ijt},$$

where α_j are defined by (25), are optimal.

More results on optimal recovery of functions and their derivatives in the periodic case and in the case when functions defined on the real line may be found in [20]~[24].

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