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# Indefinite knowledge about an object and accuracy of its recovery methods

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**Abstract**—An approach to the problem of optimal recovery of functionals and operators on classes of functions under the conditions of infinite knowledge of functions themselves is discussed. The capabilities of this approach are demonstrated in a number of examples. In the end of the paper a general result about optimal recovery of linear functionals is given.

## 1. INTRODUCTION

Many aspects of human activity are connected with the fact that a man has to judge about objects under investigations using incomplete and/or inaccurate information about them. As a rule, it is impossible to recover an object exactly from such information so an *indefiniteness* is appeared usually in the form of some region where the object may be found. Sometimes making more precise the input information we may approach to the object closer and closer (in this case one may consider the object as “knowable”) but the “price” of such cognoscibility quite often turns out excessively high.

For a long time it was assumed that the world is knowable but it is not insisted on it now since there were found fundamental bounds of cognoscibility (in mathematical logic, in quantum mechanics, and so on). On the other hand, if there is some information, then we want to restrict bounds of indefiniteness at most using this information as much as possible. For this purpose some *methods of recovery* of an object by the information which we have at our disposal are applied. If a method of recovery gives the bounds for an object which coincide with its measure of indefiniteness for a given information, then one may say about the optimality of this recovery method.

Andrei Nikolaevich Kolmogorov was interested in such problems during all his creative life and in any case he faced with them in his scientific activity (in theory of probability, in information theory, in theory of firing, and in many other problems). Several of quantities introduced by him (for example, $\varepsilon$-capacity and $\varepsilon$-entropy) are the characteristics of measures of indefiniteness and his results on extrapolation of stochastic processes led to appropriate optimal methods of recovery.

In this paper for a sufficiently extensive class of problems (quite natural from the application standpoint) the notion of optimality of recovery method from various types of information is introduced. This approach is demonstrated on a number of examples having an illustrative nature and given to show a variety of problems covering by the proposed setting.

## 2. STATEMENT OF THE PROBLEMS

The general statement of the problems of indefiniteness and recovery discussed here is in finding of values of a given functional or operator at some functions. We have two types of information about these functions. One of them is “global” described the class of functions which may occur

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and other is "local" (individual) connected with the characterization of an individual function. Usually classes connected with the properties of smoothness or analyticity of included functions. Generally the local information is in the fact that some characteristics of a function (for example, values at some points, moments, Fourier or Taylor coefficients, Fourier transform, and so on) are available to a researcher. This information may be given accurately or inaccurately. By these two types of information the estimation of indefiniteness of a function or operator value is given and a method of its recovery is constructed. We proceed now to precise statements.

Let $C$ be a set (class) and $f\colon C \to Z$ be a map where $(Z, d)$ is a metric space. The fact that an element belongs to the class $C$ is the "global" information about it. Moreover, there is the "local" (individual) information about it which is in the fact that we know a map (in general, a multivalued map which corresponds to inaccurate information) $F\colon C \to Y$ where $Y$ is some set. The map $F$ is called the *information operator*.

The problem is in *recovering of the value $f(x)$, $x \in C$, by the information $y \in F(x)$ as good as possible.*

The examples are the problems of recovering of a function value at some point from it values at other points or from Fourier coefficients, or the problem of recovering of an integral of a function, or its derivative at some point, or recovering of a function itself by the same or other information.

Let us explain the sense which we mean by the words "to recover as good as possible".

Any map $m\colon F(C) \to Z$ is called a *method of recovery*. The error of such method is the quantity

$$e(f, C, F, m) = \sup_{x \in C,\ y \in F(x)} d(f(x), m(y))$$

and the *error of optimal recovery* ($f$ on $C$ by $F$), which we denote by $E(f, C, F)$, is defined as a solution of the following extremal problem:

$$\inf e(f, C, F, m), \tag{1}$$

where the infimum is taking over all maps $m\colon F(C) \to Z$. A method $\widehat{m}$ for which the infimum in (1) is attained is called an *optimal method of recovery*. It may occur that there is a possibility to use various types of information, that is, we have a set of information maps $\mathcal{F}$. Then by the problem of choosing of optimal information we mean the problem of finding the value

$$E(f, C, \mathcal{F}) = \inf_{F \in \mathcal{F}} E(f, C, F).$$

For the first time the problem of optimal recovery was stated by S. A. Smolyak [1] for the case when $C$ is a convex balanced (that is, $C = \alpha C$ for all $\alpha$ such that $|\alpha| = 1$) subset of a linear space $X$, $Y$ is a finite-dimensional linear space, $f$ is a linear functional on $X$, and $F\colon X \to Y$ is a linear map. Smolyak proved that in this case there exists a linear method among optimal methods. Later on the problems concerned with optimal recovery were intensively developed (see [2]–[7]). The results related to the class of problems considered in the present paper may be found in [8]–[10] where, in particular, various generalization of most examples being discussed below are given.

## 3. EXAMPLES

**1. Recovery of a function at a point by its values at other points.** Denote by $W^1_\infty([-1,1])$ the class of real functions $x(\cdot)$ defined on the interval $[-1,1]$ absolutely continuous and satisfying the condition

$$|x'(t)| \le 1, \quad \text{for almost all } t \in [-1,1].$$

Let $-1 \le t_1 < \ldots < t_n \le 1$. Consider the problem of optimal recovery of the value of a function $x(\cdot) \in W^1_\infty([-1,1])$ at a point $\tau \in [-1,1]$ by its values at the points $\bar t = (t_1, \ldots, t_n)$. In accordance with the general setting we have here $C = W^1_\infty([-1,1])$, $Z = \mathbb{R}$, $f(x(\cdot)) = x(\tau)$, $Y = \mathbb{R}^n$, and $F_{\bar t}\colon C \to Y$, $F_{\bar t}x(\cdot) = (x(t_1), \ldots, x(t_n))$.

Any functions $m\colon \mathbb{R}^n \to \mathbb{R}$ are admitted as recovery methods. The error of a given method $m$ is the value

$$e(x(\tau), W^1_\infty([-1,1]), F_{\bar t}, m) = \sup_{x(\cdot)\in W^1_\infty([-1,1])} |x(\tau) - m(F_{\bar t}x(\cdot))|,$$

and the error of optimal recovery is the value

$$E(x(\tau), W^1_\infty([-1,1]), F_{\bar t}) = \inf_{m\,:\,\mathbb{R}^n\to\mathbb{R}} E(x(\tau), W^1_\infty([-1,1]), F_{\bar t}, m).$$

Let us find this value and also an optimal method of recovery.

Denote by $\alpha(t)$ the nearest point to $t$ from the set $\{t_1, \ldots, t_n\}$ (in the case when $t$ is in the middle between $t_i$ and $t_{i+1}$ for definiteness we set $\alpha(t) = t_i$). Set

$$\widehat{x}(t) = |t - \alpha(t)|.$$

It is obvious that $\widehat{x}(\cdot) \in W^1_\infty([-1,1])$, $-\widehat{x}(\cdot) \in W^1_\infty([-1,1])$, and $F_{\bar t}\widehat{x}(\cdot) = F_{\bar t}(-\widehat{x}(\cdot)) = 0$. For any method $m$ we have

$$2\widehat{x}(\tau) \le |\widehat{x}(\tau) - m(0)| + |-\widehat{x}(\tau) - m(0)| \le 2e(x(\tau), W^1_\infty([-1,1]), F_{\bar t}, m).$$

Hence

$$E(x(\tau), W^1_\infty([-1,1]), F_{\bar t}) \ge \widehat{x}(\tau).$$

Let $\alpha(\tau) = t_k$, $1 \le k \le n$. Define the method $\widehat{m}$ by the equality $\widehat{m}(y) = y_k$, $y = (y_1, \ldots, y_n)$. Then for any function $x(\cdot) \in W^1_\infty([-1,1])$ we have

$$|x(\tau) - \widehat{m}(F_{\bar t}x(\cdot))| = |x(\tau) - x(t_k)| \le |\tau - t_k| = \widehat{x}(\tau).$$

Consequently,

$$E(x(\tau), W^1_\infty([-1,1]), F_{\bar t}) = \widehat{x}(\tau)$$

and the method

$$x(\tau) \approx x(\alpha(\tau))$$

is an optimal method of recovery.

**2. Recovery of an integral of a function by its values at points.** For the same class $W^1_\infty([-1,1])$ and the same information operator $F_{\bar t}$ consider now the problem of optimal recovery of the integral

$$Ix(\cdot) = \int_{-1}^{1} x(t)\,dt.$$

As before any functions $m\colon \mathbb{R}^n \to \mathbb{R}$ are admitted as recovery methods. Here the problem of optimal recovery is in finding the value

$$E(I, W^1_\infty([-1,1]), F_{\bar t}) = \inf_{m\,:\,\mathbb{R}^n\to\mathbb{R}}\ \sup_{x(\cdot)\in W^1_\infty([-1,1])} \left| \int_{-1}^{1} x(t)\,dt - m(F_{\bar t}x(\cdot)) \right|$$

and an optimal method of recovery $\widehat{m}_0$ for which the infimum in this equality is attained.

Using for the function $\widehat{x}(\cdot)$ the same notation as in the preceding example we obtain that for any method $m$

$$2\int_{-1}^{1}\widehat{x}(t)\,dt \le \left|\int_{-1}^{1}\widehat{x}(t)\,dt - m(0)\right| + \left|\int_{-1}^{1}(-\widehat{x}(t))\,dt - m(0)\right|$$
$$\le 2\sup_{x(\cdot)\in W_\infty^1([-1,1])}\left|\int_{-1}^{1}x(t)\,dt - m(F_{\bar t}x(\cdot))\right|.$$

Thus,

$$E(I, W_\infty^1([-1,1]), F_{\bar t}) \ge \int_{-1}^{1}\widehat{x}(t)\,dt.$$

On the other hand, putting

$$\widehat{m}_0(y) = \int_{-1}^{1} y(t)\,dt,$$

where

$$y(t) = \begin{cases} y_1, & -1 \le t \le \dfrac{t_1+t_2}{2}, \\ y_i, & \dfrac{t_{i-1}+t_i}{2} < t \le \dfrac{t_i+t_{i+1}}{2},\ 2 \le i \le n-1, \\ y_n, & \dfrac{t_{n-1}+t_n}{2} < t \le 1, \end{cases}$$

we have for all $\widehat{x}(\cdot) \in W_\infty^1([-1,1])$

$$\left|\int_{-1}^{1}x(t)\,dt - \widehat{m}_0(F_{\bar t}x(\cdot))\right| = \left|\int_{-1}^{1}(x(t) - x(\alpha(t)))\,dt\right| \le \int_{-1}^{1}|t-\alpha(t)|\,dt = \int_{-1}^{1}\widehat{x}(t)\,dt.$$

Consequently,

$$E(I, W_\infty^1([-1,1]), F_{\bar t}) = \int_{-1}^{1}\widehat{x}(t)\,dt = \frac{(t_1+1)^2}{2} + \sum_{j=1}^{n-1}\frac{(t_{j+1}-t_j)^2}{4} + \frac{(1-t_n)^2}{2}$$

and

$$\int_{-1}^{1}x(t)\,dt \approx \widehat{m}_0(F_{\bar t}x(\cdot)) = \left(\frac{t_1+t_2}{2}+1\right)x(t_1) + \sum_{j=2}^{n-1}\frac{t_{j+1}-t_{j-1}}{2}x(t_j) + \left(1-\frac{t_{n-1}+t_n}{2}\right)x(t_n)$$

is an optimal method of recovery.

If it is possible to choose the system of points $\bar t = (t_1,\dots,t_n)$ at which values of function $x(\cdot) \in W_\infty^1([-1,1])$ will be calculated (in other words, it is possible to choose an input information), then it is natural to choose these points so that the value $E(I, W_\infty^1([-1,1]), F_{\bar t})$ be as small as possible. It is easy to verify that

$$\inf_{\bar t} E(I, W_\infty^1([-1,1]), F_{\bar t}) = \frac{1}{n}$$

and the optimal points (that is, the points for which the infimum is attained) are

$$\widehat{t}_j = -1 + \frac{2j-1}{n}, \quad j = 1,\dots,n.$$

Although the information about a function in the examples considered above was incomplete but it was accurate. Practically any input information contains some error. Further examples are devoted to the cases when an information about functions is given with some error.

**3. Recovery of the derivative of a function by inaccurate values at other points.** Denote by $W^2_\infty([-1,1])$ the set of functions $x(\cdot)$ defined on the interval $[-1,1]$ for which $x'(\cdot)\in W^1_\infty([-1,1])$. Let there are known approximate values of $x(-h)$ and $x(h)$, $0<h\le 1$, for a function $x(\cdot)\in W^2_\infty([-1,1])$. It is required to recover the value $x'(0)$ in the optimal way. We assume that for every function $x(\cdot)\in W^2_\infty([-1,1])$ we know the values $\widetilde{x}_{-1}$ and $\widetilde{x}_1$ such that

$$|x(jh)-\widetilde{x}_j|\le\delta,\quad j=-1,1,\tag{2}$$

where $\delta>0$ is the error of input information. Here the information operator is the multivalued map $F_{h,\delta}$ associated with every function $x(\cdot)\in W^2_\infty([-1,1])$ the set $F_{h,\delta}x(\cdot)=\{(\widetilde{x}_{-1},\widetilde{x}_1)\}$ where $\widetilde{x}_{-1}$ and $\widetilde{x}_1$ satisfy the condition (2). Consider arbitrary functions $m\colon\mathbb{R}^2\to\mathbb{R}$ as methods of recovery. The quantity

$$e(x'(0),W^2_\infty([-1,1]),F_{h,\delta},m)=\sup_{x(\cdot)\in W^2_\infty([-1,1])}\ \sup_{\substack{\widetilde{x}_{-1},\widetilde{x}_1\\ |x(jh)-\widetilde{x}_j|\le\delta,\ j=-1,1}}|x'(0)-m(\widetilde{x}_{-1},\widetilde{x}_1)|$$

is called the error of the given method $m$. We are interested in the error of optimal recovery

$$E(x'(0),W^2_\infty([-1,1]),F_{h,\delta})=\inf_{m\,:\,\mathbb{R}^2\to\mathbb{R}}e(x'(0),W^2_\infty([-1,1]),F_{h,\delta},m)$$

and in an optimal method of recovery, that is, in a method for which this infimum is attained.

Set

$$\widehat{x}(t)=\begin{cases}-\dfrac{t^2}{2}+\left(\dfrac{h}{2}+\dfrac{\delta}{h}\right)t, & 0\le t\le 1,\\[2ex] \dfrac{t^2}{2}+\left(\dfrac{h}{2}+\dfrac{\delta}{h}\right)t, & -1\le t<0.\end{cases}$$

It is easily verified that $\pm\widehat{x}(\cdot)\in W^2_\infty([-1,1])$ and $\widehat{x}(-h)=-\delta$, $\widehat{x}(h)=\delta$. For any method $m$ we have

$$2\widehat{x}'(0)\le|\widehat{x}'(0)-m(0,0)|+|-\widehat{x}'(0)-m(0,0)|\le 2e(x'(0),W^2_\infty([-1,1]),F_{h,\delta},m).$$

Consequently,

$$E(x'(0),W^2_\infty([-1,1]),F_{h,\delta})\ge\widehat{x}'(0)=\frac{h}{2}+\frac{\delta}{h}.\tag{3}$$

Consider the method

$$\widehat{m}(\widetilde{x}_{-1},\widetilde{x}_1)=\frac{\widetilde{x}_1-\widetilde{x}_{-1}}{2h}.\tag{4}$$

Taking into account that $\widetilde{x}_j=x(jh)+\delta_j$ where $|\delta_j|\le\delta$, $j=-1,1$, we have

$$\begin{aligned}e(x'(0),W^2_\infty([-1,1]),F_{h,\delta},\widehat{m})&=\sup_{x(\cdot)\in W^2_\infty}\ \sup_{|\delta_j|\le\delta,\ j=-1,1}\left|x'(0)-\frac{x(h)-x(-h)}{2h}-\frac{\delta_1-\delta_{-1}}{2h}\right|\\&\le\sup_{x(\cdot)\in W^2_\infty([-1,1])}\left|x'(0)-\frac{x(h)-x(-h)}{2h}\right|+\frac{\delta}{h}.\end{aligned}$$

Using the equalities

$$x(h)=x(0)+x'(0)h+M_1\frac{h^2}{2},$$
$$x(-h)=x(0)-x'(0)h+M_{-1}\frac{h^2}{2},$$

where $M_1, M_{-1} \in [-1, 1]$, we obtain

$$\left| x'(0) - \frac{x(h) - x(-h)}{2h} \right| = \frac{h}{4} |M_1 - M_{-1}| \le \frac{h}{2}.$$

Thus,

$$e(x'(0), W^2_\infty([-1, 1]), F_{h,\delta}, \widehat{m}) \le \frac{h}{2} + \frac{\delta}{h}.$$

Taking into account (3) we see that

$$E(x'(0), W^2_\infty([-1, 1]), F_{h,\delta}) = \frac{h}{2} + \frac{\delta}{h}$$

and the method (4) is optimal.

We may raise the question about optimization of input information by means of choosing of step $h$. Simple calculations show that

$$\min_{0<h\le 1} E(x'(0), W^2_\infty([-1, 1]), F_{h,\delta}) = \begin{cases} \sqrt{2\delta}, & \delta < 1/2, \\ \delta + 1/2, & \delta \ge 1/2, \end{cases}$$

moreover,

$$\widehat{h} = \begin{cases} \sqrt{2\delta}, & \delta < 1/2, \\ 1, & \delta \ge 1/2, \end{cases}$$

is the value of the step for which the minimum is attained.

**4. Recovery of a function by its inaccurate Fourier coefficients.** Denote by $\mathbb{T}$ the unit circle realized as the interval $[-\pi, \pi]$ with identified endpoints. We denote by $L_2(\mathbb{T})$ the set of square integrable functions $x(\cdot)$ on $\mathbb{T}$ with norm

$$\|x(\cdot)\|_{L_2(\mathbb{T})} = \left( \frac{1}{2\pi} \int_{\mathbb{T}} |x(t)|^2 \, dt \right)^{1/2}.$$

The class $W^2_2(\mathbb{T})$ is the set of $2\pi$-periodic functions $x(\cdot)$ for which the first derivative is absolutely continuous and $\|x''(\cdot)\|_{L_2(\mathbb{T})} \le 1$.

For this class we consider the recovery problem of the first derivative of a function $x(\cdot)$ in the metric $L_2(\mathbb{T})$ by the finite system of Fourier coefficients

$$x_j = \frac{1}{2\pi} \int_{\mathbb{T}} x(t) e^{-ijt} \, dt$$

given with an error. More precisely, we assume that for every function $x(\cdot) \in W^2_2(\mathbb{T})$ we know the numbers $y_j$, $|j| \le N$, such that

$$|x_j - y_j| \le \delta, \quad |j| \le N, \quad \delta > 0. \tag{5}$$

Here the information operator is the multivalued map $F^N_\delta$ which associates with every function $x(\cdot) \in W^2_2(\mathbb{T})$ the set $F^N_\delta x(\cdot) = \{y_j\}_{|j|\le N}$ where $y_j$ satisfy the condition (5). The problem is in finding the value

$$E(x'(\cdot), W^2_2(\mathbb{T}), F^N_\delta) = \inf_{m\colon \mathbb{C}^{2N+1} \to L_2(\mathbb{T})} \sup_{\substack{x(\cdot) \in W^2_2(\mathbb{T}) \\ y \in F^N_\delta x(\cdot)}} \|x'(\cdot) - m(y)(\cdot)\|_{L_2(\mathbb{T})}$$

and an appropriate optimal method.

Analogously to the forgoing it is easy to obtain the lower bound

$$E(x'(\cdot), W_2^2(\mathbb{T}), F_\delta^N) \geq \sup_{\substack{x(\cdot)\in W_2^2(\mathbb{T}) \\ |x_j|\leq\delta,\ |j|\leq N}} \|x'(\cdot)\|_{L_2(\mathbb{T})}.$$

In view of the Parseval equality the problem in the right-hand side may be written in the form (for convenience we go over to norm squared)

$$\sum_{j\in\mathbb{Z}} j^2 u_j \to \max, \quad \sum_{j\in\mathbb{Z}} j^4 u_j \leq 1, \quad 0 \leq u_j \leq \delta^2,\ |j| \leq N, \tag{6}$$

where $u_j = |x_j|^2$, $j \in \mathbb{Z}$.

This is a problem of convex programming. It can be easily verified that for finding its solution it is sufficient to find $\widehat{\lambda} \geq 0$, $\widehat{\lambda}_j \geq 0$, $|j| \leq N$, and an admissible sequence $\{\widehat{u}_j\}_{j\in\mathbb{Z}}$ such that for all $u_j \geq 0$, $j \in \mathbb{Z}$, we have

$$(a) \quad \sum_{j\in\mathbb{Z}} (-j^2 + \widehat{\lambda} j^4 + \widehat{\lambda}_j \chi_j) u_j \geq \sum_{j\in\mathbb{Z}} (-j^2 + \widehat{\lambda} j^4 + \widehat{\lambda}_j \chi_j) \widehat{u}_j$$

and

$$(b) \quad \widehat{\lambda}\Big(\sum_{j\in\mathbb{Z}} j^4 \widehat{u}_j - 1\Big) = 0, \quad \widehat{\lambda}_j(\widehat{u}_j - \delta_j^2) = 0,\ |j| \leq N,$$

where $\chi_j = 1$, if $|j| \leq N$, and zero in other cases. Let

$$p_0 = \max\Big\{\, p \in \mathbb{Z}_+ : \delta^2 \sum_{|j|\leq p} j^4 < 1,\ 0 \leq p \leq N \,\Big\}.$$

We set $\widehat{\lambda} = (p_0 + 1)^{-2}$,

$$\widehat{\lambda}_j = \begin{cases} j^2 - (p_0+1)^{-2} j^4, & |j| \leq p_0, \\ 0, & p_0 + 1 \leq |j| \leq N. \end{cases}$$

We define the sequence $\{\widehat{u}_j\}_{j\in\mathbb{Z}}$ by the equality

$$\widehat{u}_j = \begin{cases} \delta^2, & |j| \leq p_0, \\ \dfrac{1 - \delta^2 \sum_{|k|\leq p_0} k^4}{2(p_0+1)^4}, & |j| = p_0 + 1, \\ 0, & |j| > p_0 + 1. \end{cases}$$

It is easy to check that the sequence $\widehat{u} = \{\widehat{u}_j\}_{j\in\mathbb{Z}}$ is admissible and conditions $(a)$ and $(b)$ are fulfilled. Thus $\widehat{u}$ is a solution of the problem (6). Substituting $\widehat{u}$ in the functional to be maximized and extracting the square root we obtain

$$E(x'(\cdot), W_2^2(\mathbb{T}), F_\delta^N) \geq \frac{\Big(1 + \delta^2 \sum_{|j|\leq p_0} (j^2(p_0+1)^2 - j^4)\Big)^{1/2}}{p_0 + 1}. \tag{7}$$

By the analogous sufficient arguments it is easy to verify that $\widehat{u}$ is also a solution of the following problem

$$\sum_{j\in\mathbb{Z}} j^2 u_j \to \max, \quad \widehat{\lambda} \sum_{j\in\mathbb{Z}} j^4 u_j + \sum_{|j|\leq N} \widehat{\lambda}_j u_j \leq \widehat{\lambda} + \delta^2 \sum_{|j|\leq N} \widehat{\lambda}_j,\ u_j \geq 0. \tag{8}$$

Consequently, the values of the problems (6) and (8) are the same.

Put

$$\widehat{x}_j = \begin{cases} y_0, & j = 0, \\ \dfrac{\widehat{\lambda}_j}{\widehat{\lambda} j^4 + \widehat{\lambda}_j} y_j, & 1 \le |j| \le p_0, \\ 0, & |j| > p_0. \end{cases}$$

By direct verification it is easy to verify that for all $x(\cdot) \in W_2^2(\mathbb{T})$ the equality

$$\widehat{\lambda} \sum_{j\in\mathbb{Z}} j^4 |x_j - \widehat{x}_j|^2 + \sum_{|j|\le N} \widehat{\lambda}_j |x_j - \widehat{x}_j|^2 + \widehat{\lambda} \sum_{j\in\mathbb{Z}} j^4 |\widehat{x}_j|^2 + \sum_{|j|\le N} \widehat{\lambda}_j |\widehat{x}_j - y_j|^2 = \widehat{\lambda} \sum_{j\in\mathbb{Z}} j^4 |x_j|^2 + \sum_{|j|\le N} \widehat{\lambda}_j |x_j - y_j|^2$$

holds. If $x(\cdot) \in W_2^2(\mathbb{T})$ and $|x_j - y_j| \le \delta$, then putting $v_j = |x_j - \widehat{x}_j|^2$ we have

$$\widehat{\lambda} \sum_{j\in\mathbb{Z}} j^4 v_j + \sum_{|j|\le N} \widehat{\lambda}_j v_j \le \widehat{\lambda} \sum_{j\in\mathbb{Z}} j^4 |x_j|^2 + \sum_{|j|\le N} \widehat{\lambda}_j |x_j - y_j|^2 \le \widehat{\lambda} + \delta^2 \sum_{|j|\le N} \widehat{\lambda}_j.$$

Hence

$$\Big\| x'(t) - \sum_{|j|\le N} ij \widehat{x}_j e^{ijt} \Big\|^2_{L_2(\mathbb{T})} = \sum_{j\in\mathbb{Z}} j^2 v_j \le \sup\Big\{ \sum_{j\in\mathbb{Z}} j^2 u_j : \widehat{\lambda} \sum_{j\in\mathbb{Z}} j^4 u_j + \sum_{|j|\le N} \widehat{\lambda}_j u_j \le \widehat{\lambda} + \delta^2 \sum_{|j|\le N} \widehat{\lambda}_j,\ u_j \ge 0 \Big\}.$$

Since the value of the extremal problem in the right-hand side which is the problem (8) coincides with the value of the problem (6) we obtain the upper bound for the error of optimal recovery coinciding with the lower bound (7). Thus,

$$E(x'(\cdot), W_2^2(\mathbb{T}), F_\delta^N) = \frac{\Big(1 + \delta^2 \sum_{|j|\le p_0} \big(j^2 (p_0+1)^2 - j^4\big)\Big)^{1/2}}{p_0 + 1}$$

and the method

$$x'(t) \approx \sum_{|j|\le N} ij \widehat{x}_j e^{ijt} = \sum_{|j|\le p_0} ij \left(1 - \left(\frac{j}{p_0+1}\right)^2\right) y_j e^{ijt}$$

is optimal.

Note that if $p_0 < N$, then the further increase of the number of Fourier coefficients known with the same error does not decrease the error of optimal recovery. Thus, for a fixed $\delta$ the system of $2N(\delta)$ Fourier coefficients (the zero coefficient is not used in the optimal method) where

$$N(\delta) = \max\Big\{ N \in \mathbb{Z}_+ : \delta^2 \sum_{|j|\le N} j^4 < 1 \Big\}$$

allows to recover the derivative of a function from $L_2(\mathbb{T})$ with the best accuracy. We give some values of the function $N(\delta)$ and the corresponding error of optimal recovery.

| $\delta^2$ | $N(\delta)$ | $E^2\left(x'(\cdot), W_2^2(\mathbb{T}), F_\delta^{N(\delta)}\right)$ |
|---|---|---|
| $\left[\frac{1}{2}, +\infty\right)$ | 0 | 1 |
| $\left[\frac{1}{34}, \frac{1}{2}\right)$ | 1 | $\frac{1+6\delta^2}{4}$ |
| $\left[\frac{1}{196}, \frac{1}{34}\right)$ | 2 | $\frac{1+56\delta^2}{9}$ |
| $\left[\frac{1}{1446}, \frac{1}{196}\right)$ | 3 | $\frac{1+252\delta^2}{16}$ |

In the general case if

$$\Bigl(\sum_{|j|\le k+1} j^4\Bigr)^{-1/2} \le \delta < \Bigl(\sum_{|j|\le k} j^4\Bigr)^{-1/2},$$

then $N(\delta) = k$ and

$$E\left(x'(\cdot), W_2^2(\mathbb{T}), F_\delta^{N(\delta)}\right) = \frac{\left(1 + \delta^2 \sum_{|j|\le k} \left(j^2(k+1)^2 - j^4\right)\right)^{1/2}}{k+1}.$$

**5. Recovery of a function at a point by the function itself given with an error in the $L_2$-norm.** Denote by $L_2(\mathbb{R})$ the space of functions $x(\cdot)$ defined on $\mathbb{R}$ for which

$$\|x(\cdot)\|_{L_2(\mathbb{R})} = \left(\int_{\mathbb{R}} |x(t)|^2\,dt\right)^{1/2} < \infty.$$

We denote by $\mathcal{W}_2^1(\mathbb{R})$ the space of functions $x(\cdot) \in L_2(\mathbb{R})$ for which $\|x'(\cdot)\|_{L_2(\mathbb{R})} < \infty$ and by $W_2^1(\mathbb{R})$ the class of functions from $\mathcal{W}_2^1(\mathbb{R})$ for which $\|x'(\cdot)\|_{L_2(\mathbb{T})} \le 1$. For the class $W_2^1(\mathbb{R})$ we consider the problem of optimal recovery of the value $x(0)$ by the information about the function $x(\cdot)$ itself given with the error $\delta > 0$ in the norm of $L_2(\mathbb{R})$. In other words, we assume that for each function $x(\cdot) \in W_2^1(\mathbb{R})$ we know a function $y(\cdot) \in L_2(\mathbb{R})$ such that

$$\|x(\cdot) - y(\cdot)\|_{L_2(\mathbb{R})} \le \delta. \tag{9}$$

Thus, for the information operator $F_\delta$ we take here the multivalued map which associates with every function $x(\cdot) \in W_2^1(\mathbb{R})$ the set of functions $y(\cdot) \in L_2(\mathbb{R})$ satisfying the condition (9). As in the previous examples we interested in the error of optimal recovery

$$E(x(0), W_2^1(\mathbb{R}), F_\delta) = \inf_{m\colon L_2(\mathbb{R})\to\mathbb{R}} \sup_{\substack{x(\cdot)\in W_2^1(\mathbb{R}),\ y(\cdot)\in L_2(\mathbb{R})\\ \|x(\cdot)-y(\cdot)\|_{L_2(\mathbb{R})}\le\delta}} |x(0) - m(y(\cdot))|$$

and also in an optimal recovery method (a method for which the infimum is attained).

In the same way as above we prove the estimate

$$E(x(0), W_2^1(\mathbb{R}), F_\delta) \ge \sup_{\substack{x(\cdot)\in W_2^1(\mathbb{R})\\ \|x(\cdot)\|_{L_2(\mathbb{R})}\le\delta}} |x(0)|.$$

Since the function

$$\widehat{x}(t) = \sqrt{\delta} e^{-|t|/\delta}$$

belongs to the class $W_2^1(\mathbb{R})$ and, moreover, $\|\widehat{x}(\cdot)\|_{L_2(\mathbb{R})} = \delta$, we have

$$E(x(0), W_2^1(\mathbb{R}), F_\delta) \geq \widehat{x}(0) = \sqrt{\delta}.$$

To find an optimal method of recovery we use the easily verified identity

$$x(0) = \frac{1}{2\delta} \int_{\mathbb{R}} e^{-|t|/\delta} x(t)\, dt - \frac{1}{2} \int_{\mathbb{R}} e^{-|t|/\delta} x'(t) \operatorname{sign} t\, dt \tag{10}$$

which is valid for all $x(\cdot) \in \mathcal{W}_2^1(\mathbb{R})$. Consider the method

$$m(y(\cdot)) = \frac{1}{2\delta} \int_{\mathbb{R}} e^{-|t|/\delta} y(t)\, dt. \tag{11}$$

Using the identity (10) and applying the Caushy–Schwartz–Bunyakovskiĭ inequality we have

$$E(x(0), W_2^1(\mathbb{R}), F_\delta) = \sup_{x(\cdot) \in W_2^1(\mathbb{R})} \sup_{\substack{y(\cdot) \in L_2(\mathbb{R}) \\ \|x(\cdot) - y(\cdot)\|_{L_2(\mathbb{R})} \leq \delta}} \Bigg| x(0) - \frac{1}{2\delta} \int_{\mathbb{R}} e^{-|t|/\delta} x(t)\, dt$$

$$- \frac{1}{2\delta} \int_{\mathbb{R}} e^{-|t|/\delta} (y(t) - x(t))\, dt \Bigg| \leq \sup_{x(\cdot) \in W_2^1(\mathbb{R})} \left| \frac{1}{2} \int_{\mathbb{R}} e^{-|t|/\delta} x'(t) \operatorname{sign} t\, dt \right| + \frac{\sqrt{\delta}}{2} \leq \sqrt{\delta}.$$

Hence and from the corresponding lower bound it follows that the equality

$$E(x(0), W_2^1(\mathbb{R}), F_\delta) = \sqrt{\delta}$$

holds and the method (11) is optimal.

Moreover, by the fact proved above it follows that the value of the problem

$$x(0) \to \max, \quad \|x(\cdot)\|_{L_2(\mathbb{R})} \leq \delta, \quad \|x'(\cdot)\|_{L_2(\mathbb{R})} \leq 1, \tag{12}$$

is equal to $\sqrt{\delta}$. For all $x(\cdot) \in \mathcal{W}_2^1(\mathbb{R})$ the function $x(\cdot)/\|x'(\cdot)\|_{L_2(\mathbb{R})}$ is admissible in the problem (12) with $\delta = \|x(\cdot)\|_{L_2(\mathbb{R})}/\|x'(\cdot)\|_{L_2(\mathbb{R})}$. Consequently, for all $x(\cdot) \in \mathcal{W}_2^1(\mathbb{R})$ the inequality

$$\frac{|x(0)|}{\|x'(\cdot)\|_{L_2(\mathbb{R})}} \leq \frac{\|x(\cdot)\|_{L_2(\mathbb{R})}^{1/2}}{\|x'(\cdot)\|_{L_2(\mathbb{R})}^{1/2}}$$

holds, that is,

$$|x(0)| \leq \|x(\cdot)\|_{L_2(\mathbb{R})}^{1/2} \|x'(\cdot)\|_{L_2(\mathbb{R})}^{1/2}.$$

In view of the translation invariance of the norm the point 0 may be replaced by any point $\tau \in \mathbb{R}$.

The obtained inequality may be considered as some uncertainty principle for functions from $\mathcal{W}_2^1(\mathbb{R})$ which means that for a fixed value of a function at an arbitrary point the norms of the function and its derivative cannot be small simultaneously, their product is always no less than this squared value.

**6. Recovery of a function by its inaccurate values in a weighted norm.** Denote by $L_2(\mathbb{R}, t^2)$ the space of functions $x(\cdot)$ defined on $\mathbb{R}$ such that

$$\|x(\cdot)\|_{L_2(\mathbb{R}, t^2)} = \left( \int_{\mathbb{R}} t^2 |x(t)|^2\, dt \right)^{1/2} < \infty.$$

We denote by $\mathcal{W}_2^1(\mathbb{R},t^2)$ the space of functions from $L_2(\mathbb{R},t^2)$ for which $x'(\cdot)\in L_2(\mathbb{R})$. Set

$$W_2^1(\mathbb{R},t^2)=\{\,x(\cdot)\in\mathcal{W}_2^1(\mathbb{R},t^2):\|x'(\cdot)\|_{L_2(\mathbb{R})}\le 1\,\}.$$

For the class $W_2^1(\mathbb{R},t^2)$ we consider the problem of optimal recovery of a function $x(\cdot)$ by the information about its inaccurate values in the $L_2(\mathbb{R},t^2)$-norm. We assume that for any function $x(\cdot)\in W_2^1(\mathbb{R},t^2)$ we know a function $y(\cdot)\in L_2(\mathbb{R},t^2)$ such that

$$\|x(\cdot)-y(\cdot)\|_{L_2(\mathbb{R},t^2)}\le\delta. \tag{13}$$

Here for the information operator $F_\delta$ we consider the multivalued map which associates with each function $x(\cdot)\in W_2^1(\mathbb{R},t^2)$ the set of functions $y(\cdot)\in L_2(\mathbb{R},t^2)$ satisfying the condition (13). We are interested in the error of optimal recovery

$$E(x(\cdot),W_2^1(\mathbb{R},t^2),F_\delta)=\inf_{m\,:\,L_2(\mathbb{R},t^2)\to L_2(\mathbb{R})}\ \sup_{\substack{x(\cdot)\in W_2^1(\mathbb{R},t^2),\ y(\cdot)\in L_2(\mathbb{R},t^2)\\ \|x(\cdot)-y(\cdot)\|_{L_2(\mathbb{R},t^2)}\le\delta}}\|x(\cdot)-m(y)(\cdot)\|_{L_2(\mathbb{R})}$$

and in an optimal method of recovery, too.

The arguments analogous to the ones given in the previous examples lead to the inequality

$$E(x(\cdot),W_2^1(\mathbb{R},t^2),F_\delta)\ge\sup_{\substack{\|x(\cdot)\|_{L_2(\mathbb{R},t^2)}\le\delta\\ \|x'(\cdot)\|_{L_2(\mathbb{R})}\le 1}}\|x(\cdot)\|_{L_2(\mathbb{R})}.$$

To solve the extremal problem

$$\|x(\cdot)\|^2_{L_2(\mathbb{R})}\to\max,\quad \|x(\cdot)\|^2_{L_2(\mathbb{R},t^2)}\le\delta^2,\quad \|x'(\cdot)\|^2_{L_2(\mathbb{R})}\le 1, \tag{14}$$

we consider the Lagrange function

$$\mathcal{L}(x(\cdot),\lambda_1,\lambda_2)=-\int_{\mathbb{R}}x^2(t)\,dt+\lambda_1\int_{\mathbb{R}}t^2x^2(t)\,dt+\lambda_2\int_{\mathbb{R}}x'^2(t)\,dt.$$

It is easy to show that for the function $\widehat{x}(\cdot)\in W_2^1(\mathbb{R},t^2)$ to be a solution of the problem (14) it is sufficient to find $\widehat{\lambda}_1,\widehat{\lambda}_2\ge 0$ for which

$$\min_{x(\cdot)\in W_2^1(\mathbb{R},t^2)}\mathcal{L}(x(\cdot),\widehat{\lambda}_1,\widehat{\lambda}_2)=\mathcal{L}(\widehat{x}(\cdot),\widehat{\lambda}_1,\widehat{\lambda}_2)$$

and

$$\widehat{\lambda}_1\left(\int_{\mathbb{R}}t^2\widehat{x}^2(t)\,dt-\delta^2\right)=0,\quad \widehat{\lambda}_2\left(\int_{\mathbb{R}}\widehat{x}'^2(t)\,dt-1\right)=0.$$

Set $\widehat{\lambda}_1=\delta^{-1}$ and $\widehat{\lambda}_2=\delta$. Integrating by parts the first term of the Lagrange function we obtain

$$\mathcal{L}(x(\cdot),\widehat{\lambda}_1,\widehat{\lambda}_2)=\frac{1}{\delta}\int_{\mathbb{R}}\left(tx(t)+\delta x'(t)\right)^2\,dt.$$

It is obvious that the Lagrange function vanishes on the function

$$\widehat{x}(t)=\sqrt{2}\left(\frac{\delta}{\pi}\right)^{1/4}e^{-\frac{t^2}{2\delta}},$$

that is, the minimum is attained for this function. Since

$$\int_{\mathbb{R}}t^2\widehat{x}^2(t)\,dt=\delta^2,\quad \int_{\mathbb{R}}\widehat{x}'^2(t)\,dt=1,$$

$\widehat{x}(\cdot)$ is a solution of the problem (14). Thus,

$$E(x(\cdot), W_2^1(\mathbb{R}, t^2), F_\delta) \geq \left(\int_{\mathbb{R}} \widehat{x}^2(t)\, dt\right)^{1/2} = \sqrt{2\delta}.$$

It follows from the analogous sufficient arguments that the function $\widehat{x}(\cdot)$ is also a solution of the extremal problem

$$\|x(\cdot)\|^2_{L_2(\mathbb{R})} \to \max, \quad \delta^{-1}\|x(\cdot)\|^2_{L_2(\mathbb{R},t^2)} + \delta\|x'(\cdot)\|^2_{L_2(\mathbb{R})} \leq 2\delta. \tag{15}$$

We now proceed to the construction of optimal method of recovery. Set

$$\psi_n(t) = H_n\left(\frac{t}{\sqrt{\delta}}\right) e^{-\frac{t^2}{2\delta}}, \quad n = 0, 1, \ldots,$$

where $H_n(\cdot)$ are the Chebyshev–Hermite polynomials ($\{H_n(\cdot)\}_{n=0}^{\infty}$ is an orthogonal system of polynomials on $\mathbb{R}$ for the weight function $e^{-x^2}$ with leading coefficients $a_n = 2^n$). The functions $\psi_n(\cdot)$, $n = 0, 1, \ldots$, form an orthogonal basis in $L_2(\mathbb{R})$. Let $y(\cdot) \in L_2(\mathbb{R}, t^2)$ and

$$ty(t) = \sum_{n=0}^{\infty} y_n \psi_n(t).$$

Set

$$\widehat{x}(t) = \sum_{n=0}^{\infty} \widehat{x}_n \psi_n(t),$$

where

$$\widehat{x}_0 = \frac{y_1}{\sqrt{\delta}}, \quad \widehat{x}_n = \frac{(n+1)y_{n+1} + y_{n-1}/2}{\sqrt{\delta}(2n+1)}, \quad n = 1, 2, \ldots .$$

Using properties of Chebyshev–Hermite polynomials one can show that for all $z(\cdot) \in \mathcal{W}_2^1(\mathbb{R}, t^2)$ the equality

$$\frac{1}{\delta}\int_{\mathbb{R}} t^2(\widehat{x}(t) - y(t))z(t)\, dt + \delta\int_{\mathbb{R}} \widehat{x}'(t)z'(t)\, dt = 0$$

holds. It follows that for any $x(\cdot) \in W_2^1(\mathbb{R}, t^2)$

$$\begin{aligned}\frac{1}{\delta}\int_{\mathbb{R}} t^2(x(t) - \widehat{x}(t))^2\, dt + \delta\int_{\mathbb{R}} (x'(t) - \widehat{x}'(t))^2\, dt + \frac{1}{\delta}\int_{\mathbb{R}} t^2(\widehat{x}(t) - y(t))^2\, dt + \delta\int_{\mathbb{R}} \widehat{x}'^2(t)\, dt \\ = \frac{1}{\delta}\int_{\mathbb{R}} t^2(x(t) - y(t))^2\, dt + \delta\int_{\mathbb{R}} x'^2(t)\, dt.\end{aligned}$$

If $x(\cdot) \in W_2^1(\mathbb{R}, t^2)$ and $\|x(\cdot) - y(\cdot)\|_{L_2(\mathbb{R},t^2)} \leq \delta$, then putting $z(\cdot) = x(\cdot) - \widehat{x}(\cdot)$ we have

$$\frac{1}{\delta}\int_{\mathbb{R}} t^2 z^2(t)\, dt + \delta\int_{\mathbb{R}} z'^2(t)\, dt \leq \frac{1}{\delta}\int_{\mathbb{R}} t^2(x(t) - y(t))^2\, dt + \delta\int_{\mathbb{R}} x'^2(t)\, dt \leq 2\delta.$$

Hence

$$\|x(\cdot) - \widehat{x}(\cdot)\|_{L_2(\mathbb{R})} = \|z(\cdot)\|_{L_2(\mathbb{R})} \leq \sup\left\{\|x(\cdot)\|_{L_2(\mathbb{R})} : \delta^{-1}\|x(\cdot)\|^2_{L_2(\mathbb{R},t^2)} + \delta\|x'(\cdot)\|^2_{L_2(\mathbb{R})} \leq 2\delta\right\}.$$

The squared value of the extremal problem in the right-hand side coincides with the value of the problem (15) and thus with the value of the problem (14). Consequently, we have obtained the upper bound for the error of optimal recovery coincided with the lower bound. Thus,

$$E(x(\cdot), W_2^1(\mathbb{R}, t^2), F_\delta) = \sqrt{2\delta} \tag{16}$$

and the method

$$x(t) \approx \sum_{n=0}^{\infty} \widehat{x}_n \psi_n(t) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \alpha_n H_n\left(\frac{t}{\sqrt{\delta}}\right) e^{-\frac{t^2}{2\delta}},$$

where

$$\alpha_n = \frac{1}{(2n+1)2^n n!} \int_{\mathbb{R}} y(t) H_n\left(\frac{t}{\sqrt{\delta}}\right) e^{-\frac{t^2}{2\delta}} t^2\, dt,$$

is optimal.

By the same arguments as in the previous example from (16) follows the following exact inequality

$$\int_{\mathbb{R}} x^2(t)\, dt \le 2 \left(\int_{\mathbb{R}} t^2 x^2(t)\, dt\right)^{1/2} \left(\int_{\mathbb{R}} x'^2(t)\, dt\right)^{1/2}. \tag{17}$$

If we consider functions $x(\cdot)$ normalized by the condition

$$\int_{\mathbb{R}} x^2(t)\, dt = 1,$$

then from (17) follows the inequality

$$\int_{\mathbb{R}} t^2 x^2(t)\, dt \int_{\mathbb{R}} x'^2(t)\, dt \ge 1/4$$

which is known as the Heisenberg Uncertainty Principle.

## 4. THEORY

The examples given above were solved directly without using any general assertions. We did it deliberately to concentrate the reader's attention on the examples themselves. Here we give a general result connected with optimal recovery of linear functionals (we have used it, in fact, in some examples).

Using the notation from Section 2 let $C$ be a subset of real or complex linear space $X$, $X'$ be the algebraic dual of $X$, and $f = x' \in X'$, that is, $Z = \mathbb{R}$ or $\mathbb{C}$. Denote by $\langle x', x\rangle$ the value of the linear functional $x'$ at the element $x \in X$. Let $Y$ be another real or complex linear space, $Y'$ be the algebraic dual of it, and $F\colon C \to Y$ be a map (in general, a multivalued map). The problem is to recover the values of the linear functional $x'$ on the set $C$ by the information $F$.

For definiteness we assume that $X$ and $Y$ are complex linear spaces. Any map $m\colon F(C) \to \mathbb{C}$, as before, we call a *method of recovery*. The error of such method is the quantity

$$e(x', C, F, m) = \sup_{x \in C,\ y \in F(x)} |\langle x', x\rangle - m(y)| \tag{18}$$

and the *error of optimal recovery* ($x'$ on $C$ by $F$), which we denote by $E(x', C, F)$, is defined as a solution of the problem:

$$e(x', C, F, m) \to \min, \tag{19}$$

where the infimum is taking over all maps $m\colon F(C) \to \mathbb{C}$. Any method $\widehat{m}$ which is a solution of this problem is called an *optimal method of recovery*.

We associate with the problem (19) the following extremal problem

$$\mathrm{Re}\langle x', x\rangle \to \max, \quad x \in F^{-1}(0), \quad x \in C, \tag{20}$$

where $F^{-1}(y) = \{x \in C \mid y \in F(x)\}$.

The function

$$\mathcal{L}\left((x, y), \lambda_0, y'\right) = \lambda_0 \operatorname{Re}\langle x', x\rangle + \operatorname{Re}\langle y', y\rangle$$

is called the Lagrange function of the problem (20), and the number $\lambda_0$ and functional $y' \in Y'$ are the Lagrange multipliers.

**Theorem 1.** *Let the sets $C$ and $\operatorname{gr} F = \{(x, y) : x \in C,\ y \in F(x)\}$ from the problem* (20) *be convex and balanced. Then the admissible in* (20) *point $\widehat{x}$ is a solution in this problem if and only if there exists the Lagrange multiplier $\widehat{y}' \in Y'$ for which*

$$\min_{\substack{x \in C \\ y \in F(x)}} \mathcal{L}\left((x, y), -1, \widehat{y}'\right) = \mathcal{L}\left((\widehat{x}, 0), -1, \widehat{y}'\right).$$

*In this case $\widehat{y}'$ is an optimal method of recovery in the problem* (19) *and $E(x', C, F) = \operatorname{Re}\langle x', \widehat{x}\rangle$.*

It is clear from this theorem that for finding an optimal method in (19) it is sufficient to solve the problem (20) which is convex. Solving it by the standard methods of convex optimization, we find at the same time the Lagrange multipliers, that is, we find an optimal method of recovery (which turns out to be linear). From the point of view of convex duality it means that the problems (19) and (20) are dual to each other.

For optimal recovery of linear operators the lower bound of the error of optimal recovery is also reduced to solving a problem analogous to (20) but the upper bound needs individual arguments. There are some general concepts about it but we shall not dwell on them here (see [10]).

The proof of the formulated theorem may be found in [9].

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