# HOW TO RECOVER FUNCTIONS FROM INACCURATE INFORMATION

G. G. MAGARIL-IL'YAEV, K. YU. OSIPENKO

Here two problems are considered: optimal recovery of derivatives of function from inaccurate information about its spectrum and optimal recovery of the solution of the differential equation from inaccurate information about the initial data.

## 1. Optimal recovery of derivatives

In 1934 Hardy, Littlewood, and Pólya proved that for all integers $0 < k < n$ the exact inequality

$$\|x^{(k)}\|_{L_2(\mathbb{R})} \le \|x\|_{L_2(\mathbb{R})}^{1-\frac{k}{n}} \|x^{(n)}\|_{L_2(\mathbb{R})}^{\frac{k}{n}}$$

holds for all functions $x \in L_2(\mathbb{R})$ for which the $(n-1)$-st derivative is locally absolute continuous on $\mathbb{R}$ and $x^{(n)} \in L_2(\mathbb{R})$.

The Hardy–Littlewood–Pólya inequality may be considered as the solution of the following extremal problem

$$\|x^{(k)}\|_{L_2(\mathbb{R})} \to \max, \quad \|x\|_{L_2(\mathbb{R})} \le \delta_1, \quad \|x^{(n)}\|_{L_2(\mathbb{R})} \le \delta_2.$$

We consider a slightly different extremal problem which closely connected with problems of signal reconstruction

$$\|x^{(k)}\|_{L_2(\mathbb{R})} \to \max, \quad \|Fx\|_{L_2(\Delta_\sigma)} \le \delta, \quad \|x^{(n)}\|_{L_2(\mathbb{R})} \le 1,$$

where $Fx$ is the Fourier transform of $x$ and $\Delta_\sigma = [-\sigma, \sigma]$, $\sigma > 0$. Namely, we consider the problem of optimal recovery of $x^{(k)}$ knowing the Fourier transform of $x$ giving with some error on $\Delta_\sigma$.

Assume that $x \in W_2^n(\mathbb{R})$,

$$W_2^n(\mathbb{R}) = \{x \in L_2(\mathbb{R}) : x^{(n-1)} \text{ is loc. abs. cont. }, \ \|x^{(n)}\|_{L_2(\mathbb{R})} \le 1\},$$

and for any $x \in W_2^n(\mathbb{R})$ we know a function $y \in L_2(\Delta_\sigma)$ such that

$$\|Fx - y\|_{L_2(\Delta_\sigma)} \le \delta.$$

The problem is to recover $x^{(k)}$ knowing $y$.

Any method of recovery is a map $m \colon L_2(\Delta_\sigma) \to L_2(\mathbb{R})$. The error of such method is defined as follows

$$e_\sigma(m) = \sup_{\substack{x \in W_2^n(\mathbb{R}),\ y \in L_2(\Delta_\sigma) \\ \|Fx - y\|_{L_2(\Delta_\sigma)} \le \delta}} \|x^{(k)} - m(y)\|_{L_2(\mathbb{R})}.$$

We are interested in the value

$$E_{\sigma,2} = \inf_{m \colon L_2(\Delta_\sigma) \to L_2(\mathbb{R})} e_\sigma(m),$$

which is called the error of optimal recovery and in the method $\widehat{m}$, for which the infinum is attained that is in the method $\widehat{m}$ for which

$$E_{\sigma,2} = e_\sigma(\widehat{m}).$$

We call this method the optimal recovery method.

Set

$$\widehat{\sigma} = \left(\frac{n}{k}\right)^{\frac{1}{2(n-k)}} \left(\frac{2\pi}{\delta^2}\right)^{\frac{1}{2n}}, \quad \sigma_0 = \min\{\sigma, \widehat{\sigma}\},$$

$$\widehat{\alpha}(\xi) = \left(1 + \frac{n}{n-k}\left(\frac{n}{k}\right)^{\frac{k}{n-k}} \left(\frac{\xi}{\sigma_0}\right)^{2n}\right)^{-1}.$$

**Theorem 1.**

$$E_{\sigma,2} = \sigma_0^k \sqrt{\frac{n-k}{2\pi n}\left(\frac{k}{n}\right)^{\frac{k}{n-k}} \delta^2 + \sigma_0^{2(k-n)}}.$$

*For all* $\alpha$ *such that*

$$|\alpha(\xi) - \widehat{\alpha}(\xi)| \le \sqrt{\widehat{\alpha}^2(\xi) + \widehat{\alpha}(\xi)\left(\left(\frac{\xi}{\sigma_0}\right)^{2(n-k)} - 1\right)} \tag{1}$$

*the methods*

$$m(y)(t) = \frac{1}{2\pi}\int_{\Delta_\sigma} (i\xi)^k \alpha(\xi) y(\xi) e^{i\xi t}\, d\xi$$

*are optimal.*

For a fixed error of input data consider the error of optimal recovery $E_{\sigma,2}$ as a function of $\sigma$. The larger interval $(-\sigma, \sigma)$ we take the less error we have. But beginning with $\widehat{\sigma}$ the error $E_\sigma$ does not change.

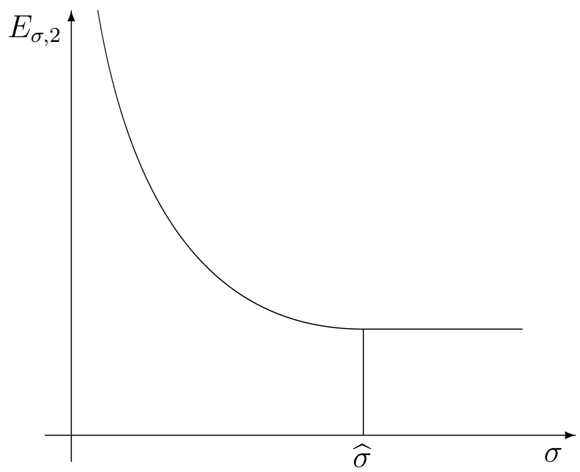


FIGURE 1

Consequently, for $\sigma > \widehat{\sigma}$ the observed information becomes partially redundant. To avoid this case the following condition

$$\delta^2\sigma^{2n} \le 2\pi\left(\frac{n}{k}\right)^{\frac{n}{n-k}} \tag{2}$$

should hold. This inequality may be considered as some "uncertain principle".

From Theorem 1 we may obtain the following family of optimal methods.

**Corollary 1.** *For all*

$$0 \le \theta \le \left(\frac{n-k}{n}\right)^{\frac{1}{2k}}\left(\frac{k}{n}\right)^{\frac{1}{2(n-k)}}$$

*the methods*

$$m(y)(t) = \frac{1}{2\pi}\int_{|\xi|\le\theta\sigma_0}(i\xi)^k y(\xi)e^{i\xi t}\,d\xi + \frac{1}{2\pi}\int_{\theta\sigma_0\le|\xi|\le\sigma_0}(i\xi)^k\alpha(\xi)y(\xi)e^{i\xi t}\,d\xi,$$

*where $\alpha$ any function satisfying* (1), *are optimal.*

Note that obtained methods do not smooth the input data on the interval $[-\theta\sigma_0,\theta\sigma_0]$.

Now let us consider the case when the error of the input data is measured in $L_\infty$-norm. Let $S$ be the Schwartz space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}$, $S'$ the dual space of distributions, and $F\colon S'\to S'$ the Fourier transform. Let

$$C_\infty^n = \{\, x\in X_\infty^n : \|x^{(n)}\|_{L_2(\mathbb{R})}\le 1\,\},$$

where

$$X_\infty^n = \{\, x\in S' : Fx\in L_\infty(\mathbb{R}),\ x^{(n)}\in L_2(\mathbb{R})\,\}.$$

Assume that for any $x\in C_\infty$ we know a function $y\in L_\infty(\mathbb{R})$ such that

$$\|Fx-y\|_{L_\infty(\Delta_\sigma)}\le\delta.$$

The problem again is to recover $x^{(k)}$ knowing $y$. Now we are interested in the value

$$E_{\sigma,\infty} = \inf_{m\colon L_\infty(\Delta_\sigma)\to L_2(\mathbb{R})}\ \sup_{\substack{x\in C_\infty^n,\ y\in L_\infty(\Delta_\sigma)\\ \|Fx-y\|_{L_\infty(\Delta_\sigma)}\le\delta}}\|x^{(k)}-m(y)\|_{L_2(\mathbb{R})}$$

and in the optimal method $\widehat{m}$, that is, in the method for which the lower bound ia attained.

**Theorem 2.** *Set*

$$\widehat{\sigma}_\infty = (\pi(2n+1))^{\frac{1}{2n+1}}\delta^{-\frac{2}{2n+1}},\quad \widetilde{\sigma}_0=\min(\sigma,\widehat{\sigma}).$$

*Then*

$$E_{\sigma,\infty}=\begin{cases}\sqrt{\sigma^{-2(n-k)}+\dfrac{2\delta^2(n-k)}{\pi(2k+1)(2n+1)}\sigma^{2k+1}}, & \sigma<\widehat{\sigma}_\infty,\\ \sqrt{\dfrac{2n+1}{2k+1}}\left(\dfrac{1}{\pi(2n+1)}\right)^{\frac{n-k}{2n+1}}\delta^{\frac{2(n-k)}{2n+1}}, & \sigma\ge\widehat{\sigma}_\infty,\end{cases}$$

*and the method*

$$\widehat{m}(y)(t)=\frac{1}{2\pi}\int_{-\widetilde{\sigma}_0}^{\widetilde{\sigma}_0}(i\xi)^k\left(1-\left(\frac{\xi}{\widetilde{\sigma}_0}\right)^{2(n-k)}\right)y(\xi)e^{i\xi t}\,d\xi$$

*is optimal.*

It follows from this theorem that for a given $\delta$, starting from $\widehat{\sigma}_\infty$, further extension of the interval on which the Fourier transform of a function from $C^n_\infty$ is given with error $\delta$ does not result in a decrease in the recovery error. In other words, if the relation

$$\delta^2\sigma^{2n+1}\le\pi(2n+1) \tag{3}$$

between the input data and the size of the interval on which the data is measured is violated, then the available information turns out to be redundant. Inequality (3) is an analog of uncertain principle (2) in this case.

## 2. Optimal recovery of the solution of the heat equation

Now we consider the problem of optimal recovery of the solution of the heat equation from inaccurate observations of the solution at the time moments $t_1,\dots,t_n$.

Let $u$ be the solution of the heat equation in $\mathbb{R}^d$

$$\begin{aligned}&u_t=\Delta u,\\ &u\big|_{t=0}=f(x),\quad f\in L_2(\mathbb{R}^d).\end{aligned}$$

Assume that we know functions $y_j\in L_2(\mathbb{R}^d)$, $j=1,\dots,n$, such that

$$\|u(t_j,\cdot)-y_j(\cdot)\|_{L_2(\mathbb{R}^d)}\le\delta_j,\quad j=1,\dots,n.$$

What is the best way to use this information to recover the temperature distribution at the time $\tau\ne t_j$, $1\le j\le n$, that is to recover the function $u(\tau,\cdot)$?

We admit as recovery methods arbitrary maps $m\colon (L_2(\mathbb{R}^d))^n\to L_2(\mathbb{R}^d)$. For a fixed method $m$ the quantity

$$e_\tau(L_2(\mathbb{R}^d),\delta,m)=\sup_{\substack{f,y_1,\dots,y_n\in L_2(\mathbb{R}^d)\\ \|u(t_j,\cdot)-y_j(\cdot)\|_{L_2(\mathbb{R}^d)}\le\delta_j,\ j=1,\dots,n}}\|u(\tau,\cdot)-m(y)(\cdot)\|_{L_2(\mathbb{R}^d)},$$

where $u$ is the solution of the heat equation with the initial function $f$, $\delta=(\delta_1,\dots,\delta_n)$, and $y=(y_1,\dots,y_n)$, is called the error of the method $m$.

We are interested in the value

$$E_\tau(L_2(\mathbb{R}^d),\delta)=\inf_{m\colon (L_2(\mathbb{R}^d))^n\to L_2(\mathbb{R}^d)} e_\tau(L_2(\mathbb{R}^d),\delta,m),$$

which is called the error of optimal recovery and in the method $\widehat{m}$, for which the infinum is attained that is in the method $\widehat{m}$ for which

$$E_\tau(L_2(\mathbb{R}^d),\delta)=e_\tau(L_2(\mathbb{R}^d),\delta,\widehat{m}).$$

We call this method the optimal recovery method.

To formulate the result we consider the set

$$M=\operatorname{co}\{\,(t_j,\log 1/\delta_j),\ 1\le j\le n\,\}+\{\,(t,0)\mid\ t\ge 0\,\},$$

where $\operatorname{co}A$ is a convex hull of $A$. Define the function $\theta(t)$, $t\in[t_1,\infty)$ as follows

$$\theta(t)=\max\{\,y:(t,y)\in M\,\}.$$

It is clear that $\theta$ is a polygonal line on $[t_1,\infty)$. Let $t_{s_j}$, $j=1,\dots,r$, be points of break of $\theta$.

For $\tau\in(t_{s_j},t_{s_{j+1}})$ put

$$\lambda_{s_j}=\frac{t_{s_{j+1}}-\tau}{t_{s_{j+1}}-t_{s_j}}\left(\frac{\delta_{s_{j+1}}}{\delta_{s_j}}\right)^{\frac{2(\tau-t_{s_j})}{t_{s_{j+1}}-t_{s_j}}},\qquad \lambda_{s_{j+1}}=\frac{\tau-t_{s_j}}{t_{s_{j+1}}-t_{s_j}}\left(\frac{\delta_{s_j}}{\delta_{s_{j+1}}}\right)^{\frac{2(t_{s_{j+1}}-\tau)}{t_{s_{j+1}}-t_{s_j}}}$$

**Theorem 3.** *For all $\tau\ge t_1$*

$$E_\tau(L_2(\mathbb{R}^d),\delta)=e^{-\theta(\tau)}.$$

*If $\tau\in(t_{s_j},t_{s_{j+1}})$, then for all $\gamma_j$ such that*

$$\text{(4)}\quad \lambda_{s_{j+1}}|\gamma_j(\xi)|^2e^{2|\xi|^2(t_{s_j}-\tau)}+\lambda_{s_j}|1-\gamma_j(\xi)|^2e^{2|\xi|^2(t_{s_{j+1}}-\tau)}\le\lambda_{s_j}\lambda_{s_{j+1}},$$

*all methods*

$$m(y)(t)=(K_j*y_{s_j})(t)+(L_{j+1}*y_{s_{j+1}})(t),$$

*where*

$$FK_j(\xi)=\gamma_j(\xi)e^{|\xi|^2(t_{s_j}-\tau)},\quad FL_{j+1}(\xi)=(1-\gamma_j(\xi))e^{|\xi|^2(t_{s_{j+1}}-\tau)},$$

*are optimal.*

*For $\tau=t_{s_j}$, $j=1,\dots,r$, methods $m(y)(t)=y_{s_j}(t)$ are optimal and for $\tau>t_{s_r}$ the method*

$$m(y)=F^{-1}(e^{-|\xi|^2(\tau-t_{s_r})}Fy_{s_r}(\xi))(x)$$

*is optimal.*

Condition (4) may be rewritten in the form

$$\left|\gamma_j(\xi) - \frac{\mu_1}{\mu_1+\mu_2}\right| \le \frac{\sqrt{\mu_1\mu_2}\sqrt{\mu_1+\mu_2-1}}{\mu_1+\mu_2},$$

where

$$\mu_1 = \lambda_{s_j} e^{-2|\xi|^2(t_{s_j}-\tau)}, \quad \mu_2 = \lambda_{s_{j+1}} e^{-2|\xi|^2(t_{s_{j+1}}-\tau)}.$$

It can be shown that $\mu_1+\mu_2 \ge 1$ for all $\xi \in \mathbb{R}^d$. Thus, $\gamma_j(\xi)$ may be chosen from the interval

$$\left[\frac{\mu_1}{\mu_1+\mu_2} - \frac{\sqrt{\mu_1\mu_2}\sqrt{\mu_1+\mu_2-1}}{\mu_1+\mu_2}, \frac{\mu_1}{\mu_1+\mu_2} + \frac{\sqrt{\mu_1\mu_2}\sqrt{\mu_1+\mu_2-1}}{\mu_1+\mu_2}\right].$$

Note that optimal method of recovery uses not more than two observations. To find these observation we have to construct the set $M$ and the polygonal line $\theta$. Then we have to find the nearest points of break of $\theta$ to the point $\tau$. The observations at these points are those that use in optimal method of recovery.

Note also that we can make more precise points of observation which are not on the polygonal line. Suppose that for some $t_m$, $t_{s_j} < t_m < t_{s_{j+1}}$ and

$$\theta(t_m) > \log 1/\delta_m.$$

Then optimal recovery method gives the error less than $\delta_m$. Indeed

$$\|u(t_m,\cdot) - \widehat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)} \le e^{-\theta(t_m)} < \delta_m.$$

## References


[1] Magaril-Il'yaev G. G., Osipenko K. Yu. Optimal recovery of functions and their derivatives from Fourier coefficients prescribed with an error, *Mat. Sb.* **193** (2002), 79–100; English transl. in *Sbornic: Mathematics* **193** (2002), 387–407.

[2] Magaril-Il'yaev G. G., Osipenko K. Yu. Optimal recovery of functions and their derivatives from inaccurate information about a spectrum and inequalities for derivatives, *Funkc. analiz i ego prilozh.* **37** (2003), 51–64; English transl. in *Funct. Anal and Its Appl.*, **37** (2003), 203–214.

[3] Magaril-Il'yaev G. G., Osipenko K. Yu. Optimal recovery of the solution of the heat equation from inaccurate data, *Mat. Sb.*, **200**, 5 (2009) 37–54; English transl. in *Sbornic: Mathematics* 200 (2009), 665–682.

[4] Magaril-Il'yaev G. G., Osipenko K. Yu. On recovery of convolution type operators from inaccurate information, *Trudy Mat. Inst. Steklov*, **269** (2010), 181–192.

[5] Magaril-Il'yaev G. G., Osipenko K. Yu. On optimal harmonic synthesis from inaccurate spectral data, *Funkc. analiz i ego prilozh.*, 44:3 (2010), 76–79; English transl. in *Funct. Anal and Its Appl.*, 44:3 (2010), 223–225.

[6] Magaril-Il'yaev G. G., Osipenko K. Yu. Hardy–Littlewood–Pólya inequality and recovery of derivatives from inaccurate data, *Dokl. Akad. Nauk*, **438**, 3 (2011), 300–302; English transl. in *Dokl. Math.*, **83**, 3 (2011).



Moscow State Institute of Radio Engineering, Electronics and Automation (Technology University)

MATI — Russian State Technological University