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ON BEST HARMONIC SYNTHESIS OF PERIODIC FUNCTIONS

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Abstract. In this paper, we construct optimal methods of recovery of periodic functions from a known (exact or inexact) finite family of their Fourier coefficients. The proposed approach to constructing recovery methods is compared with the approach based on the Tikhonov regularization method.

Introduction

This paper is concerned with construction of best (optimal) methods of recovery of functions from their approximately given Fourier coefficients. Such methods are built simultaneously for an entire class of functions, and this is what determines their important specific feature—they do not use, in general, all the Fourier coefficients available for measurement (exact or inexact), and those which are used are somehow "smoothed." This is in full agreement with what happens in engineering practice related to digital signal processing: high frequencies are thrown away, and low ones are filtered in some way.

The approach presented here to the definition of an optimal method goes back conceptually to A. N. Kolmogorov's works on finding a best subspace among all the subspaces of fixed dimension approximating a given class of functions. This approach, which might have been entitled "the Kolmogorov regularization," is a certain alternative to the regularization in the sense of A. N. Tikhonov, which deals with individual objects, does not take into account concrete values of the measurement error (which may well not be close to zero), and is not concerned with the problem of best methods.

The paper is structured as follows. We start by considering one example of Tikhonov regularization in the problem of recovery of a function at a point from inexactly given Fourier coefficients. This example is taken from one classical graduate calculus text [1]. We put forward our own variant of the solution of this problem, which resides in the Kolmogorov regularization. We give exact solutions for a number of optimal recovery problems for functions and their derivatives in the mean square metric from a finite family of Fourier coefficients given exactly or inexactly. Similar problems were studied in [2–8], *viz.* the problem of optimal recovery of functions (periodic or defined on $\mathbb{R}^d$) and of operators of these functions from inexactly given spectral data.

1. On One Problem of Recovery of a Function from Inexactly Given Fourier Coefficients

In [1], the following problem was addressed. Let a 2π-periodic function $x(\cdot)$ be such that its Fourier series

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$$

converges to $x(\cdot)$ uniformly, where

$$\begin{aligned} a_k &= a_k\big(x(\cdot)\big) = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\cos kt\,dt, \quad k = 0, 1, 2, \dots, \\ b_k &= b_k\big(x(\cdot)\big) = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\sin kt\,dt, \quad k = 1, 2, \dots . \end{aligned} \tag{1}$$

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Suppose that, instead of the exact values of Fourier coefficients of a function $x(\cdot)$, we are given their approximate values $\tilde{a}_k$ and $\tilde{b}_k$ such that

$$\frac{(a_0-\tilde{a}_0)^2}{2}+\sum_{k=1}^{\infty}\big((a_k-\tilde{a}_k)^2+(b_k-\tilde{b}_k)^2\big)\le\delta^2,$$

where $\delta>0$. The problem is to recover the value of the function $x(\cdot)$ at some point τ from this information.

It is easily proved that, however rapidly the given series may converge to $x(\tau)$ and however small $\delta>0$ may be, it is possible to specify numbers $\tilde{a}_k$ and $\tilde{b}_k$ such that the sum of the series

$$\frac{\tilde{a}_0}{2}+\sum_{k=1}^{\infty}(\tilde{a}_k\cos kt+\tilde{b}_k\sin kt)$$

differs from $x(\tau)$ by any preassigned number (or even diverge).

The following approach is proposed for solving the recovery problem of a function $x(\cdot)$ at a point τ. As an approximate value of $x(\tau)$ one considers the series

$$\frac{\tilde{a}_0}{2}+\sum_{k=1}^{\infty}\frac{1}{1+\alpha k^2}(\tilde{a}_k\cos k\tau+\tilde{b}_k\sin k\tau),$$

where α is of the same order of smallness as δ (for example, one may take $\alpha=\delta$). It is shown that if $x(\cdot)\in L_2(\mathbb{T})$ and if $x(\cdot)$ is continuous at τ, then this series converges to $x(\tau)$ as $\alpha\to 0$ (here and henceforth, $\mathbb{T}$ will denote the closed interval $[-\pi,\pi]$ with the endpoints identified).

The concluding remark of [1] states that "should we, with the aim at getting the fullest overall picture about a physical process of interest, unboundedly refine the accuracy of an instrument, or should the right approach to this objective rather involve development of such mathematical methods for analyzing measurements that are capable, within the *available measurement accuracy* of frequency characteristics, of extracting the *maximum amount of information* about the process under study" (emphasis added)?

The approach that we call Kolmogorov regularization does require some *a priori* information about the function $x(\cdot)$. Consequently, under this approach, one has the benefit of taking into account the given measurement accuracy and may pose the problem of finding the best method among all possible ones.

Now we proceed to the precise statement. We let $\mathcal{W}_2^1(\mathbb{T})$ denote the space of absolutely continuous 2π-periodic functions $x(\cdot)$ for which the derivative $\dot{x}(\cdot)$ lies in $L_2(\mathbb{T})$. The norm of a function $x(\cdot)$ in $L_2(\mathbb{T})$ is defined as follows:

$$\|x(\cdot)\|_{L_2(\mathbb{T})}=\left(\frac{1}{\pi}\int_{-\pi}^{\pi}|x(t)|^2\,dt\right)^{1/2}.$$

If $x(\cdot)\in\mathcal{W}_2^1(\mathbb{T})$, then at each point $t\in\mathbb{T}$ the function $x(\cdot)$ can be expanded into a Fourier series, which converges to $x(\cdot)$ uniformly.

In the space $\mathcal{W}_2^1(\mathbb{T})$ we consider the class of functions

$$W_2^1(\mathbb{T})=\big\{x(\cdot)\in\mathcal{W}_2^1(\mathbb{T})\colon\|\dot{x}(\cdot)\|_{L_2(\mathbb{T})}\le 1\big\}.$$

As usual, l_2 denotes the space of square summable real sequences with the inner product

$$\langle y,y'\rangle=\frac{a_0a_0'}{2}+\sum_{k=1}^{\infty}y_ky_k',$$

where $y=(y_0,y_1,\dots)$, $y'=(y_0',y_1',\dots)$; the corresponding norm is as follows:

$$\|y\|_{l_2}=\left(\frac{a_0^2}{2}+\sum_{k=1}^{\infty}y_k^2\right)^{1/2}.$$

If a function $x(\cdot)$ belongs to the space $\mathcal{W}_2^1(\mathbb{T})$, then its Fourier coefficients are known to lie in l_2. Let $F\colon \mathcal{W}_2^1(\mathbb{T}) \to l_2$ be the Fourier transform of $x(\cdot)$; i.e., $Fx(\cdot) = \big(a_0\big(x(\cdot)\big), a_1\big(x(\cdot)\big), b_1\big(x(\cdot)\big), \dots\big)$ is the sequence of Fourier coefficients of a function $x(\cdot)$.

Assume that about each function $x(\cdot) \in W_2^1(\mathbb{T})$ we know approximate values of its Fourier coefficients. Namely, we know the vector

$$y = (\tilde{a}_0, \tilde{a}_1, \tilde{b}_1 \dots) \in l_2$$

such that

$$\|Fx(\cdot) - y\|_{l_2} \le \delta,$$

where $\delta > 0$.

Any recovery method $x(\tau)$ is assumed to associate with a vector (observation) y a number, which in accordance with this method is some approximation to $x(\tau)$. So, any method is some function $\varphi\colon l_2 \to \mathbb{R}$. By the *error* of a given method we shall understand the quantity

$$e(W_2^1(\mathbb{T}), F, \delta, \varphi) = \sup_{\substack{x(\cdot)\in W_2^1(\mathbb{T}),\ y\in l_2 \\ \|Fx(\cdot)-y\|_{l_2}\le\delta}} |x(\tau) - \varphi(y)|.$$

We shall be concerned both with the quantity

$$E(W_2^1(\mathbb{T}), F, \delta) = \inf_{\varphi\colon l_2\to\mathbb{R}} e(W_2^1(\mathbb{T}), F, \delta, \varphi)$$

known as the *optimal recovery error*, and with the *optimal recovery method* $\hat{\varphi}$ at which the infimum is attained; i.e.,

$$E(W_2^1(\mathbb{T}), F, \delta) = e(W_2^1(\mathbb{T}), F, \delta, \hat{\varphi}).$$

Theorem 1. *For any $\delta > 0$,*

$$E(W_2^1(\mathbb{T}), F, \delta) = (\hat{a} + \delta^2)\bigg(\sum_{k=1}^{\infty} \frac{k^2}{(1+\hat{a}k^2)^2}\bigg)^{1/2},$$

where $\hat{a} = \hat{a}(\delta)$ is a unique solution of the equation

$$\frac{\dfrac{1}{2} + \sum\limits_{k=1}^{\infty} \dfrac{1}{(1+ak^2)^2}}{\sum\limits_{k=1}^{\infty} \dfrac{k^2}{(1+ak^2)^2}} = \delta^2.$$

Moreover, the method

$$\hat{\varphi}(y) = \frac{\tilde{a}_0}{2} + \sum_{k=1}^{\infty} \frac{1}{1+\hat{a}k^2}(\tilde{a}_k \cos k\tau + \tilde{b}_k \sin k\tau)$$

is optimal.

As is evident from the statement of the theorem, for any $\delta > 0$ the regularization method of [1] is optimal in the class $W_2^1(\mathbb{T})$ with $\alpha = \hat{a}(\delta)$. Moreover, the minimal estimation error $x(\tau)$ is given by the quantity $E(W_2^1(\mathbb{T}), F, \delta)$, which tends to zero as $\delta \to 0$.

We also note that the information about a function $x(\cdot) \in W_2^1(\mathbb{T})$ that we are given a vector $y \in l_2$ such that $\|Fx(\cdot) - y\|_{l_2} \le \delta$ is equivalent, by the Parseval equality, to the information that we are given a function $y(\cdot) \in L_2(\mathbb{T})$ such that $\|x(\cdot) - y(\cdot)\|_{L_2(\mathbb{T})} \le \delta$.

Proof of Theorem 1. We first claim that the optimal recovery error $E(W_2^1(\mathbb{T}), F, \delta)$ is not smaller than the value of the problem

$$x(\tau) \to \max, \quad x(\cdot) \in W_2^1(\mathbb{T}), \quad \|Fx(\cdot)\|_{l_2} \le \delta, \tag{2}$$

i.e., not smaller than the supremum of the functional to be maximized under given constraints.

Indeed, let $\varphi\colon l_2 \to \mathbb{R}$ be an arbitrary recovery method, $x(\cdot) \in W_2^1(\mathbb{T})$ (and hence $-x(\cdot) \in W_2^1(\mathbb{T})$), and let $\|Fx(\cdot)\|_{l_2} \le \delta$. We have

$$2x(\tau) \le \big|x(\tau) - \varphi(0) - \big(-x(\tau) - \varphi(0)\big)\big| \le |x(\tau) - \varphi(0)| + |-x(\tau) - \varphi(0)| \\ \le 2 \sup_{\substack{x(\cdot)\in W_2^1(\mathbb{T}),\\ \|Fx(\cdot)\|_{l_2}\le\delta}} |x(\tau) - \varphi(0)| \le 2 \sup_{\substack{x(\cdot)\in W_2^1(\mathbb{T}),\ y\in l_2,\\ \|Fx(\cdot)-y\|_{l_2}\le\delta}} |x(\tau) - \varphi(y)| = 2e(W_2^1(\mathbb{T}), F, \delta, \varphi).$$

The result required is obtained by taking, first, the supremum on the left over all such $x(\cdot)$, and second, on the right over all methods φ.

Now let us find the value of problem (2) (thereby obtaining a lower estimate for the optimal recovery error). To this aim, it is convenient to rewrite the problem in terms of the Fourier coefficients. If $x(\cdot) \in \mathcal{W}_2^1(\mathbb{T})$, then, by the Parseval equality,

$$\|x(\cdot)\|^2_{L_2(\mathbb{T})} = \|Fx(\cdot)\|^2_{l_2} = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2),$$
$$\|\dot{x}(\cdot)\|^2_{L_2(\mathbb{T})} = \|F\dot{x}(\cdot)\|^2_{l_2} = \sum_{k=1}^{\infty} k^2(a_k^2 + b_k^2),$$

where $a_k = a_k\big(x(\cdot)\big)$, $k \in \mathbb{Z}_+$, and $b_k = b_k\big(x(\cdot)\big)$, $k \in \mathbb{N}$. Now problem (2) can be rewritten as

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\tau + b_k \sin k\tau) \to \max, \quad \sum_{k=1}^{\infty} k^2(a_k^2 + b_k^2) \le 1, \\ \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \le \delta^2. \tag{3}$$

Note that problem (3) *qua* a problem on the l_2-sequence $(a_0, a_1, b_1, \dots)$ (in what follows, we denote such sequence by $\{a_k, b_k\}$) for which the sequence $\{ka_k, kb_k\}$ also lies in l_2 (the space of such sequences will be denoted by l_2^1) is equivalent to problem (2) in the sense that if $\{a_k, b_k\} \in l_2^1$, then there exists a unique function $x(\cdot) \in \mathcal{W}_2^1(\mathbb{T})$ for which $\{a_k, b_k\}$ is the sequence of its Fourier coefficients. The constraints in (3) are carried over into the constraints in (2), and $x(\tau)$ is the functional to be maximized in (3).

If we find a solution of problem (3), then we shall find its value. This is a convex problem. Let us employ sufficiency conditions for existence of a solution. The Lagrange function of problem (3) is as follows:

$$\mathcal{L}(\{a_k, b_k\}, \lambda_1, \lambda_2) = -\frac{a_0}{2} - \sum_{k=1}^{\infty}(a_k \cos k\tau + b_k \sin k\tau) + \lambda_1 \sum_{k=1}^{\infty} k^2(a_k^2 + b_k^2) + \lambda_2\bigg(\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)\bigg).$$

If we find an admissible sequence $\{\hat{a}_k, \hat{b}_k\}$ for (3) (i.e., a sequence for which the constraints are satisfied) and Lagrange multipliers $\hat{\lambda}_1 \ge 0$ and $\hat{\lambda}_2 \ge 0$ such that

$$\text{(a)} \quad \min_{\{a_k,b_k\}\in l_2^1} \mathcal{L}(\{a_k, b_k\}, \hat{\lambda}_1, \hat{\lambda}_2) = \mathcal{L}(\{\hat{a}_k, \hat{b}_k\}, \hat{\lambda}_1, \hat{\lambda}_2),$$
$$\text{(b)} \quad \hat{\lambda}_1\bigg(\sum_{k=1}^{\infty} k^2(\hat{a}_k^2 + \hat{b}_k^2) - 1\bigg) = 0, \quad \hat{\lambda}_2\bigg(\frac{\hat{a}_0^2}{2} + \sum_{k=1}^{\infty}(\hat{a}_k^2 + \hat{b}_k^2) - \delta^2\bigg) = 0,$$

then $\{\hat{a}_k, \hat{b}_k\}$ is a solution of problem (3).

This can be easily checked. Indeed, for any admissible sequence $\{a_k, b_k\}$, using conditions (a) and (b), we have that

$$
\begin{aligned}
&-\frac{a_0}{2}-\sum_{k=1}^{\infty}(a_k\cos k\tau+b_k\sin k\tau)\geq -\frac{a_0}{2}-\sum_{k=1}^{\infty}(a_k\cos k\tau+b_k\sin k\tau)\\
&+\hat\lambda_1\bigg(\sum_{k=1}^{\infty}k^2(a_k^2+b_k^2)-1\bigg)+\hat\lambda_2\bigg(\frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)-\delta^2\bigg)\\
&=\mathcal{L}(\{a_k,b_k\},\hat\lambda_1,\hat\lambda_2)-\hat\lambda_1-\hat\lambda_2\delta^2\geq\mathcal{L}(\{\hat a_k,\hat b_k\},\hat\lambda_1,\hat\lambda_2)-\hat\lambda_1-\hat\lambda_2\delta^2\\
&=\frac{\hat a_0}{2}-\sum_{k=1}^{\infty}(\hat a_k\cos k\tau+\hat b_k\sin k\tau)+\hat\lambda_1\bigg(\sum_{k+1}^{\infty}k^2(\hat a_k^2+\hat b_k^2)-1\bigg)\\
&+\hat\lambda_2\bigg(\frac{\hat a_0^2}{2}+\sum_{k=1}^{\infty}(\hat a_k^2+\hat b_k^2)-\delta^2\bigg)=-\frac{\hat a_0}{2}-\sum_{k=1}^{\infty}(\hat a_k\cos k\tau+\hat b_k\sin k\tau),
\end{aligned}
$$

i.e., $\{\hat a_k, \hat b_k\}$ is a solution of problem (3).

Proceeding heuristically, we next try to understand what form should a sequence $\{\hat a_k, \hat b_k\}$ and Lagrange multipliers $\hat\lambda_1 \geq 0$ and $\hat\lambda_2 \geq 0$ have if they satisfy the conditions (a) and (b). The Lagrange function *qua* a function of a sequence $\{a_k, b_k\}$ is smooth, the derivative of this function vanishing at a point $\{\hat a_k, \hat b_k\}$, as is seen from condition (a). Formally calculating this derivative, we see that any sequence $\{a_k, b_k\}$ must satisfy the following identity:

$$
-\frac{a_0}{2}-\sum_{k=1}^{\infty}(a_k\cos k\tau+b_k\sin k\tau)+2\hat\lambda_1\sum_{k=1}^{\infty}k^2(\hat a_k a_k+\hat b_k b_k)+2\hat\lambda_2\bigg(\frac{\hat a_0 a_0}{2}+\sum_{k=1}^{\infty}(\hat a_k a_k+\hat b_k b_k)\bigg)=0. \quad (4)
$$

Hence, taking sequences of the form $(1, 0, \ldots), (0, 1, 0, \ldots), \ldots$, we obtain that

$$
\hat a_0=\frac{1}{2\hat\lambda_2},\quad \hat a_k=\frac{\cos k\tau}{2(\hat\lambda_1 k^2+\hat\lambda_2)},\quad \hat b_k=\frac{\sin k\tau}{2(\hat\lambda_1 k^2+\hat\lambda_2)},\quad k\in\mathbb{N}. \quad (5)
$$

If we assume that $\hat\lambda_1 > 0$ and $\hat\lambda_2 > 0$, then, in order to satisfy condition (b), it is necessary that the bracketed expressions in this condition vanish; i.e.,

$$
\sum_{k=1}^{\infty}k^2(\hat a_k^2+\hat b_k^2)=\frac{1}{4}\sum_{k=1}^{\infty}\frac{k^2}{(\hat\lambda_1 k^2+\hat\lambda_2)^2}=1
$$

and

$$
\frac{\hat a_0^2}{2}+\sum_{k=1}^{\infty}(\hat a_k^2+\hat b_k^2)=\frac{1}{4}\bigg(\frac{1}{2\hat\lambda_2^2}+\sum_{k=1}^{\infty}\frac{1}{(\hat\lambda_1 k^2+\hat\lambda_2)^2}\bigg)=\delta^2.
$$

Denoting $a = \hat\lambda_1/\hat\lambda_2$ and dividing the second equation by the first one gives

$$
\frac{\dfrac{1}{2}+\sum\limits_{k=1}^{\infty}\dfrac{1}{(1+ak^2)^2}}{\sum\limits_{k=1}^{\infty}\dfrac{k^2}{(1+ak^2)^2}}=\delta^2. \quad (6)
$$

We shall make the above argument rigorous. The expression on the left of (6) defines a function f (of variable a) on $(0, \infty)$. We claim that, for any $\delta > 0$, there exists a unique $\hat a = \hat a(\delta) > 0$ such that $f(\hat a) = \delta^2$. To this aim, we first check that $\lim\limits_{a\to 0} f(a) = 0$ and $\lim\limits_{a\to+\infty} f(a) = +\infty$.

Let $\varepsilon > 0$ and $n \geq 6$ be such that $1/n < \varepsilon$. Since, clearly,

$$\lim_{a\to 0}\left(\frac{1}{2} + \sum_{k=1}^{n} \frac{1}{(1+ak^2)^2}\right) = \frac{1}{2} + n < 2n+1$$

and

$$\lim_{a\to 0}\sum_{k=1}^{n} \frac{k^2}{(1+ak^2)^2} = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} > n(2n+1),$$

there exists an $a_0 > 0$ such that, for all $0 < a < a_0$,

$$\frac{1}{2} + \sum_{k=1}^{n} \frac{1}{(1+ak^2)^2} < 2n+1, \qquad \sum_{k=1}^{n} \frac{k^2}{(1+ak^2)^2} > n(2n+1).$$

It follows that

$$\frac{1}{2} + \sum_{k=1}^{n} \frac{1}{(1+ak^2)^2} < \frac{1}{n}\sum_{k=1}^{n} \frac{k^2}{(1+ak^2)^2}.$$

Next, it is clear that

$$\sum_{k=n+1}^{\infty} \frac{1}{(1+ak^2)^2} < \frac{1}{n^2}\sum_{k=n+1}^{\infty} \frac{k^2}{(1+ak^2)^2} < \frac{1}{n}\sum_{k=n+1}^{\infty} \frac{k^2}{(1+ak^2)^2}.$$

Adding these inequalities and dividing one by the other, we see that, for all $0 < a < a_0$,

$$f(a) = \frac{\frac{1}{2} + \sum\limits_{k=1}^{\infty} \frac{1}{(1+ak^2)^2}}{\sum\limits_{k=1}^{\infty} \frac{k^2}{(1+ak^2)^2}} < \frac{1}{n} < \varepsilon,$$

i.e., $\lim\limits_{a\to 0} f(a) = 0$.

On the other hand, for any $a > 0$,

$$f(a) = \frac{\frac{a^2}{2} + \sum\limits_{k=1}^{\infty} \frac{1}{(a^{-1}+k^2)^2}}{\sum\limits_{k=1}^{\infty} \frac{k^2}{(a^{-1}+k^2)^2}} > \frac{a^2}{2\sum\limits_{k=1}^{\infty} \frac{1}{k^2}},$$

which immediately implies that $\lim\limits_{a\to+\infty} f(a) = +\infty$.

To prove the uniqueness we need to show that f is strictly increasing on $(0, \infty)$. To this aim it suffices to show that f is strictly increasing on any interval $[a_0, a_1]$, where $0 < a_0 < a_1 < \infty$. On each such interval the standard conditions for term-by-term differentiation of the series involved in the definition of f are satisfied, and hence, for any $a \in [a_0, a_1]$,

$$f'(a) = \frac{2g(a)}{\left(\sum\limits_{k=1}^{\infty} \frac{k^2}{(1+ak^2)^2}\right)^2},$$

where

$$g(a) = -\sum_{k=1}^{\infty} \frac{k^2}{(1+ak^2)^3}\sum_{k=1}^{\infty} \frac{k^2}{(1+ak^2)^2} + \left(\frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{(1+ak^2)^2}\right)\sum_{k=1}^{\infty} \frac{k^4}{(1+ak^2)^3}$$

$$> -\sum_{k=1}^{\infty} \frac{k^2}{(1+ak^2)^3}\sum_{k=1}^{\infty} \frac{k^2}{(1+ak^2)^2} + \sum_{k=1}^{\infty} \frac{1}{(1+ak^2)^2}\sum_{k=1}^{\infty} \frac{k^4}{(1+ak^2)^3} = -\sum_{j,k=1}^{\infty} \alpha_{jk} + \sum_{j,k=1}^{\infty} \beta_{jk}$$

and

$$\alpha_{jk} = \frac{j^2k^2}{(1+aj^2)^3(1+ak^2)^2}, \quad \beta_{jk} = \frac{k^4}{(1+aj^2)^2(1+ak^2)^3}.$$

Hence,

$$-\alpha_{jk} + \beta_{kj} = \frac{j^2(j^2-k^2)}{(1+aj^2)^3(1+ak^2)^2}, \quad \alpha_{kj} + \beta_{jk} = \frac{k^2(k^2-j^2)}{(1+ak^2)^3(1+aj^2)^2},$$

and now

$$-\alpha_{jk} + \beta_{kj} - \alpha_{kj} + \beta_{jk} = \frac{j^2-k^2}{(1+aj^2)^2(1+ak^2)^2}\left(\frac{j^2}{1+aj^2} - \frac{k^2}{1+ak^2}\right) = \frac{(j^2-k^2)^2}{(1+aj^2)^3(1+ak^2)^3} \ge 0.$$

This proves that $g(a) > 0$, and so $f'(a) > 0$ for any $a \in (0,\infty)$, completing the proof of uniqueness.

Let $\delta > 0$, and let $\hat{a} = \hat{a}(\delta)$ be a unique solution of the equation $f(a) = \delta^2$. Further, let

$$\hat{\lambda}_2 = \hat{\lambda}_2(\hat{a}) = \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{k^2}{(1+\hat{a}k^2)^2}\right)^{1/2},$$

$\hat{\lambda}_1 = \hat{\lambda}_1(\hat{a}) = \hat{a}\hat{\lambda}_2$, and let $\hat{a}_0$, $\hat{a}_k$, $\hat{b}_k$, $k \in \mathbb{N}$, be defined by (5) with these $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

It is readily seen that the sequence $\{\hat{a}_k, \hat{b}_k\}$ lies in l_2^1. Next, by elementary calculation,

$$\sum_{k=1}^{\infty} k^2(\hat{a}_k^2 + \hat{b}_k^2) = 1, \quad \frac{\hat{a}_0^2}{2} + \sum_{k=1}^{\infty}(\hat{a}_k^2 + \hat{b}_k^2) = \delta^2, \tag{7}$$

proving that $\{\hat{a}_k, \hat{b}_k\}$ is admissible for problem (3).

The second and third sums in (4) are the inner products of, respectively, the sequences $\{k\hat{a}_k, k\hat{b}_k\}$, $\{ka_k, kb_k\}$ and $\{\hat{a}_k, \hat{b}_k\}$, $\{a_k, b_k\}$, and hence are well defined. Substituting $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{a}_0$, $\hat{a}_k$, $\hat{b}_k$, $k \in \mathbb{N}$, into (4), we see, after elementary calculation, that this equality is satisfied for any sequence $\{a_k, b_k\} \in l_2^1$.

Let a function $\hat{x}(\cdot) \in \mathcal{W}_2^1(\mathbb{T})$ be such that $\{\hat{a}_k, \hat{b}_k\}$ is the sequence of its Fourier coefficients. Using (4), it follows by the Parseval equality that, for any function $x(\cdot) \in \mathcal{W}_2^1(\mathbb{T})$,

$$x(\tau) = 2\hat{\lambda}_1\langle F\dot{\hat{x}}(\cdot), F\dot{x}(\cdot)\rangle + 2\hat{\lambda}_2\langle F\hat{x}(\cdot), Fx(\cdot)\rangle. \tag{8}$$

Substituting here the function $\hat{x}(\cdot)$ for $x(\cdot)$, we see that

$$\hat{x}(\tau) = (\hat{a} + \delta^2)\left(\sum_{k=1}^{\infty} \frac{k^2}{(1+\hat{a}k^2)^2}\right)^{1/2}.$$

The quantity on the left is the value of the functional to be maximized in problem (2) on the function $\hat{x}(\cdot)$, and hence the value of the problem itself is not smaller than this quantity.

As shown above, the optimal recovery error $E(W_2^1(\mathbb{T}), F, \delta)$ is not smaller than the value of problem (2), and hence,

$$E(W_2^1(\mathbb{T}), F, \delta) \ge (\hat{a} + \delta^2)\left(\sum_{k=1}^{\infty} \frac{k^2}{(1+\hat{a}k^2)^2}\right)^{1/2}. \tag{9}$$

Note that the function $\hat{x}(\cdot)$ is actually a solution of problem (2), but this will play no role here and so we do not dwell on this.

Now let us estimate from above the quantity $E(W_2^1(\mathbb{T}), F, \delta)$ and check that the method $\hat{\varphi}$ from the statement of the theorem is optimal. To estimate the error of this method we consider $x(\cdot) \in W_2^1(\mathbb{T})$ and $y = (\tilde{a}_0, \tilde{a}_1, \tilde{b}_1, \ldots) \in l_2$ such that $\|Fx(\cdot) - y\|_{l_2} \le \delta$. It is easily verified that $\hat{\varphi}(y) = 2\hat{\lambda}_2\langle F\hat{x}(\cdot), y\rangle$.

By (8), and using the Cauchy–Schwarz inequality, the Parseval equality, equalities (7), and conditions on $x(\cdot)$ and y, we have that

$$\begin{aligned}|x(\tau)-\hat\varphi(y)| &= |2\hat\lambda_1\langle F\dot{\hat x}(\cdot),F\dot x(\cdot)\rangle+2\hat\lambda_2\langle F\hat x(\cdot),Fx(\cdot)-y\rangle| \\ &\le 2\hat\lambda_1\|F\dot{\hat x}(\cdot)\|_{l_2}\|F\dot x(\cdot)\|_{l_2}+2\hat\lambda_2\|F\hat x(\cdot)\|_{l_2}\|Fx(\cdot)-y\|_{l_2} \\ &\le 2\hat\lambda_2\hat a+2\hat\lambda_2\delta^2=(\hat a+\delta^2)\biggl(\sum_{k=1}^{\infty}\frac{k^2}{(1+\hat a k^2)^2}\biggr)^{1/2}.\end{aligned}$$

Hence, by (9) the method $\hat\varphi$ is optimal, giving the desired expression for the quantity $E(W_2^1(\mathbb{T}),F,\delta)$. $\square$

2. Optimal Recovery of Functions and Their Derivatives from Finitely Many Fourier Coefficients

In the previous section, we were given approximate values of Fourier coefficients in the l_2-metric. From practical considerations, it is more natural when there a possibility to approximately measure each Fourier coefficients of a function from a certain finite family of coefficients. In this section, we shall be concerned with this setting, and the function and its derivatives will be recovered not at a point, but rather "entirely" in the metric of $L_2(\mathbb{T})$. This gives rise to an interesting phenomenon that not all the approximate Fourier coefficients are used by the optimal recovery method. In order to demonstrate that this phenomenon is related not only to the presence of errors in input data, we first consider the case where we are given a finite family of exactly measured Fourier coefficients.

2.1. Recovery in the Mean-Square Metric from Exact Values of Fourier Coefficients. Assume that n is natural. We let $\mathcal{W}_2^n(\mathbb{T})$ denote the space of 2π-periodic functions $x(\cdot)$ for which the $(n-1)$th derivative is absolutely continuous and $x^{(n)}(\cdot)\in L_2(\mathbb{T})$. In the space $\mathcal{W}_2^n(\mathbb{T})$, we consider the class of functions

$$W_2^n(\mathbb{T})=\{x(\cdot)\in\mathcal{W}_2^n(\mathbb{T})\mid \|x^{(n)}(\cdot)\|_{L_2(\mathbb{T})}\le 1\}.$$

We pose the following problem. Let $A\subset\mathbb{Z}_+=\{0,1,2,\dots\}$, and let $B\subset\mathbb{N}=\{1,2,\dots\}$ be finite sets (of which one may possibly be empty). Assume that about each function $x(\cdot)\in W_2^n(\mathbb{T})$ we know its Fourier coefficients $\{a_k\}_{k\in A}$ and $\{b_k\}_{k\in B}$; i.e., corresponding to $x(\cdot)$ there is a tuple $F_{A,B}x(\cdot)=(\{a_k\}_{k\in A},\{b_k\}_{b\in B})$ of N numbers, where $N=\operatorname{card}A+\operatorname{card}B$. How can we best recover functions from $W_2^n(\mathbb{T})$ and their rth derivatives, $1\le r\le n-1$, in the $L_2(\mathbb{T})$-metric from this information? We proceed as follows. Any candidate method φ for recovering $x^{(r)}(\cdot)$ $(0\le r\le n-1)$ from the tuple $F_{A,B}x(\cdot)$ should associate a function $\varphi\bigl(F_{A,B}x(\cdot)\bigr)(\cdot)\in L_2(\mathbb{T})$ with this tuple; i.e., φ is a mapping from $\mathbb{R}^N$ into $L_2(\mathbb{T})$.

By the error of a method φ we understand the quantity

$$e(D^r,W_2^n(\mathbb{T}),F_{A,B},\varphi)=\sup_{x(\cdot)\in W_2^n(\mathbb{T})}\bigl\|x^{(r)}(\cdot)-\varphi\bigl(F_{A,B}x(\cdot)\bigr)(\cdot)\bigr\|_{L_2(\mathbb{T})}$$

(here D^r is the rth order differential operator and D^0 is the identity operator), which is the quantity delivering the maximum deviation on the class $W_2^n(\mathbb{T})$ of the function $x^{(r)}(\cdot)$ from the function, which "recovers" this function in accordance with this method.

As above, we will be interested in the method with smallest error. To be more precise, we are concerned with the quantity

$$E(D^r,W_2^n(\mathbb{T}),F_{A,B})=\inf_{\varphi\colon\mathbb{R}^N\to L_2(\mathbb{T})}e(D^r,W_2^n(\mathbb{T}),F_{A,B},\varphi),$$

which will be called the *optimal recovery error*; the methods on which the infimum is attained will be called *optimal recovery methods*; i.e.,

$$E(D^r,W_2^n(\mathbb{T}),F_{A,B})=e(D^r,W_2^n(\mathbb{T}),F_{A,B},\hat\varphi).$$

With the sets A and B we will associate the number

$$k_0 = k_0(A, B) = \min\Big\{ \min_{k\in\mathbb{N}\setminus A} k,\ \min_{k\in\mathbb{N}\setminus B} k \Big\}.$$

Also let χ_r be the function on $\mathbb{Z}_+$ that takes value 1 at zero and vanishes at other points.

Theorem 2. *If $0 \notin A$, then*

$$E(D^0, W_2^n(\mathbb{T}), F_{A,B}) = +\infty.$$

If $1 \le r \le n-1$ or $r = 0$ and $0 \in A$, then

$$E(D^r, W_2^n(\mathbb{T}), F_{A,B}) = \frac{1}{k_0^{n-r}}.$$

Moreover, for any finite tuples of numbers $\alpha = \{\alpha_k\}_{k\in A}$ and $\beta = \{\beta_k\}_{k\in B}$ such that

$$|\alpha_k - 1| \le \left(\frac{k}{k_0}\right)^{n-r}, \quad k \in A, \quad |\beta_k - 1| \le \left(\frac{k}{k_0}\right)^{n-r}, \quad k \in B,$$

the method

$$\hat{\varphi}_{\alpha,\beta}\big(F_{A,B}x(\cdot)\big)(t) = \frac{a_0}{2}\chi_r + \sum_{k\in A\setminus\{0\}} k^r \alpha_k a_k \cos\Big(kt + \frac{\pi r}{2}\Big) + \sum_{k\in B} k^r \beta_k b_k \sin\Big(kt + \frac{\pi r}{2}\Big)$$

is optimal.

A few comments on this result may be made.

(1) The condition $E(D^0, W_2^n(\mathbb{T}), F_{A,B}) = +\infty$ for $0 \notin A$ means that by no means can one recover functions from the class $W_2^n(\mathbb{T})$, and so the method is optimal in this sense.
(2) All optimal methods are linear and there exists an optimal method using the Fourier coefficients only with numbers up to $k_0 - 1$ (it is easily verified that for $k \ge k_0$ the coefficients α_k and β_k may be taken to be zero). Moreover, if $k_0 = 1$ and $r \ge 1$, then the optimal method is zero.
(3) Among optimal methods there are "natural" ones, when $\alpha_k = \beta_k = 1$, i.e., in the Fourier series we substitute the known Fourier coefficients.

Proof of Theorem 2. Let $0 \le r \le n-1$. As in the proof of Theorem 1, we start with estimating from below the quantity $E(D^r, W_2^n(\mathbb{T}), F_{A,B})$. A similar argument shows that

$$E(D^r, W_2^n(\mathbb{T}), F_{A,B}) \ge \sup_{\substack{x(\cdot)\in W_2^n(\mathbb{T}),\\ F_{A,B}x(\cdot)=0}} \|x^{(r)}(\cdot)\|_{L_2(\mathbb{T})}. \tag{10}$$

Let $r = 0$. If $0 \notin A$, then any constant function $x(\cdot)$ satisfies the conditions $x(\cdot) \in W_2^n(\mathbb{T})$ and $F_{A,B}x(\cdot) = 0$, whereas the quantity $\|x(\cdot)\|_{L_2(\mathbb{T})}$ can be made arbitrary large. Now $E(D^0, W_2^n(\mathbb{T}), F_{A,B}) = +\infty$ by (10).

Now assume that $0 \in A$. Setting $k_0 = \min\{ k \in \mathbb{N} \setminus A\}$, we consider the function $t \mapsto x_0(t) = k_0^{-n}\cos k_0 t$. It is easily checked that $x_0(\cdot) \in W_2^n(\mathbb{T})$, $F_{A,B}x_0(\cdot) = 0$, and $\|x_0(\cdot)\|_{L_2(\mathbb{T})} = k_0^{-n}$. If now we set $k_0 = \min\{k \in \mathbb{N} \setminus B\}$, then one needs to deal with the function $t \mapsto k_0^{-n}\sin k_0 t$, which has similar properties. Consequently, the right-hand side of (10) is not smaller than k_0^{-n}, and hence,

$$E(D^0, W_2^n(\mathbb{T}), F_{A,B}) \ge \frac{1}{k_0^n}. \tag{11}$$

Now we estimate from above the quantity $E(D^0, W_2^n(\mathbb{T}), F_{A,B})$ and check the optimality of the methods $\hat{\varphi}_{\alpha,\beta}$ from the statement of the theorem.

Let $x(\cdot) \in W_2^n(\mathbb{T})$. Using the Parseval equality, we see that

$$\begin{aligned}
&\big\|x(\cdot) - \varphi_{\alpha,\beta}\big(F_{A,B}x(\cdot)\big)(\cdot)\big\|^2_{L_2(\mathbb{T})} \\
&\quad = \sum_{k\in A\setminus\{0\}} (1-\alpha_k)^2 a_k^2 + \sum_{k\in B}(1-\beta_k)^2 b_k^2 + \sum_{k\in\mathbb{N}\setminus A} a_k^2 + \sum_{k\in\mathbb{N}\setminus B} b_k^2 \\
&\quad \le \max_{k\in A\setminus\{0\}} \frac{(1-\alpha_k)^2}{k^{2n}} \sum_{k\in A\setminus\{0\}} k^{2n}a_k^2 + \max_{k\in B}\frac{(1-\beta_k)^2}{k^{2n}}\sum_{k\in B} k^{2n}b_k^2 + \frac{1}{k_0^{2n}}\sum_{k\in\mathbb{N}\setminus A} k^{2n}a_k^2 + \frac{1}{k_0^{2n}}\sum_{k\in\mathbb{N}\setminus B} k^{2n}b_k^2.
\end{aligned}$$

Taking into account the conditions on the vectors α and β, we see that, for any method $\hat{\varphi}_{\alpha,\beta}$ with such α and β,

$$\big\|x(\cdot) - \hat{\varphi}_{\alpha,\beta}\big(F_{A,B}x(\cdot)\big)(\cdot)\big\|^2_{L_2(\mathbb{T})} \le \frac{1}{k_0^{2n}}\sum_{k\in\mathbb{N}} k^{2n}(a_k^2+b_k^2) \le \frac{1}{k_0^{2n}}$$

for all $x(\cdot) \in W_2^n(\mathbb{T})$. Hence,

$$e(D^0, W_2^n(\mathbb{T}), F_{A,B}, \hat{\varphi}_{\alpha,\beta}) \le \frac{1}{k_0^n}.$$

Comparing this with estimate (11), we obtain the conclusion of the theorem for the case $r=0$ and $0\in A$.

Now assume that $1 \le r \le n-1$. Considering the same functions $t \mapsto k_0^{-n}\cos k_0 t$ and $t\mapsto k_0^{-n}\sin k_0 t$ as before, we see that the quantity on the right in (10) is not smaller than $k_0^{-(n-r)}$, and hence,

$$E(D^r, W_2^n(\mathbb{T}), F_{A,B}) \ge \frac{1}{k_0^{n-r}}. \tag{12}$$

The error of the method $\hat{\varphi}_{\alpha,\beta}$ is estimated in a precisely similarly way as for the case $r=0$. □

2.2. Recovery in the Mean-Square Metric from Fourier Coefficients Given with Some Error. Here we are concerned with the problem of recovery of a function and its derivatives on the same class $W_2^n(\mathbb{T})$, with the same sets A and B, but instead of the exact values of the Fourier coefficients $a_k = a_k(x(\cdot))$, $k\in A$, and $b_k = b_k(x(\cdot))$, $k\in B$, of a function $x(\cdot)\in W_2^n(\mathbb{T})$, we will be given only their approximate values; i.e., the numbers $\{\tilde{a}_k\}_{k\in A}$ and $\{\tilde{b}_k\}_{k\in B}$ such that

$$|a_k - \tilde{a}_k| \le \delta, \quad k\in A, \quad |b_k - \tilde{b}_k| \le \delta, \quad k \in B.$$

This can be written in a more convenient form. Let l_∞^N be the space $\mathbb{R}^N$ with norm $\|y\|_\infty = \max\limits_{0\le j\le N-1}|y_j|$, where $y = (y_0, y_1, \dots, y_{N-1})$.

We shall assume, for definiteness, that

$$F_{A,B}x(\cdot) = (a_{k_0}, a_{k_1}, \dots, a_{k_{N_1}}, b_{l_1}, \dots, b_{l_{N_2}}),$$

where $k_0 < \dots < k_{N_1}$ and $l_1 < \dots < l_{N_2}$, $N_1 + 1 + N_2 = N$. Now we may say that we know a vector $y = (y_0, \dots, y_N)$ such that $\|F_{A,B}x(\cdot) - y\|_\infty \le \delta$.

By the *error* of a method $\varphi\colon \mathbb{R}^N \to L_2(\mathbb{T})$ in this case we shall mean the quantity

$$e(D^r, W_2^n(\mathbb{T}), F_{A,B}, \delta, \varphi) = \sup_{\substack{x(\cdot)\in W_2^n(\mathbb{T}),\ y\in l_\infty^N,\\ \|F_{A,B}x(\cdot)-y\|_\infty\le\delta}} \|x^{(r)}(\cdot) - \varphi(y)(\cdot)\|_{L_2(\mathbb{T})}.$$

The *optimal recovery error* is, by definition, the quantity

$$E(D^r, W_2^n(\mathbb{T}), F_{A,B}, \delta) = \inf_{\varphi\colon l_\infty^N \to L_2(\mathbb{T})} e(D^r, W_2^n(\mathbb{T}), F_{A,B}, \delta, \varphi).$$

We continue to call a method $\hat{\varphi}$ on which the infimum is attained an *optimal recovery method.*

We set

$$\hat{p} = \hat{p}(\delta) = \max\Big\{p\in\mathbb{Z}_+ \colon 2\delta^2\sum_{k=0}^{p} k^{2n} < 1\Big\}$$

and $p_0 = p_0(A, B, \delta) = \min\{\hat{p},\ k_0 - 1\}$, where $k_0 = k_0(A, B)$ and the function χ_r on $\mathbb{Z}_+$ are defined before Theorem 2.

Theorem 3. *If $0 \notin A$, then*

$$E(D^0, W_2^n(\mathbb{T}), F_{A,B}, \delta) = +\infty.$$

If $1 \le r \le n-1$ or $r = 0$ and $0 \in A$, then

$$E(D^r, W_2^n(\mathbb{T}), F_{A,B}, \delta) = \sqrt{\frac{1}{(p_0+1)^{2(n-r)}} + \frac{\delta^2}{2}\chi_r + 2\delta^2 \sum_{k=1}^{p_0} k^{2r} \left(1 - \left(\frac{k}{p_0+1}\right)^{2(n-r)}\right)}$$

and the method

$$\hat{\varphi}(\{\tilde{a}_k\}_{k\in A}, \{\tilde{b}_k\}_{k\in B})(t) = \frac{\tilde{a}_0}{2}\chi_r + \sum_{k=1}^{p_0} \left(1 - \left(\frac{k}{p_0+1}\right)^{2(n-r)}\right) k^r \left(\tilde{a}_k \cos\left(kt + \frac{\pi r}{2}\right) + \tilde{b}_k \sin\left(kt + \frac{\pi r}{2}\right)\right)$$

is optimal.

It is worth noting that if $\hat{p} \le k_0 - 1$, then the Fourier coefficients with numbers exceeding $\hat{p}$ can be neglected—they are not used in the optimal method.

We also note that for $\delta = 0$ we formally obtain the value of the optimal recovery error from the preceding theorem, because in this way it is natural to assume that $p_0 = k_0 - 1$. Moreover, for $\delta = 0$ we obtain one of the methods indicated in Theorem 2: this is indeed so when

$$\alpha_k = \beta_k = 1 - \left(\frac{k}{k_0}\right)^{2(n-r)}, \quad k = 1, \dots, k_0 - 1,$$

and the remaining α_k and β_k are zero.

Proof of Theorem 3. Let $0 \le r \le n-1$. Using the same trick as in the proof of Theorem 1, one easily shows that the optimal recovery error is estimated as follows:

$$E(D^r, W_2^n(\mathbb{T}), F_{A,B}, \delta) \ge \sup_{\substack{x(\cdot)\in W_2^n(\mathbb{T}),\\ \|F_{A,B}x(\cdot)\|_\infty \le \delta}} \|x^{(r)}(\cdot)\|_{L_2(\mathbb{T})}. \tag{13}$$

Let $r = 0$. We note that if $0 \notin A$, then any constant function $x(\cdot)$ satisfies the conditions $x(\cdot) \in W_2^n(\mathbb{T})$ and $\|F_{A,B}x(\cdot)\|_\infty \le \delta$ ($F_{A,B}x(\cdot)$ is the zero vector). By doing so, the right-hand side in (13) can be made arbitrarily large; i.e., $E(D^0, W_2^n(\mathbb{T}), F_{A,B}, \delta) = +\infty$.

Now let $0 \in A$. In order to find the value of the quantity on the right-hand side in (13), we consider the following extremal problem:

$$\|x(\cdot)\|_{L_2(\mathbb{T})} \to \max, \quad x(\cdot) \in W_2^n(\mathbb{T}), \quad \|F_{A,B}x(\cdot)\|_\infty \le \delta. \tag{14}$$

By finding a solution to this problem, we shall find the value of the above quantity.

Passing to the Fourier coefficients and using the Parseval equality, we find that the squared value of problem (14) is equal to the value of the problem

$$\begin{gathered} \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \to \max, \quad \sum_{k=1}^{\infty} k^{2n}(a_k^2 + b_k^2) \le 1, \\ a_k^2 \le \delta^2, \quad k \in A, \quad b_k^2 \le \delta^2, \quad k \in B, \end{gathered} \tag{15}$$

where $a_k = a_k(x(\cdot))$, $k \in \mathbb{Z}_+$, and $b_k = b_k(x(\cdot))$, $k \in \mathbb{N}$, and $x(\cdot) \in \mathcal{W}_2^n(\mathbb{T})$.

In analogy with the above procedure for problems (2) and (3), we note that problem (15) *qua* a problem on l_2-sequences $\{a_k, b_k\}$ such that the sequence $\{k^n a_k, k^n b_k\}$ also lies in l_2 (we denote the set of all such sequences by l_2^n) is equivalent to problem (14) with $\|x(\cdot)\|_{L_2(\mathbb{T})}$ replaced by $\|x(\cdot)\|^2_{L_2(\mathbb{T})}$.

Problem (15) is convex, its Lagrange function is as follows:

$$\mathcal{L}(\{a_k,b_k\},\lambda,\{\lambda_k\}_{k\in A},\{\mu_k\}_{k\in B}) = -\frac{a_0}{2} - \sum_{k=1}^{\infty}(a_k^2+b_k^2) + \lambda\sum_{k=1}^{\infty}k^{2n}(a_k^2+b_k^2) + \sum_{k\in A}\lambda_k a_k^2 + \sum_{k\in B}\mu_k b_k^2.$$

If there exist an admissible sequence $\{\hat{a}_k,\hat{b}_k\}$ for problem (15) and Lagrange multipliers $\hat{\lambda}\ge 0$, $\hat{\lambda}_k\ge 0$, $k\in A$, and $\hat{\mu}_k\ge 0$, $k\in B$, such that

$$\text{(a)}\qquad \min_{\{a_k,b_k\}\in l_2^n}\mathcal{L}(\{a_k,b_k\},\hat{\lambda},\{\hat{\lambda}_k\}_{k\in A},\{\hat{\mu}_k\}_{k\in B}) = \mathcal{L}(\{\hat{a}_k,\hat{b}_k\},\hat{\lambda},\{\hat{\lambda}_k\}_{k\in A},\{\hat{\mu}_k\}_{k\in B}),$$

$$\text{(b)}\quad \hat{\lambda}\left(\sum_{k=1}^{\infty}k^{2n}(\hat{a}_k^2+\hat{b}_k^2)-1\right)=0,\quad \hat{\lambda}_k(\hat{a}_k^2-\delta^2)=0,\quad k\in A,\quad \hat{\mu}_k(\hat{b}_k^2-\delta^2)=0,\quad k\in B,$$

then $\{\hat{a}_k,\hat{b}_k\}$ is a solution of problem (15). This can be verified by exactly similar arguments as in the previous section.

Now we need to produce an admissible sequence $\{\hat{a}_k,\hat{b}_k\}$ for problem (15) and Lagrange multipliers $\hat{\lambda}\ge 0$, $\hat{\lambda}_k\ge 0$, $k\in A$, and $\hat{\mu}_k\ge 0$, $k\in B$, to satisfy conditions (a) and (b).

Let $p_0=\hat{p}<k_0-1$. We set $\hat{a}_k=\delta$, $k=0,1,\dots,p_0$, $\hat{b}_k=\delta$, $k=1,\dots,p_0$, $\hat{a}_{p_0+1}=\hat{b}_{p_0+1}=\alpha$, where

$$\alpha = \frac{\sqrt{\dfrac{1}{2}-\delta^2\sum\limits_{k=0}^{p_0}k^{2n}}}{(p_0+1)^n}$$

and $\hat{a}_k=\hat{b}_k=0$ if $k>p_0+1$.

We claim that $\alpha\le\delta$. Indeed, if it were not so, then we would have

$$1-2\delta^2\sum_{k=0}^{p_0}k^{2n} > 2\delta^2(p_0+1)^{2n}$$

or, what is the same,

$$2\delta^2\sum_{k=0}^{p_0+1}k^{2n}<1,$$

which contradicts the equality $p_0=\hat{p}$.

Further,

$$\sum_{k=1}^{\infty}k^{2n}(\hat{a}_k^2+\hat{b}_k^2) = 2\delta^2\sum_{k=1}^{p_0}k^{2n} + \frac{1-2\delta^2\sum\limits_{k=0}^{p_0}k^{2n}}{(p_0+1)^{2n}}(p_0+1)^{2n}=1,$$

and so $\{\hat{a}_k,\hat{b}_k\}$ is an admissible sequence for problem (15).

Now let $p_0=k_0-1$. If $k_0=\min_{k\in\mathbb{N}\setminus A}k$ ($k_0=\min\limits_{k\in\mathbb{N}\setminus B}k$), then we set $\hat{a}_k=\delta$, $k=0,1,\dots,p_0$, $\hat{b}_k=\delta$, $k=1,\dots,p_0$, $\hat{a}_{p_0+1}=\sqrt{2}\alpha$ ($\hat{b}_{p_0+1}=\sqrt{2}\alpha$), and the remaining coefficients are zero.

That these sequences are admissible for problem (15) is verified by the same argument as in the previous case, but now $p_0+1=k_0\notin A$, and hence the inequality $|\sqrt{2}\alpha|\le\delta$ is no longer supposed to hold.

We set $\hat{\lambda}=(p_0+1)^{-2n}$, $\hat{\lambda}_0=1/2$, $\hat{\lambda}_k=\hat{\mu}_k=1-\hat{\lambda}k^{2n}$, $k=1,\dots,p_0$, and $\hat{\lambda}_k=\hat{\mu}_k=0$, $k>p_0$. It is easily checked that all the Lagrange multipliers are nonnegative. Let us see whether condition (a) is satisfied. Given any sequences $\{a_k, b_k\} \in l_2^n$, we have

$$\begin{aligned}
&\mathcal{L}(\{a_k, b_k\}, \hat\lambda, \{\hat\lambda_k\}_{k\in A}, \{\hat\mu_k\}_{k\in B}) \\
&= \sum_{k=1}^{p_0}(-1+\hat\lambda k^{2n}+\hat\lambda_k)a_k^2 + \sum_{k=1}^{p_0}(-1+\hat\lambda k^{2n}+\hat\mu_k)b_k^2 + \sum_{k=p_0+1}^{\infty}(-1+\hat\lambda k^{2n})(a_k^2+b_k^2) \\
&= \sum_{k=p_0+1}^{\infty}(-1+\hat\lambda k^{2n})(a_k^2+b_k^2) = \sum_{k=p_0+2}^{\infty}(-1+\hat\lambda k^{2n})(a_k^2+b_k^2).
\end{aligned}$$

The expression on the right is nonnegative by the definition of $\hat\lambda$. On the other hand, the above sequences $\{\hat a_k, \hat b_k\}$ are zero starting from some number $p_0 + 2$, and hence the Lagrange function vanishes on these sequences. This verifies condition (a).

Condition (b) is verified by elementary arguments.

Thus, the sequences $\{\hat a_k, \hat b_k\}$ are solutions of problem (15) (in the corresponding cases), and the value of this problem in each of these cases is as follows:

$$\frac{\hat a_0^2}{2} + \sum_{k=1}^{\infty}(\hat a_k^2 + \hat b_k^2) = \frac{\delta^2}{2} + 2\delta^2 p_0 + \frac{1-2\delta^2\sum\limits_{k=1}^{p_0}k^{2n}}{(p_0+1)^{2n}} = \frac{\delta^2}{2} + \hat\lambda + 2\delta^2\sum_{k=1}^{p_0}(1-\hat\lambda k^{2n}) = \hat\lambda + \delta^2\bigg(\hat\lambda_0 + 2\sum_{k=1}^{p_0}\hat\lambda_k\bigg).$$

Hence, in view of (13),

$$E(D^0, W_2^n(\mathbb{T}), F_{A,B}, \delta) \ge \sqrt{\hat\lambda + \delta^2\bigg(\hat\lambda_0 + 2\sum_{k=1}^{p_0}\hat\lambda_k\bigg)}. \tag{16}$$

We now handle the upper estimate and next proceed to construct optimal methods. If a method $\varphi\colon \mathbb{R}^N \to L_2(\mathbb{T})$ is optimal, this means that its error, i.e., the value of the problem

$$\|x(\cdot) - \varphi(y)(\cdot)\|_{L_2(\mathbb{T})} \to \max, \quad \|F_{A,B}x(\cdot) - y\|_\infty \le \delta, \quad y \in l_\infty^N, \quad x(\cdot) \in W_2^n(\mathbb{T}), \tag{17}$$

is equal to $E(D^0, W_2^n(\mathbb{T}), F_{A,B}, \delta)$.

With each tuples $\alpha = (\alpha_1, \dots, \alpha_{p_0})$ and $\beta = (\beta_1, \dots, \beta_{p_0})$ we associate the method $\varphi_{\alpha,\beta}\colon \mathbb{R}^N \to L_2(\mathbb{T})$ defined by

$$\varphi_{\alpha,\beta}(y)(t) = \frac{y_0}{2} + \sum_{k=1}^{p_0}(\alpha_k y_k \cos kt + \beta_k y_{N_1+k}\sin kt),$$

where $y = (y_0, y_1, \dots, y_{N-1})$. We shall seek optimal methods among the methods of this form (if $p_0 = 0$, then the sum in the definition of the method is zero).

Passing to the Fourier coefficients and using the Parseval equality, we see that the squared value of problem (17) for the method $\varphi_{\alpha,\beta}$ is equal to the value of the problem

$$\begin{gathered}
\frac{(a_0-y_0)^2}{2} + \sum_{k=1}^{p_0}\big((a_k-\alpha_k y_k)^2 + (b_k - \beta_k y_{N_1+k})^2\big) + \sum_{k=p_0+1}^{\infty}(a_k^2+b_k^2) \to \max, \\
y = (y_0, \dots, y_{N-1}) \in \mathbb{R}^N, \quad \|F_{A,B}x(\cdot) - y\|_\infty \le \delta, \quad \sum_{k=1}^{\infty}k^{2n}(a_k^2+b_k^2) \le 1, \quad (18)
\end{gathered}$$

where $\{a_k, b_k\} \in l_2^n$.

Using the Cauchy–Schwarz inequality, we estimate the terms under the first summation sign in the functional to be maximized, taking into account that $\hat{\lambda} > 0$ and $\hat{\lambda}_k > 0$, $k = 1, \dots, p_0$. Hence,

$$(a_k - \alpha_k y_k)^2 = \left(\frac{1-\alpha_k}{\sqrt{\hat{\lambda}}\,k^n}\sqrt{\hat{\lambda}}\,k^n a_k + \frac{\alpha_k}{\sqrt{\hat{\lambda}_k}}\sqrt{\hat{\lambda}_k}\,(a_k - y_k)\right)^2 \le \left(\frac{(1-\alpha_k)^2}{\hat{\lambda}k^{2n}} + \frac{\alpha_k^2}{\hat{\lambda}_k}\right)\left(\hat{\lambda}k^{2n}a_k^2 + \hat{\lambda}_k(a_k - y_k)^2\right)$$

and similarly,

$$(b_k - \beta_k y_{N_1+k})^2 \le \left(\frac{(1-\beta_k)^2}{\hat{\lambda}k^{2n}} + \frac{\beta_k^2}{\hat{\lambda}_k}\right)\left(\hat{\lambda}k^{2n}b_k^2 + \hat{\lambda}_k(b_k - y_{N_1+k})^2\right).$$

Adding these estimates and denoting

$$S_{\alpha,\beta} = \max_{1 \le k \le p_0}\left(\frac{(1-\alpha_k)^2}{\hat{\lambda}k^{2n}} + \frac{\alpha_k^2}{\hat{\lambda}_k},\ \frac{(1-\beta_k)^2}{\hat{\lambda}k^{2n}} + \frac{\beta_k^2}{\hat{\lambda}_k}\right),$$

we have that

$$\begin{aligned}
&\sum_{k=1}^{p_0}\left((a_k - \alpha_k y_k)^2 + (b_k - \beta_k y_{N_1+k})^2\right) \\
&\quad \le S_{\alpha,\beta}\left(\hat{\lambda}\sum_{k=1}^{p_0}k^{2n}(a_k^2 + b_k^2) + \sum_{k=1}^{p_0}\hat{\lambda}_k\left((a_k - y_k)^2 + (b_k - y_{N_1+k})^2\right)\right) \\
&\quad \le S_{\alpha,\beta}\left(\hat{\lambda}\sum_{k=1}^{p_0}k^{2n}(a_k^2 + b_k^2) + 2\delta^2\sum_{k=1}^{p_0}\hat{\lambda}_k\right).
\end{aligned}$$

Further, if $k \ge p_0 + 1$, then $k^{-2n} \le (p_0 + 1)^{-2n} = \hat{\lambda}$, and so

$$\sum_{p_0+1}^{\infty}(a_k^2 + b_k^2) = \sum_{p_0+1}^{\infty}\frac{1}{k^{2n}}k^{2n}(a_k^2 + b_k^2) \le \hat{\lambda}\sum_{p_0+1}^{\infty}k^{2n}(a_k^2 + b_k^2).$$

If tuples α and β are such that $S_{\alpha,\beta} \le 1$, then from the above estimates it follows that the functional to be maximized in (18) is not larger than the quantity

$$\begin{aligned}
\frac{\delta^2}{2} + \hat{\lambda}\sum_{k=1}^{p_0}k^{2n}(a_k^2 + b_k^2) + 2\delta^2\sum_{k=1}^{p_0}\hat{\lambda}_k + \hat{\lambda}\sum_{p_0+1}^{\infty}k^{2n}(a_k^2 + b_k^2) &= \frac{\delta^2}{2} + \hat{\lambda}\sum_{k=1}^{\infty}k^{2n}(a_k^2 + b_k^2) + 2\delta^2\sum_{k=1}^{p_0}\hat{\lambda}_k \\
&\le \frac{\delta^2}{2} + \hat{\lambda} + 2\delta^2\sum_{k=1}^{p_0}\hat{\lambda}_k = \hat{\lambda} + \delta^2\left(\hat{\lambda}_0 + 2\sum_{k=1}^{p_0}\hat{\lambda}_k\right),
\end{aligned}$$

i.e.,

$$e(D^0, W_2^n(\mathbb{T}), F_{A,B}, \delta, \varphi_{\alpha,\beta}) \le \sqrt{\hat{\lambda} + \delta^2\left(\hat{\lambda}_0 + 2\sum_{k=1}^{p_0}\hat{\lambda}_k\right)}.$$

Together with estimate (16), this means that the method $\varphi_{\alpha,\beta}$ is optimal.

We claim that there exist tuples α and β for which $S_{\alpha,\beta} \le 1$. For each $k = 1, \dots, p_0$, we choose α_k and β_k so that they maximize the bracketed expressions in the definition of $S_{\alpha,\beta}$. It is easily checked that the minimum is attained at the points

$$\alpha_k = \beta_k = \frac{\hat{\lambda}_k}{\hat{\lambda}_k + \hat{\lambda}k^{2n}} = 1 - \left(\frac{k}{p_0 + 1}\right)^{2n}, \quad k = 1, \dots, p_0. \tag{19}$$

Now, for such α and β,

$$S_{\alpha,\beta} = \max_{1 \le k \le p_0}\frac{1}{\hat{\lambda}_k + \hat{\lambda}k^{2n}} = 1.$$

Thus, if the vectors $\alpha = (\alpha_1, \dots, \alpha_{p_0})$ and $\beta = (\beta_1, \dots, \beta_{p_0})$ are defined by (19), then the corresponding method is optimal. In reference to the notation $\tilde{a}_k$ and $\tilde{b}_k$, this is precisely the method indicated in the theorem. This completes the proof of the theorem for $r = 0$.

For $1 \le r \le n-1$ the scheme of the proof is the same as for the case $r = 0$, so we limit ourselves to a sketch. To find the quantity on the right-hand side in (13), we consider an analogue of problem (14), where the functional to be maximized is $\|x^{(r)}(\cdot)\|_{L_2(\mathbb{T})}$. In terms of the Fourier coefficients, the squared functional assumes the form $\sum_{k\in\mathbb{N}} k^{2r}(a_k^2 + b_k^2)$. Next we define in the natural way the Lagrange function for the analogue of problem (15). Going further, we employ sufficient conditions for the minimum, the corresponding Lagrange multipliers being as follows: $\hat{\lambda} = (p_0+1)^{-2(n-r)}$, $\hat{\lambda}_0 = 0$ (if $0 \in A$), $\hat{\lambda}_k = \hat{\mu}_k = k^{2r} - \hat{\lambda} k^{2n}$, $k = 1, \dots, p_0$, and $\hat{\lambda}_k = \hat{\mu}_k = 0$, $k > p_0$. An upper estimate for the optimal recovery error and construction of optimal methods follow the same lines as above, the optimal methods being sought among methods of the form

$$\varphi_{\alpha,\beta}(y)(t) = \sum_{k=0}^{p_0} k^r \left(\alpha_k y_k \cos\left(kt + \frac{\pi r}{2}\right) + \beta_k y_{N_1+k} \sin\left(kt + \frac{\pi r}{2}\right)\right). \qquad \square$$

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