ON EXACT VALUES OF n-WIDTHS FOR CLASSES DEFINED BY CYCLIC VARIATION DIMINISHING OPERATORS

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Abstract. In the paper the unique approach to the problems of calculation of exact values of n-widths in the uniform metric is suggested for the classes of 2π-periodic functions defined by operators (not necessary linear) having certain oscillating properties. This approach allows to obtain exact results of n-widths both for classes of functions represented by the convolution with cyclic variation diminishing kernels and some classes of analytic functions which are not represented in the form of such convolution.

Introduction

For classes of smooth periodic functions many extremal problems of approximation theory can be solved due to analysis of oscillating properties of functions from these classes. The approach based on such analysis leaded to investigation of classes of functions represented in the form of the convolution with cyclic variation diminishing kernels. For such classes sufficiently general results concerning optimal recovery, optimal quadratures formulas, n-widths, Kolmogorov inequalities, and several others were obtained (see [1]–[3]).

However some classes of analytic functions are not represented in the form of the convolution with cyclic variation diminishing kernels. Neveretheless for these classes there are very closed results to the smooth case (see [4]–[6]). In this paper for the problem of calculation exact values of n-widths the unique approach served both as the smooth and analytic cases is suggested which based on the introduction of special class of operators (in general, nonlinear) having the cyclic variation diminishing property.

1. Basic definitions

Recall definitions of some n-widths. Let A be a subset of a normed linear space X. *The Kolmogorov n-width* is the number

$$d_n(A,X):=\inf_{X_n}\sup_{x\in A}\inf_{y\in X_n}\|x-y\|,$$

where X_n are any n-dimensional subspaces of X.

The linear n-width is the number

$$\lambda_n(A,X):=\inf_{Y}\inf_{P_n}\sup_{x\in A}\|x-P_nx\|,$$

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where Y are any normed linear spaces containing A and P_n are bounded linear operators mapping Y into X, whose range is n or less.

The Gel'fand n-width is defined by following

$$d^n(A,X) := \inf_Y \inf_{Y^n} \sup_{x\in A\cap Y^n} \|x\|,$$

where Y has the same sense as in the definition of the linear n-width and Y^n are any subsets of Y of codimension n (here it is assumed that $0\in A$).

The information n-width of a set A is the number

$$i_n(A,X) := \inf_{\substack{Y\supset A\\ l_1,\dots,l_n\in Y^*}} \inf_{\mathcal{L}:\mathbb{R}^n\to X} \sup_{x\in A} \|x-\mathcal{L}(l_1x,\dots,l_nx)\|_X, \tag{1}$$

where Y as above are any normed linear spaces containing A. Linear functionals which realizes the lower bound in (1) will be called optimal functionals for the corresponding information n-width.

Lemma 1. *Let A be a centrally symmetric set containing zero. Then*

$$d^n(A,X) \le i_n(A,X) \le \lambda_n(A,X). \tag{2}$$

Proof. The inequality

$$i_n(A,X) \le \lambda_n(A,X)$$

immediately follows from the definitions of the information and linear n-widths. Let us prove the lower bound. Let $Y\supset A$ and $l_1,\dots,l_n\in Y^*$. For all $\varepsilon>0$ there exists $x_\varepsilon\in A$ for which $l_1x_\varepsilon=\ldots=l_nx_\varepsilon=0$ and

$$\sup_{\substack{x\in A\\ l_1x=\ldots=l_nx=0}} \|x\|_X \le \|x_\varepsilon\|_X+\varepsilon.$$

For all $\mathcal{L}$ we have

$$\|x_\varepsilon-\mathcal{L}(0,\dots,0)\|_X + \|-x_\varepsilon-\mathcal{L}(0,\dots,0)\|_X \ge 2\|x_\varepsilon\|_X.$$

Therefore

$$\sup_{x\in A}\|x-\mathcal{L}(l_1x,\dots,l_nx)\|_X \ge \|x_\varepsilon\|_X \ge \sup_{\substack{x\in A\\ l_1x=\ldots=l_nx=0}} \|x\|_X-\varepsilon \ge d^n(A,X)-\varepsilon.$$

Hence by virtue of the arbitrariness of $l_1\dots,l_n$, $\mathcal{L}$, and $\varepsilon>0$ we obtain

$$i_n(A,X)\ge d^n(A,X). \quad \square$$

Denote by L_p, $1\le p\le\infty$, the space of real 2π-periodic functions for which

$$\|x\|_p := \left(\int_{\mathbb{T}} |x(t)|^p\,dt\right)^{1/p} < \infty, \quad 1\le p<\infty,$$
$$\|x\|_\infty := \operatorname{vrai\,sup}_{t\in\mathbb{T}} |x(t)| < \infty, \quad p=\infty.$$

Set
$$BL_\infty := \{\, x \in L_\infty \mid \|x\|_\infty \le 1 \,\}.$$

Denote by $\mathcal{F}$ the set of operators $F\colon BL_\infty \to L_\infty$ (in general, nonlinear) for which

(1) for all $x \in BL_\infty$ $F(-x) = -Fx$;
(2) for all $\alpha \in \mathbb{T}$ $P_\alpha \circ F = F \circ P_\alpha$, where $(P_\alpha x)(\cdot) = x(\cdot + \alpha)$;
(3) F is continuous operator as an operator acting from the subset BL_∞ of the space L_1 into L_1 (i.e., $\|Fx_m - Fx\|_1 \to 0$ as $\|x_m - x\|_1 \to 0$).

For a finite, symmetric with respect to zero set $M \subset \mathbb{Z}$ denote by $\mathcal{T}_M$ the set of real trigonometric polynomials from $\operatorname{span}\{e^{ikt}\}_{k\in M}$, $\mathcal{T}_\emptyset := \{0\}$. Let $G \in \mathcal{F}$ and M_0, M be finite, symmetric with respect to zero sets of integers. We shall denote by $CVD(M_0, M, G)$ the set of operators $F \in \mathcal{F}$ for which the following conditions are hold:

(1) if x_1, x_2 are different functions from BL_∞ such that $Gx_1 \perp \mathcal{T}_M$, $Gx_2 \perp \mathcal{T}_M$, and $p \in \mathcal{T}_{M_0}$, then
$$S(p + Fx_1 - Fx_2) \le S(x_1 - x_2),$$
where $S(f)$ is the number of sign changes of 2π-periodic function f on the period (about the definition of $S(f)$ for $f \in L_\infty$ see [1, p. 41]);
(2) for all $0 < \rho < 1$ and all $x \in BL_\infty$ such that $\|x\|_\infty \le 1$ and $Gx \perp \mathcal{T}_M$ there exists a function $x_\rho \in BL_\infty$ such that $\|x_\rho\|_\infty < 1$, $Gx_\rho \perp \mathcal{T}_M$, and $\rho Fx = Fx_\rho$.
(3) the set $\{Fx \mid x \in BL_\infty,\ Gx \perp \mathcal{T}_M\}$ is convex.
(4) if $x \in BL_\infty$ and $Gx \perp \mathcal{T}_M$, then $Fx \perp \mathcal{T}_M$.

The last condition is valid, for example, if G is the identical operator and F is the convolution one, or if $G = F$. These two cases are the basic ones in the examples considered below.

For a function $f \in C(\mathbb{T})$ denote by $\operatorname{dist} f$ the maximum length of subinterval of $\mathbb{T}$ on which the function f has no zeros. Set
$$(f * g)(t) := \frac{1}{2\pi}\int_{\mathbb{T}} f(t-s)g(s)\,ds.$$
Denote by $\mathcal{K}(M, \delta)$ the class of kernels $\Omega \in L_1$ for which for all $x \in L_\infty$ and $p \in \mathcal{T}_M$ such that $x \perp \mathcal{T}_M$, $x \not\equiv 0$, and $\operatorname{dist}(p + \Omega * x) < \delta$ the inequality
$$S(p + \Omega * x) \le S(x)$$
holds, moreover, if $\Omega * x \in C^2(\mathbb{T})$, then
$$Z_2(p + \Omega * x) \le S(x),$$
where Z_2 is the number of zeros of a function when multiple zeros are counted twice and intervals on which a function identically equals zero are discarded. We shall also suppouse that $c_j(\Omega) \ne 0$, $j \notin M$, where
$$c_j(f) := \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-ijt}\,dt, \quad j = 0, \pm 1, \dots\ .$$

Suppouse that

$$(3)\qquad \Omega_j \in \mathcal{K}(M_j,\delta_j), \quad j=1,\dots,k, \qquad F\in CVD(M_0,M,G), \quad M=\bigcup_{j=0}^{k} M_j.$$

We shall interest in exact values of n-widths for classes

$$\mathcal{F}_\infty := \mathcal{F}_\infty(\Phi,M) := \{\, p+\Phi x \mid p\in \mathcal{T}_M,\ x\in BL_\infty,\ Gx \perp \mathcal{T}_M \,\},$$

where $\Phi x = \Omega_k * \dots * \Omega_1 * Fx$.

2. Exact values of n-widths on classes $\mathcal{F}_\infty$

Let $n\in\mathbb{N}$ and

$$\Theta_n := \{\, \theta \mid \theta=(\theta_1,\dots,\theta_n),\ 0\le\theta_1\le\dots\le\theta_n<2\pi \,\}.$$

For $\eta\in\Theta_{2n}$ put

$$h_\eta(t) := (-1)^j, \quad t\in[\eta_{j-1},\eta_j), \quad j=1,\dots,2n+1,$$

where $\eta_0 := 0$, $\eta_{2n+1} := 2\pi$. Denote by $h_n(t)$ the function $h_\eta(t)$ when $\eta_j = (j-1)\pi/n$, $j=1,\dots,2n$.

We will suppouse conditions (3) are fulfilled. Set

$$\delta := \min_{1\le j\le k} \delta_j, \qquad m := \begin{cases} \sup\{j \mid j\in M\}, & M\ne\emptyset,\\ -1, & M=\emptyset. \end{cases}$$

Lemma 2. *For all $n>\max\{m,2\pi/\delta\}$ the inequality*

$$\inf_{\substack{p\in\mathcal{T}_M\\ \eta\in\Theta_{2n},\ Gh_\eta\perp\mathcal{T}_M}} \|p+\Phi h_\eta\|_\infty = \|\Phi h_n\|_\infty$$

holds.

Proof. Set

$$G_\sigma(t) = \frac{\sqrt{2\pi}}{\sigma}\sum_{j\in\mathbb{Z}} \exp\left(-\frac{(t-2\pi j)^2}{2\sigma^2}\right) = 1+2\sum_{j=1}^{\infty} e^{-j^2\sigma^2/2}\cos jt, \quad \sigma>0.$$

The following properties of the kernel G_σ are well known (see [7]): for all $f\in L_\infty$ $G_\sigma * f$ are analytic functions and furthermore

$$\lim_{\sigma\to 0}\|G_\sigma * f\|_\infty = \|f\|_\infty, \qquad Z(G_\sigma * f)\le S(f),$$

where $Z(g)$ is the number of zeros of a function g with regard to multiplicities. Suppose that there exist $p\in\mathcal{T}_M$ and $\eta\in\Theta_{2n}$ for which $Gh_\eta\perp\mathcal{T}_M$ and

$$\|p+\Phi h_\eta\|_\infty < \|\Phi h_n\|_\infty.$$

Then for sufficiently small σ

$$\|p_\sigma + \Phi_\sigma h_\eta\|_\infty < \|\Phi_\sigma h_n\|_\infty,$$

where $p_\sigma = G_\sigma * p \in \mathcal{T}_M$, $\Phi_\sigma x = G_\sigma * \Phi x$. In view of properties of the convolution and operator F we have

$$P_{\pi/n}(\Phi_\sigma h_n) = \Phi_\sigma(P_{\pi/n} h_n) = \Phi_\sigma(-h_n) = -\Phi_\sigma h_n. \tag{4}$$

Hence it follows the existence of the points $0 \le t_1 < \ldots < t_{2n} < 2\pi$ such that

$$(\Phi_\sigma h_n)(t_j) = \varepsilon(-1)^j \|\Phi_\sigma h_n\|_\infty, \quad j = 1, \ldots, 2n, \tag{5}$$

where $\varepsilon = 1$ or -1. Thus for any $\alpha \in \mathbb{T}$

$$2n \le S(P_\alpha(\Phi_\sigma h_n) - p_\sigma - \Phi_\sigma h_\eta) = S(\Phi_\sigma(P_\alpha h_n) - p_\sigma - \Phi_\sigma h_\eta). \tag{6}$$

If for some α $h_\eta = P_\alpha h_n$, then

$$2n \le S(P_\alpha(\Phi_\sigma h_n) - p_\sigma - \Phi_\sigma h_\eta) = S(p) \le 2m < 2n.$$

Thus for all α $P_\alpha h_n \ne h_\eta$.

From the fact that $c_s(\Omega_j) \ne 0$, $s \notin M_j$ it follows the existence of trigonometric polynomials $p_j \in \mathcal{T}_{M_j}$, $j = 0, \ldots, k$ for which

$$p_\sigma + \Phi_\sigma h_\eta = p_k + \Omega_k * (p_{k-1} + \ldots + \Omega_1 * G_\sigma * (p_0 + F h_\eta) \ldots).$$

Set

$$\begin{gathered} f_j := p_j + \Omega_j * (p_{j-1} + \ldots + \Omega_1 * G_\sigma * (p_0 + F h_\eta) \ldots), \quad j = 1, \ldots, k, \\ g_j := \Omega_j * \ldots * \Omega_1 * G_\sigma * F h_n, \quad j = 1, \ldots, k, \\ f_0 := G_\sigma * (p_0 + F h_\eta), \qquad g_0 := G_\sigma * F h_n, \\ \mu_j := \frac{\|f_j\|_\infty}{\|g_j\|_\infty}, \quad j = 0, \ldots, k-1, \qquad \mu := \max_{0 \le j \le k-1} \mu_j. \end{gathered}$$

Let us prove that $\mu_0 \ge 1$. Assume the contrary. Then by the analogy with the previous arguments (see (4)–(6)) it can be shown that for all $\alpha \in \mathbb{T}$

$$S(P_\alpha g_0 - f_0) \ge 2n.$$

Having choosed $\alpha = -\eta_1$, we have

$$\begin{aligned} 2n \le S(G_\sigma * F(P_\alpha h_n) - G_\sigma * (p_0 + F h_\eta)) &\le S(F(P_\alpha h_n) - p_0 - F h_\eta) \\ &\le S(h_n(\cdot + \alpha) - h_\eta(\cdot)) \le 2(n-1). \end{aligned}$$

Thus it is proved that $\mu \ge 1$. Let $\mu = \mu_s$, $0 \le s \le k-1$. Choose $\alpha \in \mathbb{T}$ so that the difference

$$P_\alpha g_s - \mu_s^{-1} f_s$$

has a multiple zero. By the analogy with equalities (4), we obtain

$$P_{\pi/n} g_j = -g_j. \tag{7}$$

Hence it follows that

$$\operatorname{dist}(P_\alpha g_j - \mu_s^{-1} f_j) \le \frac{2\pi}{n} < \delta.$$

Consequently by virtue of properties of kernels Ω_j as $s > 0$ we have

$$\begin{aligned} 2n \le S(P_\alpha g_k - \mu_s^{-1} f_k) \le \ldots \le S(P_\alpha g_s - \mu_s^{-1} f_s) < Z_2(P_\alpha g_s - \mu_s^{-1} f_s) \\ \le S(P_\alpha g_{s-1} - \mu_s^{-1} f_{s-1}) \le \ldots \le S(P_\alpha g_0 - \mu_s^{-1} f_0) \\ \le S(F(P_\alpha h_n) - \mu_s^{-1}(p_0 + F h_\eta)). \end{aligned} \tag{8}$$

If $\mu_s > 1$, then in view of property (2) of operators from the set $CVD(M_0, M, G)$ there exists a function $h^* \in BL_\infty$ such that $\|h^*\|_\infty < 1$, $Gh^* \perp \mathcal{T}_M$ and $\mu_s^{-1} F h_\eta = Fh^*$. For $\mu_s = 1$ set $h^* = h_\eta$. Thus

$$S(F(P_\alpha h_n) - \mu_s^{-1}(p_0 + F h_\eta)) \le S(h_n(\cdot + \alpha) - h^*(\cdot)) \le 2n$$

what contradicts with (8). If $s = 0$, then inequalities (8) are replaced with the following ones

$$\begin{aligned} 2n \le S(P_\alpha g_k - \mu_0^{-1} f_k) \le \ldots \le S(P_\alpha g_0 - \mu_0^{-1} f_0) < Z_2(P_\alpha g_0 - \mu_0^{-1} f_0) \\ \le S(F(P_\alpha h_n) - \mu_s^{-1}(p_0 + F h_\eta)). \end{aligned}$$

Thus it is proved that

$$\|p + \Phi h_\eta\|_\infty \ge \|\Phi h_n\|_\infty.$$

From the equality

$$P_{\pi/n} G h_n = -G h_n$$

it follows that Gh_n is a $2\pi/n$-periodic function. Therefore $Gh_n \perp \mathcal{T}_M$. $\square$

Set $\mathcal{T}_n := \mathcal{T}_{[-n,n]\cap\mathbb{Z}}$

$$\begin{aligned} a_j(f) &:= \frac{1}{\pi} \int_{\mathbb{T}} f(t) \cos jt\, dt, \quad j = 0, 1, \ldots\ , \\ b_j(f) &:= \frac{1}{\pi} \int_{\mathbb{T}} f(t) \sin jt\, dt, \quad j = 1, 2, \ldots\ , \\ I_{2n-1}(f) &:= (a_0(f), a_1(f), b_1(f), \ldots, a_{n-1}(f), b_{n-1}(f)). \end{aligned}$$

In what follows we shall assume that for $k = 0$ $F(BL_\infty) \subset C(\mathbb{T})$ and, moreover, the operator F is continuous as an operator acting from the subset BL_∞ of the space L_1 into $C(\mathbb{T})$ (i.e., $\|Fx_m - Fx\|_\infty \to 0$ as $\|x_m - x\|_1 \to 0$).

Lemma 3. *For all $n > \max\{m, 2\pi/\delta\}$ there exists a linear operator $\mathcal{L}\colon \mathbb{T}^{2n-1} \to \mathcal{T}_{n-1}$ for which*

$$\sup_{f\in\mathcal{F}_\infty} \|f - \mathcal{L}(I_{2n-1}(f))\|_\infty = \|\Phi h_n\|_\infty.$$

Proof. First we prove that

$$\sup_{\substack{f\in\mathcal{F}_\infty\\ I_{2n-1}(f)=0}} \|f\|_\infty = \|\Phi h_n\|_\infty. \tag{9}$$

Assume that there exists a function $f \in \mathcal{F}_\infty$ for which $I_{2n-1}(f) = 0$ and $\|f\|_\infty > \|\Phi h_n\|_\infty$. Let $f = \Phi x$, $x \in BL_\infty$. Then for sufficiently small σ $\|\Phi_\sigma x\|_\infty > \|\Phi_\sigma h_n\|_\infty$. Having leaved the same notation for g_j as in Lemma 2, set

$$f_j := \Omega_j * \dots * \Omega_1 * G_\sigma * Fx, \quad j = 1, \dots, k, \qquad f_0 := G_\sigma * Fx,$$
$$\nu_j := \frac{\|f_j\|_\infty}{\|g_j\|_\infty}, \quad j = 0, \dots, k, \qquad \nu := \max_{0\le j\le k} \nu_j.$$

Let $\nu = \nu_s$, $0 \le s \le k$. By virtue of our assumptions $\nu_s > 1$. Choose $\alpha \in \mathbb{T}$ so that the difference

$$P_\alpha g_s - \nu_s^{-1} f_s$$

has a multiple zero. From equalities (7) it follows that the functions g_j have the period $2\pi/n$. Consequently, $I_{2n-1}(g_j) = 0$. Thus

$$I_{2n-1}(P_\alpha g_s - \nu_s^{-1} f_s) = 0.$$

Since the trigonometric system is the Chebyshev system hence it follows (see [1, p. 41]) that

$$S(P_\alpha g_s - \nu_s^{-1} f_s) \ge 2n.$$

By virtue of the fact that $\operatorname{dist}(P_\alpha g_s - \nu_s^{-1} f_s) \le 2\pi/n < \delta$ we have

$$\begin{aligned}2n \le S(P_\alpha g_s - \nu_s^{-1} f_s) < Z_2(P_\alpha g_s - \nu_s^{-1} f_s) \le S(P_\alpha g_{s-1} - \nu_s^{-1} f_{s-1})\\ \le \dots \le S(P_\alpha g_0 - \nu_s^{-1} f_0) \le S(F(P_\alpha h_n) - \nu_s^{-1} Fx)\\ \le S(h_n(\cdot + \alpha) - x^*(\cdot)) = 2n,\end{aligned}$$

where x^* is defined by equality $\nu_s^{-1} Fx = Fx^*$, $\|x^*\|_\infty < 1$, $Gx^* \perp \mathcal{T}_M$. The obtained contradiction proves equality (9).

Now consider the problem of optimal recovery of the value $f(0)$ on the class $\mathcal{F}_\infty$ with the information $I_{2n-1}(f)$. From general results concerning the problems of optimal recovery the existence of linear optimal method of recovery follows, that is, there exist such numbers $\alpha_0, \alpha_1, \beta_1, \dots, \alpha_{n-1}, \beta_{n-1}$ that

$$\sup_{f\in\mathcal{F}_\infty} |f(0) - \alpha_0 a_0(f) - \sum_{j=1}^{n-1} (\alpha_j a_j(f) + \beta_j b_j(f))| = \sup_{\substack{f\in\mathcal{F}_\infty\\ I_{2n-1}(f)=0}} \|f\|_\infty = \|\Phi h_n\|_\infty. \tag{10}$$

Let g be an arbitrary function from $\mathcal{F}_\infty$. For $t \in \mathbb{T}$ put $f_t(\tau) := g(t+\tau)$. Since $f_t \in \mathcal{F}_\infty$, $a_0(f_t) = a_0(g)$, and

$$\begin{aligned} a_j(f_t) &= \quad a_j(g)\cos jt + b_j(g)\sin jt, \\ b_j(f_t) &= -a_j(g)\sin jt + b_j(g)\cos jt, \end{aligned} \qquad j = 1, 2, \dots ,$$

upon putting

$$\mathcal{L}(I_{2n-1}(g)) := \alpha_0 a_0(g) \\ + \sum_{j=1}^{n-1}\Big((\alpha_j \cos jt - \beta_j \sin jt) a_j(g) + (\alpha_j \sin jt + \beta_j \cos jt) b_j(g)\Big),$$

from (10) for $\tau = 0$ we find

$$|g(t) - \mathcal{L}(I_{2n-1}(g))| \le \|\Phi h_n\|_\infty.$$

In view of arbitrariness of $t \in \mathbb{T}$ we have

$$\|g - \mathcal{L}(I_{2n-1}(g))\|_\infty \le \|\Phi h_n\|_\infty.$$

Since for $g = \Phi h_n$ the last inequality turns into equality the conclusion of the lemma is proved. $\square$

We now prove the basic result of the paper.

Theorem 4. *For all $n > \max\{m, 2\pi/\delta\}$ the equalities*

$$d_{2n}(\mathcal{F}_\infty, L_\infty) = \lambda_{2n}(\mathcal{F}_\infty, L_\infty) = d^{2n}(\mathcal{F}_\infty, L_\infty) = i_{2n}(\mathcal{F}_\infty, L_\infty) \\ = d_{2n-1}(\mathcal{F}_\infty, L_\infty) = \lambda_{2n-1}(\mathcal{F}_\infty, L_\infty) = d^{2n-1}(\mathcal{F}_\infty, L_\infty) = i_{2n-1}(\mathcal{F}_\infty, L_\infty) \\ = \|\Phi h_n\|_\infty$$

hold. Moreover, the Fourier coefficients $a_0(f), a_1(f), b_1(f), \dots, a_{n-1}(f), b_{n-1}(f)$ are optimal functionals for the values i_{2n-1} and i_{2n}.

Proof. First we prove the lower bound for the Kolmogorov and Gel'fand $2n$-widths. Set

$$S^{2n} := \left\{ \xi = (\xi_1, \dots, \xi_{2n+1}) \in \mathbb{R}^{2n+1} \;\Big|\; \sum_{s=1}^{2n+1} |\xi_s| = 2\pi \right\},$$

$$\tau_0(\xi) := 0, \qquad \tau_j(\xi) := \sum_{s=1}^{j} |\xi_s|, \quad j = 1, \dots, 2n+1.$$

For $\xi \in S^{2n}$ define the functions

$$g_\xi(t) := \operatorname{sign} \xi_j, \quad \tau_{j-1}(\xi) \le t < \tau_j(\xi), \quad j = 1, \dots, 2n+1, \qquad f_\xi := \Phi g_\xi.$$

Let X_{2n} be an arbitrary $2n$-dimensional subspace of L_q, $1 < q < \infty$. Assume that $X_{2n} = \operatorname{span}\{f_1, \dots, f_{2n}\}$ and

$$f_\xi^* := \sum_{j=1}^{2n} \alpha_j(\xi) f_j$$

is the best approximation to f_ξ by the subspace X_{2n}. Set

$$E(\mathcal{F}_\infty, X_{2n}) := \sup_{f \in \mathcal{F}_\infty} \inf_{g \in X_{2n}} \|f - g\|_q.$$

If $\mathcal{T}_M \not\subset X_{2n}$, then $E(\mathcal{F}_\infty, X_{2n}) = \infty$. We will assume that $\mathcal{T}_M \subset X_{2n}$, $\dim \mathcal{T}_M = r$, and $\{f_j\}_{j=1}^r$ is a basis of $\mathcal{T}_M$. Put $I_M(f) := \{a_j(f), b_j(f)\}_{j \in M \cap \mathbb{Z}_+}$. Consider the mapping

$$\alpha(\xi) := (I_M(Gg_\xi), \alpha_{r+1}(\xi), \dots, \alpha_{2n}(\xi)).$$

The mapping $\alpha\colon S^{2n} \to \mathbb{R}^{2n}$ is a continuous and odd one. Therefore by the Borsuk Theorem there exists $\xi^* \in S^{2n}$ for which $\alpha(\xi^*) = 0$. We have

$$\begin{aligned} E(\mathcal{F}_\infty, X_{2n}) \geq \inf_{g \in X_{2n}} \|f_{\xi^*} - g\|_q = \|\Phi g_{\xi^*} - \sum_{j=1}^{r} \alpha_j(\xi^*) f_j\|_q \\ \geq \inf_{\substack{p \in \mathcal{T}_M \\ \eta \in \Theta_{2n},\ Gh_\eta \perp \mathcal{T}_M}} \|p + \Phi h_\eta\|_q. \end{aligned}$$

Consequently

$$d_{2n}(\mathcal{F}_\infty, L_q) \geq \inf_{\substack{p \in \mathcal{T}_M \\ \eta \in \Theta_{2n},\ Gh_\eta \perp \mathcal{T}_M}} \|p + \Phi h_\eta\|_q.$$

By passing to the limit as $q \to \infty$ and using Lemma 2, we find

$$d_{2n}(\mathcal{F}_\infty, L_\infty) \geq \inf_{\substack{p \in \mathcal{T}_M \\ \eta \in \Theta_{2n},\ Gh_\eta \perp \mathcal{T}_M}} \|p + \Phi h_\eta\|_\infty = \|\Phi h_n\|_\infty.$$

Now let us obtain the lower bound for the Gel'fand $2n$-widths. Let Y be some normed linear space containing F_∞ and

$$X^{2n} = \{\, f \in Y \mid \langle l_j, f \rangle = 0,\ j = 1, \dots, 2n,\ l_j \in Y^* \,\}.$$

Consider the mapping $J\colon \mathcal{T}_M \to \mathbb{R}^{2n}$ defined by the equality

$$Jp = (l_1 p, \dots, l_{2n} p).$$

If $\operatorname{Ker} J \neq 0$, then

$$\sup_{f \in \mathcal{F}_\infty \cap X^{2n}} \|f\|_\infty = \infty.$$

If $\operatorname{Ker} J = 0$, then among the functionals $l_1, \dots, l_{2n}$ there exist r linearly independent functionals on $\mathcal{T}_M$. Assume that $l_1, \dots, l_r$ are such functionals. Then the other functionals can be represented on $\mathcal{T}_M$ in the form

$$\langle l_j, p\rangle = \sum_{s=1}^{r} c_{js}\langle l_s, p\rangle, \quad j = r+1, \dots, 2n.$$

Set

$$L_j := l_j - \sum_{s=1}^{r} c_{js} l_s, \quad j = r+1, \dots, 2n$$

and consider the mapping

$$\alpha(\xi) := (I_M(Gf_\xi), \langle L_{r+1}, f_\xi\rangle, \dots, \langle L_{2n}, f_\xi\rangle).$$

By the Borsuk Theorem there exists $\xi^* \in S^{2n}$ for which $\alpha(\xi^*) = 0$. Since $\operatorname{Ker} J = 0$ there exists a trigonometric polynomial $p^* \in \mathcal{T}_M$ such that

$$\langle l_s, p^*\rangle = -\langle l_s, f_{\xi^*}\rangle, \quad s = 1, \dots, r.$$

For $r+1 \le j \le 2n$ we have

$$\langle l_j, p^* + f_{\xi^*}\rangle = \left\langle L_j + \sum_{s=1}^{r} c_{js} l_s, p^* + f_{\xi^*}\right\rangle = \sum_{s=1}^{r} c_{js}\langle l_s, p^* + f_{\xi^*}\rangle = 0.$$

Consequently

$$p^* + f_{\xi^*} \in X^{2n}.$$

Thus

$$\sup_{f \in \mathcal{F}_\infty \cap X^{2n}} \|f\|_\infty \ge \|p^* + f_{\xi^*}\|_\infty \ge \inf_{\substack{p \in \mathcal{T}_M \\ \eta \in \Theta_{2n},\ Gh_\eta \perp \mathcal{T}_M}} \|p + \Phi h_\eta\|_\infty = \|\Phi h_n\|_\infty.$$

Hence

$$d^{2n}(\mathcal{F}_\infty, L_\infty) \ge \|\Phi h_n\|_\infty.$$

From inequalities (2) and monotonicity of n-widths it follows that it remains to estimate the upper bound for $\lambda_{2n-1}(\mathcal{F}_\infty, L_\infty)$. This estimation follows from Lemma 3. $\square$

3. Examples of classes $\mathcal{F}_\infty$

For $\Omega \in L_1$ set

$$W_M(\Omega) := \{p + \Omega * x \mid p \in \mathcal{T}_M,\ x \in BL_\infty,\ x \perp \mathcal{T}_M\}.$$

We shall call a function $\Omega \in L_1$ a *cyclic variation diminishing kernel* if for all $x \in L_\infty$ such that $x \perp \mathcal{T}_M$, $x \not\equiv 0$, and for all $p \in \mathcal{T}_M$ the inequality

$$S(p + \Omega * x) \le S(x)$$

holds. It is evident that operator $Fx = \Omega * x$ belongs to the class $CVD(M, M, Id)$ (Id is identical operator). Consequently

$$W_M(\Omega) = \mathcal{F}_\infty(\Phi, M),$$

where $\Phi x = Fx$ ($k = 0$, $G = Id$).

For $M = \emptyset$ cyclic variation diminishing kernels are called *cyclic variation diminishing density functions* or *CVD kernels*. For classes of functions represented in the form of convolution with such type kernels Theorem 4 was proved by A. Pinkus [1, p. 179].

Let $Q(D)$ be a differential polynomial with fixed real coefficients

$$Q(D) = D^{\deg Q} + \sum_{m=0}^{\deg Q - 1} a_m D^m, \quad D = \frac{d}{dx}.$$

Set

$$\Omega_Q(t) := \sum_{\substack{m\in\mathbb{Z} \\ Q(im)\neq 0}} \frac{e^{imt}}{Q(im)}.$$

Let

$$M = \{\, m \in \mathbb{Z} \mid Q(im) = 0 \,\}.$$

Then

$$W_M(\Omega_Q) = W_\infty^Q := \{\, f \mid f^{(\deg Q-1)} \text{ abs. cont., } \|Q(D)f\|_\infty \leq 1 \,\}.$$

Denote by $h(Q)$ the maximum of imaginary part of zeros of the polynomial Q. From the papers [8] and [3] it follows that when $h(Q) \leq 1/2$ the kernel Ω_Q is a cyclyc variation diminishing kernel (in this case $M = 0$ if $Q(0) = 0$ and $M = \emptyset$ if $Q(0) \neq 0$). Thus for $h(Q) \leq 1/2$ Theorem 4 holds for the class W_∞^Q.

If $Q(D) = D^r$, then the class W_∞^Q coincides with the Sobolev class W_∞^r for which the exact values of studying n-widths were obtained by V. M. Tikhomirov [9].

There exist polynomials Q whose zeros are arbitrary far from the real axis but the corresponding kernels Ω_Q are cyclic variation diminishing kernels. For example, for

$$Q(D) = D(D^2 + 1^2)\dots(D^2 + m^2) \tag{11}$$

the kernel Ω_Q are a cyclic variation diminishing kernel with $M = \{0, \pm 1, \dots \pm m\}$ (see [10]).

In the general case a polynomial $Q(D)$ can be represented in the form

$$Q(D) = \prod_{j=1}^{k} Q_j(D), \tag{12}$$

where $Q_j(D)$ are differential polynomials with real coefficients such that $\deg Q_j \leq 2$. From the paper [3] it follows that $\Omega_{Q_j} \in \mathcal{K}(M_j, \delta_j)$, where

$$M_j = \{\, m \in \mathbb{Z} \mid Q_j(im) = 0 \,\}, \qquad \delta_j = \pi / h(Q_j). \tag{13}$$

Therefore, upon putting $\Phi x = \Omega_{Q_k} * \ldots * \Omega_{Q_1} * x$, we have

$$W^Q_\infty = \mathcal{F}_\infty(\Phi, M), \qquad M = \bigcup_{j=1}^k M_j \quad (F = G = Id,\ M_0 = \emptyset).$$

Denote by h^β_∞ (H^β_∞) the class of real-valued 2π-periodic functions analytically continued in the strip $S_\beta := \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \beta\}$ and satisfy in it the condition

$$|\operatorname{Re} f(z)| \le 1 \qquad (|f(z)| \le 1).$$

It is well known (see, for example, [11, p. 269]) that

$$h^\beta_\infty = \{\, K_\beta * x \mid x \in BL_\infty \,\},$$

where

$$K_\beta(t) = 1 + 2\sum_{m=1}^\infty \frac{\cos mt}{\operatorname{ch} m\beta}.$$

The kernel K_β is a CVD-kernel (see [1, p. 62]). Thus

$$h^\beta_\infty = W_\emptyset(K_\beta).$$

Consider now the class H^β_∞. In view of the fact that the function $w = \dfrac{4}{\pi}\operatorname{arctg} z$ conformally maps the interior of the unit disk on the strip $|\operatorname{Re} w| < 1$ we have

$$f(\cdot) \in H^\beta_\infty \Longleftrightarrow \frac{4}{\pi}\operatorname{arctg} f(\cdot) \in h^\beta_\infty.$$

Therefore

$$H^\beta_\infty = \{\, \varphi(K_\beta * x) \mid x \in BL_\infty \,\},$$

where $\varphi(w) = \operatorname{tg}\dfrac{\pi}{4}w$. Since $\operatorname{sign}\varphi(w) = \varphi(\operatorname{sign} w)$ for $w \in [-1, 1]$ it is not difficult to see that the operator

$$Fx = \varphi(K_\beta * x)$$

belongs to the set $CVD(\emptyset, M, F)$ for all M. Thus

$$H^\beta_\infty = \mathcal{F}_\infty(\Phi, \emptyset), \qquad \Phi x = Fx \quad (k = 0,\ G = F).$$

For a differential polynomial with real coefficients $Q(D)$ denote by $h^{Q,\beta}_\infty$ ($H^{Q,\beta}_\infty$) the class of real-valued 2π-periodic functions analytically continued in the strip S_β and satisfy the condition

$$Q(D)f \in h^\beta_\infty \qquad (Q(D)f \in H^\beta_\infty).$$

By notation (12) and (13) we have

$$h^{Q,\beta}_\infty = \mathcal{F}(\Phi_1, M), \qquad H^{Q,\beta}_\infty = \mathcal{F}(\Phi_2, M),$$

where $\Phi_1 x = \Omega_{Q_k} * \ldots * \Omega_{Q_1} * K_\beta * x$, $\Phi_2 x = \Omega_{Q_k} * \ldots * \Omega_{Q_1} * \varphi(K_\beta * x)$, $M = \bigcup_{j=1}^k M_j$ (in the first case $G = Id$ and in the second one $G = F$).

Thus from Theorem 4 it follows

Theorem 5. *For all* $n > 2h(Q)$

$$\begin{aligned} d_{2n}(W, L_\infty) &= \lambda_{2n}(W, L_\infty) = d^{2n}(W, L_\infty) = i_{2n}(W, L_\infty) \\ &= d_{2n-1}(W, L_\infty) = \lambda_{2n-1}(W, L_\infty) = d^{2n-1}(W, L_\infty) = i_{2n-1}(W, L_\infty) \\ &= \begin{cases} \|\Omega_Q * h_n\|_\infty, & W = W_\infty^Q, \\ \|\Omega_Q * K_\beta * h_n\|_\infty, & W = h_\infty^{Q,\beta}, \\ \|\Omega_Q * \varphi(K_\beta * h_n)\|_\infty, & W = H_\infty^{Q,\beta}. \end{cases} \end{aligned}$$

Moreover, the Fourier coefficients $a_0(f), a_1(f), b_1(f), \dots, a_{n-1}(f), b_{n-1}(f)$ *are optimal functionals for the values* i_{2n-1} *and* i_{2n}.

For the class W_∞^Q (for the Kolmogorov, linear, and Gel'fand n-widths) the conclusion of this theorem was obtained in the paper [3]. The even n-widths of the class $H_\infty^{Q,\beta}$ for $Q(D) = D^r$ were obtained in the paper [6].

The idea of the construction of a general theory for smooth and analytic cases was expressed many times by V. M. Tikhomirov whom the author is obliged for helpful discussions.

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