ON BEST QUADRATURE FORMULAS ON HARDY–SOBOLEV CLASSES

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Abstract. For functions from Hardy–Sobolev classes defined as the set of functions analytic in the unit disk and satisfying the condition $|f^{(r)}(z)| \le 1$ best quadrature formulas which used values of functions and their derivatives at the fixed system of points from the interval $(-1,1)$ are constructed. For the periodic Hardy–Sobolev class $H^r_{\infty,\beta}$ which is defined as the set of 2π-periodic functions analytic in the strip $|\operatorname{Im} z| < \beta$ and satisfying the condition $|f^{(r)}(z)| \le 1$ it is proved that the rectangle formula is best for the equidistant system of points and the error of this formula is calculated. Best quadrature formulas on the class $H_{p,\beta}$ which is defined in the similar way as the class $H_{\infty,\beta}$ but the boundary values of functions are taken in L_p-norm are constructed. An optimal method of recovery of functions from H^r_p using the Taylor information $f(0), f'(0), \dots, f^{(n+r-1)}(0)$ is obtained, too.

Introduction

Let X be a linear space over the field $K = \mathbb{R}$ or $\mathbb{C}$, W a subset of X, and $L, l_1, \dots, l_n$ linear functionals on X. The problem of optimal recovery of functional L on the set W from the values of the information operator $Ix = (l_1x, \dots, l_nx)$, $x \in W$, is the problem of finding the value

$$e(L,W,I) := \inf_{S\colon K^n \to K} \sup_{x \in W} |Lx - S(Ix)| \tag{1}$$

and a method S for which the infimum in (1) is attained (if such method exists) which is called an optimal method of recovery.

Optimal recovery problems beginning with the paper [1] are studied by many authors (see [2]–[5] and the literature cited there). We mention here only one result which was proved in [1] for the real space and in [6] for the complex one: for a convex balanced set W among optimal methods of recovery there exists a linear method and the equality

$$e(L,W,I) = \sup_{\substack{x \in W \\ Ix = 0}} |Lx| \tag{2}$$

holds. Any element x_0 for which the supremum in (2) is attained we call extremal.

This research was carried out with the financial support of the Russian Foundation for Basic Research (grants №99-01-01181 and №00–15–96109).

The problem (2) often turns out more simple than the problem of finding an optimal recovery method. In this connection in [7] it was proposed a method allowing to obtain an optimal method of recovery if there exists some parametrization of extremal element in the problem (2). Here this method is used for obtaining best quadrature formulas and an optimal recovery method by Taylor information on the Hardy–Sobolev classes.

We shall call the Hardy–Sobolev class H_p^r the set of functions f analytic in the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the condition

$$\sup_{0<\rho<1} \frac{1}{2\pi} \int_0^{2\pi} |f^{(r)}(\rho e^{i\theta})|^p \, d\theta \le 1, \quad 1 \le p < \infty,$$
$$\sup_{z \in D} |f^{(r)}(z)| \le 1, \quad p = \infty.$$

We call the periodic Hardy–Sobolev class $H_{p,\beta}^r$ the set of 2π-periodic functions f analytic in the strip $S_\beta := \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$ and satisfying the condition

$$\sup_{0 \le \eta < \beta} \left(\frac{1}{4\pi} \int_0^{2\pi} \left(|f^{(r)}(t+i\eta)|^p + |f^{(r)}(t-i\eta)|^p \right) \, dt \right)^{1/p} \le 1,$$
$$\sup_{z \in S_\beta} |f^{(r)}(z)| \le 1.$$

For $r = 0$ we denote the appropriated classes by H_p and $H_{p,\beta}$.

In §1 for H_∞^r and the information operator

$$(3) \qquad If = (f(x_1), \dots, f^{(\nu_1-1)}(x_1), \dots, f(x_n), \dots, f^{(\nu_n-1)}(x_n)),$$

where $x_1, \dots, x_n$ are distinct points from the interval $(-1, 1)$ and $\nu_1, \dots, \nu_n$ are even, a linear optimal method of integration (a best quadrature formula) for the integral

$$\int_{-1}^1 f(x)p(x) \, dx$$

in which $p(x)$ is a nonnegative weight function is constructed.

In §2 for equidistant system of points a best quadrature formula for the class $H_{\infty,\beta}^r$ is constructed. It is proved that this formula is the rectangle formula and its error is found. For $r = 0$ best quadrature formulas on the classes H_∞ and $H_{\infty,\beta}$ were investigated in [8]–[10].

In §3 best quadrature formulas on the class $H_{p,\beta}$ by the information operator (3) in which $x_1, \dots, x_n$ are distinct points from $\mathbb{T} := [0, 2\pi)$ are constructed. The similar problem in the non-periodic case was solved in [11, p. 175]. In §4 an optimal method of recovery of functions from H_p^r by the information operator $If = (f(0), f'(0), \dots, f^{(n+r-1)}(0))$ is constructed. In this problem optimal methods were previously investigated in [12] ($p = \infty$, $r = 0$), [2]

($p=\infty$, $r=1$), [13] ($1\le p\le\infty$, $r=0$), [14] ($p=\infty$, $r\in\mathbb{Z}_+$, many-dimensional case), [11, p. 65] ($1\le p\le\infty$, $r=0$, many-dimensional case).

We need the following result from [7].

Theorem 1. *Let X be a real linear space, W a convex centrally symmetric set from X, and x_0 an extremal element in the problem of optimal recovery of a linear functional L on the set W by the values of linear functionals $l_1x,\dots,l_nx$. Let every $M=(t_1,\dots,t_n)\in\mathbb{R}^n$ from some neighborhood of $M_0\in\mathbb{R}^n$ associates with $x(M)\in W$ where $x(M_0)=x_0$. Then if the functions $\varphi(M):=Lx(M)$, $\varphi_j(M):=l_jx(M)$, $j=1,\dots,n$, have continuous partial derivatives with respect to all variables in a neighborhood of M_0 and the determinant of the matrix*

$$J(M)=\begin{pmatrix}\dfrac{\partial\varphi_1}{\partial t_1} & \dots & \dfrac{\partial\varphi_n}{\partial t_1}\\ \dots & \dots & \dots\\ \dfrac{\partial\varphi_1}{\partial t_n} & \dots & \dfrac{\partial\varphi_n}{\partial t_n}\end{pmatrix}$$

does not vanish at M_0, then the method

$$Lx\approx\sum_{j=1}^n C_jl_jx,$$

where $C_1,\dots,C_n$ are solutions of the system

$$J(M_0)\mathbf{C}=\operatorname{grad}\varphi\Big|_{M_0}$$

in which $\mathbf{C}=(C_1,\dots,C_n)$, is the unique linear optimal method of recovery.

1. Best quadrature formulas on the class H^r_∞

Consider the problem of optimal recovery (1) for $W=H^r_\infty$,

$$Lf=\int_{-1}^1 f(x)p(x)\,dx, \tag{4}$$

where $p(x)$ is a nonnegative weight function, and the information operator I defined by (3). Put

$$N:=\sum_{j=1}^n\nu_j. \tag{5}$$

First, we prove some auxiliary assertions. Recall that the system of real functions $\{u_k(t)\}_{k=0}^m$ m times continuously differentiable on the interval (c,d) is called ET-system, if every generalized polynomial

$$P(t)=\sum_{k=0}^m C_ku_k(t),\quad \sum_{k=0}^m C_k^2\ne0,$$

has at most m zeros on (c,d) with regard to the algebraic multiplicity.

The Blaschke product of degree n is a function of the form

$$B(z)=\lambda\prod_{j=1}^{n}\frac{z-z_j}{1-\overline{z}_j z},$$

where $|\lambda|=1$, a $z_j\in D$, $j=1,\dots,n$. For $\mu_j\in\mathbb{N}$, $j=1,\dots,n$, and $\alpha_j\in(-1,1)$ set

$$W_j(x):=\frac{x-\alpha_j}{1-\alpha_j x},\quad W(x):=\prod_{j=1}^{m}\left(\frac{x-\alpha_j}{1-\alpha_j x}\right)^{\mu_j}.$$

Lemma 1. *The system of functions*

$$g_{jk}(x):=W(x)\left(W_j^{-k}(x)-W_j^{k}(x)\right),\qquad k=1,\dots\mu_j,\ j=1,\dots,m, \tag{6}$$

is a ET-system on $(-1,1)$.

Proof. Consider the generalized polynomial

$$P(x)=\sum_{j=1}^{m}\sum_{k=1}^{\mu_j}C_{jk}g_{jk}(x),\qquad \sum_{j=1}^{m}\sum_{k=1}^{\mu_j}C_{jk}^2\neq 0.$$

In view of the fact that $W_j(\pm1)=\pm1$ this generalized polynomial may be written in the form

$$P(x)=a_0\frac{(1-x^2)x^l\prod_{j=1}^{s}(x-a_j)}{\prod_{j=1}^{m}(1-\alpha_j x)^{2\mu_j}},$$

where $a_0,a_1,\dots,a_s\neq 0$. Since $W_j(x^{-1})=W_j^{-1}(x)$ we have

$$P(x^{-1})=\frac{1}{W(x)}\sum_{j=1}^{m}\sum_{k=1}^{\mu_j}C_{jk}\left(W_j^{k}(x)-W_j^{-k}(x)\right)=-W^2(x)P(x).$$

¿From the last equality it is easy to obtain that with every zero $a_j\neq 0$ of P coincides the zero of this function a_j^{-1} with the same multiplicity, and moreover, $l+s/2=\sum_{j=1}^{m}\mu_j-1$. Thus the generalized polynomial P has at most $\sum_{j=1}^{m}\mu_j-1$ zeros on the interval $(-1,1)$ with regard to the algebraic multiplicity. $\square$

For functions f analytic in the unit disk set $T_0 f:=f$ and

$$(T_r f)(z):=\int_0^z\frac{(z-\zeta)^{r-1}}{(r-1)!}f(\zeta)\,d\zeta,\quad r\in\mathbb{N}. \tag{7}$$

Obviously, $(T_r f)^{(r)}=f$ and consequently, $T_r f\in H_\infty^r$ for all $f\in H_\infty$.

Let

$$\sum_{j=1}^{m}\mu_j+r=N.$$

Define the functions $\omega_1, \ldots, \omega_N$ by the equality

$$(8) \quad (\omega_1(z), \ldots, \omega_N(z)) := (1, z, \ldots, z^{r-1}, \\ (T_r g_{11})(z), \ldots, (T_r g_{1\mu_1})(z), \ldots, (T_r g_{m1})(z), \ldots, (T_r g_{m\mu_m})(z)).$$

Set

$$(9) \qquad (a_{j1}, \ldots, a_{jN}) := I\omega_j, \quad j = 1, \ldots, N, \quad A := \{a_{jk}\}_{j,k=1}^{N}.$$

Lemma 2. $\det A \neq 0$.

Proof. If $\det A = 0$, then there exist $C_1, \ldots, C_N$ not all equal zero for which the function

$$F(z) := \sum_{j=1}^{N} C_j \omega_j(z)$$

has at least N zeros on the interval $(-1, 1)$ counting multiplicities. In this case by Rolle's theorem $F^{(r)}$ must have at least $N - r$ zeros on the same interval. Since

$$F^{(r)}(z) = C_{r+1} g_{11}(z) + \ldots + C_N g_{m\mu_m}(z),$$

by Lemma 1 it follows that $C_{r+1} = \ldots = C_N = 0$, but then $C_1 = \ldots = C_r = 0$. The contradiction so obtained proves that $\det A \neq 0$. $\square$

Denote by $H_\infty^{r,\mathbb{R}}$ the set of functions from H_∞^r real on the interval $(-1, 1)$.

Proposition 1. *Let $-1 < x_1 < x_2 < \ldots < x_n < 1$. Then for all even $\nu_1, \ldots, \nu_n$ there exists a function $F \in H_\infty^{r,\mathbb{R}}$ of the form*

$$F = P_{r-1} + T_r W,$$

where P_{r-1} is a polynomial of degree $r-1$ and W is a Blaschke product of degree $N - r$

$$W(z) = \prod_{j=1}^{m} \left(\frac{z - \alpha_j}{1 - \alpha_j z} \right)^{\mu_j}, \quad \sum_{j=1}^{m} \mu_j = N - r,$$

$x_1 \leq \alpha_1 < \ldots < \alpha_m \leq x_n$, such that $IF = 0$ and

$$\sup_{\substack{f \in H_\infty^{r,\mathbb{R}} \\ If=0}} \int_{-1}^{1} f(x) p(x)\, dx = \int_{-1}^{1} F(x) p(x)\, dx.$$

Proof. It follows from [15] that there exists a function $F \in H_\infty^{r,\mathbb{R}}$ normalized by the condition $F(1) > 0$ for which $IF = 0$ and $F^{(r)}$ is a Blaschke product of degree $N - r$. Moreover, in the same paper it was proved that for all $x \in (-1, 1)$ the equality

$$(10) \qquad \sup_{\substack{f \in H_\infty^{r,\mathbb{R}} \\ If=0}} |f(x)| = |F(x)|$$

holds. By Rolle's theorem it follows that F has no other zeros on the interval $(-1, 1)$ except the zeros at the points $x_1, \ldots, x_n$ with even multiplicities $\nu_1, \dots, \nu_n$. Thus in view of the normalization $F(1) > 0$ for all $x \in (-1,1)$, $F(x) \ge 0$. Taking into account (10) we obtain the assertion of the proposition. $\square$

Theorem 2. *Let $-1 < x_1 < x_2 < \dots < x_n < 1$, $\nu_1, \dots, \nu_n$ be even numbers, W a Blaschke product from Proposition* 1, *and g_{jk}, ω_j, and the matrix A be defined by* (6), (8), *and* (9), *respectively. Then the method*

$$\int_{-1}^{1} f(x)p(x)\,dx \approx \sum_{j=1}^{n} \sum_{k=0}^{\nu_j-1} c_{jk} f^{(k)}(x_j), \tag{11}$$

in which c_{jk} are defined by the system

$$A\mathbf{c} = \mathbf{d}, \tag{12}$$

where $\mathbf{c} = (c_{10}, \dots, c_{1,\nu_1-1}, \dots, c_{n0}, \dots, c_{n,\nu_n-1})$, $\mathbf{d} = (d_1, \dots, d_N)$,

$$d_j = \int_{-1}^{1} \omega_j(x)p(x)\,dx, \quad j = 1, \dots, N,$$

is optimal on the class H^r_∞.

Proof. First, we prove that the method (11) is optimal on the class $H^{r,\mathbb{R}}_\infty$. Put $W_{j0}(z) := 1$, $j = 1, \dots, m$, and

$$W_{j,k+1}(z) := \frac{W_j(z)W_{jk}(z) + \varepsilon_{j,k+1}}{1 + \varepsilon_{j,k+1}W_j(z)W_{jk}(z)}, \quad j = 1, \dots, m,\ k = 0, \dots, \mu_j - 1.$$

For all $\varepsilon_{j1}, \dots, \varepsilon_{j,\mu_j} \in (-1,1)$, $W_{j,\mu_j} \in H_\infty$. Set

$$f_P(z) := \sum_{j=0}^{r-1} a_j z^j + (T_r W_P)(z),$$

where $P = (a_0, \dots, a_{r-1}, \varepsilon_{11}, \dots, \varepsilon_{1,\mu_1}, \dots, \varepsilon_{m1}, \dots, \varepsilon_{m,\mu_m}) \in \mathbb{R}^N$ and

$$W_P(z) = \prod_{j=1}^{m} W_{j,\mu_j}(z).$$

Let the polynomial P_{r-1} from Proposition 1 be of the form

$$P_{r-1}(z) = \sum_{j=0}^{r-1} a_j^0 z^j.$$

Then in view of Proposition 1 for $P = P_0 := (a_0^0, \dots, a_{r-1}^0, 0, \dots, 0)$ the function f_{P_0} is extremal in the problem of optimal recovery of the integral (4) on the class $H^{r,\mathbb{R}}_\infty$ by the information (3). Define the function $\varphi_1, \dots, \varphi_N$ by the equality

$$(\varphi_1(P), \dots, \varphi_N(P)) := If_P.$$

It is easily seen that at the point P_0 we have

$$\left(\frac{\partial\varphi_1}{\partial a_j},\dots,\frac{\partial\varphi_N}{\partial a_j}\right) = I\omega_{j+1}, \quad 0\le j\le r-1,$$
$$\left(\frac{\partial\varphi_1}{\partial \varepsilon_{jk}},\dots,\frac{\partial\varphi_N}{\partial \varepsilon_{jk}}\right) = I(T_r g_{jk}), \quad 1\le j\le m,\ 1\le k\le \mu_j.$$

Putting

$$\varphi(P)=\int_{-1}^{1} f_P(x)p(x)\,dx,$$

it is easy to verify that at the point P_0

$$\frac{\partial\varphi}{\partial a_j}=\int_{-1}^{1} x^j p(x)\,dx, \quad 0\le j\le r-1,$$
$$\frac{\partial\varphi}{\partial \varepsilon_{jk}}=\int_{-1}^{1} (T_r g_{jk})(x)p(x)\,dx, \quad 1\le j\le m,\ 1\le k\le \mu_j.$$

By Theorem 1, taking into account Lemma 2, it follows now that the coefficients of optimal method for the class $H_\infty^{r,\mathbb{R}}$ are defined from the system (12). Now we prove that the constructed method (we denote it by S) is also optimal for the class H_∞^r. Assume that there exists a function $f_0\in H_\infty^r$ for which

$$|Lf_0-S(If_0)|>e(L,H_\infty^r,I).$$

Then the function $\overline{f_0(\overline{z})}\in H_\infty^r$ also satisfies this inequality. Since the class H_∞^r is balanced without loss of generality we may assume that $Lf_0-S(If_0)>0$. Consequently, for the function

$$g(z):=\frac{f_0(z)+\overline{f_0(\overline{z})}}{2}\in H_\infty^{r,\mathbb{R}}$$

we have

$$Lg-S(Ig)>e(L,H_\infty^r,I)\ge e(L,H_\infty^{r,\mathbb{R}},I)$$

which is impossible in view of optimality of the method S on the class $H_\infty^{r,\mathbb{R}}$. □

2. The periodic case

We construct now an optimal method of integration for the integral

$$Lf=\int_{\mathbb{T}} f(x)\,dx$$

on the class $H_{\infty,\beta}^r$ by the information operator

$$If=\left(f(0),f\left(\frac{2\pi}{n}\right),\dots,f\left(\frac{2(n-1)\pi}{n}\right)\right). \tag{13}$$

For sufficiently general conditions on the class of functions it can be proved that the rectangle formula is an optimal method of integration using the information operator (13). Let $\mathcal{H}$ be a convex and balanced class of continuous on the whole real axis 2π-periodic functions f such that for all real constants C and a, $f(x)+C \in \mathcal{H}$ and $f(x+a) \in \mathcal{H}$.

Lemma 3. *The rectangle formula*

$$\int_{\mathbb{T}} f(x)\,dx \approx \frac{2\pi}{n} \sum_{j=0}^{n-1} f\left(\frac{2j\pi}{n}\right) \tag{14}$$

is an optimal method of integration on the class $\mathcal{H}$ and for its error the equality

$$e(L,\mathcal{H},I) = 2\pi \sup_{f\in\mathcal{H}_n} |f(0)|$$

holds where $\mathcal{H}_n$ is the set of $2\pi/n$-periodic functions from $\mathcal{H}$ for which

$$\int_0^{2\pi/n} f(x)\,dx = 0. \tag{15}$$

If functions from $\mathcal{H}$ are differentiable, then the rectangle formula is also an optimal method of integration for the information operator

$$I_1 f = \left(f(0), f'(0), f\left(\frac{2\pi}{n}\right), f'\left(\frac{2\pi}{n}\right), \right.$$
$$\left. \dots, f\left(\frac{2(n-1)\pi}{n}\right) f'\left(\frac{2(n-1)\pi}{n}\right)\right).$$

Proof. It was proved in [16] (see also [17, p. 208]) that

$$\sup_{f\in\mathcal{H}} \left| \int_{\mathbb{T}} f(x)\,dx - \frac{2\pi}{n} \sum_{j=0}^{n-1} f\left(\frac{2j\pi}{n}\right) \right| = 2\pi \sup_{f\in\mathcal{H}_n} |f(0)|.$$

Thus

$$e(L,\mathcal{H},I) \le 2\pi \sup_{f\in\mathcal{H}_n} |f(0)|.$$

On the other hand, for all $\varepsilon > 0$ there exists a function $g \in \mathcal{H}_n$ for which

$$|g(0)| > \sup_{f\in\mathcal{H}_n} |f(0)| - \varepsilon.$$

In view of the properties of the class $\mathcal{H}_n$ we may assume that

$$g(0) = -\max_{x\in[0,2\pi/n)} |g(x)|.$$

Consider the function

$$f_0(x) := g(x) - g(0).$$

Since $f_0 \in \mathcal{H}$ and $If_0 = 0$, from (2) we have

$$e(L,\mathcal{H},I) \ge \left| \int_{\mathbb{T}} f_0(x)\,dx \right| = 2\pi|g(0)| > 2\pi \sup_{f\in\mathcal{H}_n} |f(0)| - 2\pi\varepsilon.$$

Hence

$$e(L,\mathcal{H},I) = 2\pi \sup_{f\in\mathcal{H}_n} |f(0)|$$

and the rectangle formula is an optimal method of integration for the information operator I.

In the case when functions from $\mathcal{H}$ are differentiable for the proof of optimality of rectangle formula for the information operator I_1 it suffices to note that $I_1 f_0 = 0$ and in view of (2)

$$e(L, \mathcal{H}, I) \geq e(L, \mathcal{H}, I_1).$$

□

Theorem 3. *For all $r \geq 1$ the rectangle formula* (14) *is an optimal method of integration on the class $H^r_{\infty,\beta}$ for the information operators I and I_1 and for its error the equalities*

$$e(L, H^r_{\infty,\beta}, I) = e(L, H^r_{\infty,\beta}, I_1) = \frac{2\pi^2}{\sqrt{\lambda}\Lambda n^r} \times \sum_{m=0}^{\infty} \frac{(-1)^{m(r+1)}}{(2m+1)^r \sinh((2m+1)2n\beta)} = \frac{4\pi}{n^r} e^{-\beta n} + O\left(\frac{e^{-5\beta n}}{n^r}\right)$$

hold where

$$\lambda = 4e^{-2\beta n} \left(\frac{\sum_{m=0}^{\infty} e^{-4\beta n m(m+1)}}{1 + 2\sum_{m=1}^{\infty} e^{-4\beta n m^2}} \right)^2$$

and

$$\Lambda = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\lambda^2 t^2)}}$$

is the complete elliptic integral of the first kind for the modulus λ.

Proof. For the information operators I and I_1 the optimality of rectangle formula on the class $H^r_{\infty,\beta}$ immediately follows from Lemma 3. It remains to find the value

$$\sup_{f \in H^r_{\infty,\beta,n}} |f(0)|,$$

where $H^r_{\infty,\beta,n}$ is the set of functions f from $H^r_{\infty,\beta}$ with the period $2\pi/n$ satisfying the condition (15). Set

$$a_j(f) := \frac{1}{\pi} \int_{\mathbb{T}} f(x) \cos jx \, dx, \quad j = 0, 1, \dots ,$$
$$b_j(f) := \frac{1}{\pi} \int_{\mathbb{T}} f(x) \sin jx \, dx, \quad j = 1, 2, \dots .$$

Obviously,

$$\sup_{f \in H^r_{\infty,\beta,n}} |f(0)| \leq \sup_{\substack{f \in H^r_{\infty,\beta} \\ a_0(f)=a_1(f)=b_1(f)=\ldots=a_{n-1}(f)=b_{n-1}(f)=0}} |f(0)|. \tag{16}$$

The value in the right hand side of (16) was calculated in [18]. It attains for the function

$$\varphi^{\beta}_{n,r}(z) := \begin{cases} \Phi^{\beta}_{n,r}\left(z + \dfrac{\pi}{2n}\right), & r = 2l, \\ \Phi^{\beta}_{n,r}(z), & r = 2l+1, \end{cases}$$

where

$$\Phi^{\beta}_{n,r} := D_r * \Phi_{n,0}, \quad r \geq 1, \quad \Phi^{\beta}_{n,0}(z) := \sqrt{\lambda}\,\mathrm{sn}\left(\frac{2n\Lambda}{\pi}z, \lambda\right),$$

$$D_r(t) = 2\sum_{m=1}^{\infty} \frac{\cos(mt - \pi r/2)}{m^r}, \quad r = 1, 2, \dots\ ,$$

is the Bernoulli kernel, and

$$(f * g)(z) := \frac{1}{2\pi}\int_{\mathbb{T}} f(z-t)g(t)\,dt.$$

It was shown in [19] that

$$\Phi^{\beta}_{n,r}(z) = \frac{\pi}{\sqrt{\lambda}\Lambda n^r}\sum_{m=0}^{\infty} \frac{\sin((2m+1)nz - \pi r/2)}{(2m+1)^r \sinh((2m+1)2n\beta)}.$$

Thus $\varphi^{\beta}_{n,r} \in H^r_{\infty,\beta,n}$ and consequently,

$$\begin{aligned} \sup_{f \in H^r_{\infty,\beta,n}} |f(0)| &\geq |\varphi^{\beta}_{n,r}(0)| \\ &= \frac{\pi}{\sqrt{\lambda}\Lambda n^r}\sum_{m=0}^{\infty} \frac{(-1)^{m(r+1)}}{(2m+1)^r \sinh((2m+1)2n\beta)}. \end{aligned}$$

To obtain the asymptotics of the error it remains to use the well-known equality (see, for example, [20])

$$\Lambda = \frac{\pi}{2}\left(1 + 2\sum_{m=1}^{\infty} e^{-4\beta n m^2}\right)^2.$$

$\square$

3. Best quadrature formulas on the classes $H_{p,\beta}$

Consider now the problem of constructing of optimal integration method for the integral

$$Lf = \int_{\mathbb{T}} f(t)p(t)\,dt,$$

where $p(t)$ is a nonnegative weight function, for the class $H_{p,\beta}$ by the information operator (3) in which $x_1, \dots, x_n$ are distinct points from $\mathbb{T}$.

Put

$$k = 4e^{-\beta}\left(\frac{\sum_{m=0}^{\infty} e^{-2\beta m(m+1)}}{1+2\sum_{m=1}^{\infty} e^{-2\beta m^2}}\right)^2. \tag{17}$$

Denote by K and K' the complete elliptic integrals of the first kind for the moduli k and $k'=\sqrt{1-k^2}$, respectively (the equality (17) is equivalent to $\pi K'/K = 2\beta$). For the strip S_β a 2π-periodic Blaschke product with zeros at the points x_j with even multiplicities is the function (see [10])

$$B(t) = k^{N/2}\prod_{j=1}^{n} \operatorname{sn}^{\nu_j}\left(\frac{K}{\pi}(t-x_j), k\right),$$

where N is defined by (5).

Denote by $H_{p,\beta}^{\mathbb{R}}$ the set of functions from the class $H_{p,\beta}$ real on the real axis.

Lemma 4. *Let $\nu_1, \dots, \nu_n$ be even numbers and $1 \le p \le \infty$. Then*

1) *there exists the unique function $g_{B,p} \in H_{p,\beta}^{\mathbb{R}}$ for which*

$$e(L, H_{p,\beta}, I) = \int_{\mathbb{T}} g_{B,p}(t)B(t)p(t)\,dt,$$

2) *$g_{B,p}$ does not vanish in the strip S_β and $g_{B,p}(t) > 0$ for $t \in \mathbb{T}$,*
3) *for $1 \le p < \infty$ and almost all $t \in \mathbb{T}$ the eqality*

$$e(L, H_{p,\beta}, I)|g_{B,p}(t+i\beta)|^p = \int_{\mathbb{T}} g_{B,p}(\tau)B(\tau)K_\beta(t-\tau)p(\tau)\,d\tau \tag{18}$$

holds, where

$$K_\beta(t) = \frac{2\Lambda}{\pi}\operatorname{dn}\left(\frac{\Lambda}{\pi}t, \lambda\right)$$

and Λ is the complete elliptic integral of the first kind for the modulus λ which is defined by the condition $\pi\Lambda'/\Lambda = \beta$.

Proof. It follows from [21] that in the problem

$$P_1 := \sup_{f\in H_{p,\beta}^{\mathbb{R}}} \int_{\mathbb{T}} |f(t)|B(t)p(t)\,dt$$

there exists the unique function $g_{B,p} \in H_{p,\beta}^{\mathbb{R}}$ normalized by the condition $g_{B,p}(0) > 0$ for which this supremum is attained. Moreover, from the same paper it follows that this function does not vanish in the strip S_β (and consequently, $g_{B,p}(t) > 0$ for $t \in \mathbb{T}$) and for all $1 \le p < \infty$

$$P_1|g_{B,p}(t \pm i\beta|^p = \int_{\mathbb{T}} |g_{B,p}(\tau)|B(\tau)K_\beta(t-\tau)p(\tau)\,d\tau.$$

Since every function $f \in H_{p,\beta}$ for which $If = 0$ may be represented in the form

$$f(z) = B(z)g(z), \quad g \in H_{p,\beta},$$

in view of (2)

$$e(L, H_{p,\beta}, I) = \sup_{f \in H_{p,\beta}} \left| \int_{\mathbb{T}} f(t)B(t)p(t)\,dt \right| =: P_2.$$

Similar to the method which was used in the proof of Theorem 2 it is easy to show that

$$e(L, H_{p,\beta}, I) = e(L, H^{\mathbb{R}}_{p,\beta}, I).$$

Since

$$P_1 \geq e(L, H^{\mathbb{R}}_{p,\beta}, I) \geq \int_{\mathbb{T}} g_{B,p}(t)B(t)p(t)\,dt = P_1,$$

we have

$$P_1 = e(L, H^{\mathbb{R}}_{p,\beta}, I) = P_2.$$

□

For $p = \infty$ and even $\nu_1, \dots, \nu_n$ it is obvious that $g_{B,p}(z) \equiv 1$.

Let $p = 2$. The space of 2π-periodic functions $\mathcal{H}_{2,\beta}$ analytic in the strip S_β and satisfying the condition

$$\sup_{0 \leq \eta < \beta} \frac{1}{4\pi} \int_{\mathbb{T}} \left(|f(t+i\eta)|^2 + |f(t-i\eta)|^2\right)\,dt < \infty$$

is a Hilbert space with the inner product

$$(f, g)_{\mathcal{H}_{2,\beta}} = \frac{1}{4\pi} \int_{\Gamma} f(\xi)\overline{g(\xi)}\,d\xi,$$

where $\Gamma = [i\beta, 2\pi + i\beta] \cup [-i\beta, 2\pi - i\beta]$. It follows from [18] that for all $f \in \mathcal{H}_{2,\beta}$ and any $t \in \mathbb{T}$ the equality

$$f(t) = (f, g_t)_{\mathcal{H}_{2,\beta}}$$

holds, where

$$g_t(z) = \frac{2K}{\pi}\,\mathrm{dn}\left(\frac{K}{\pi}(t - z), k\right)$$

and K is the complete elliptic integral of the first kind for the modulus k defined by the condition $K'/K = 2\beta/\pi$.

We have

$$\begin{aligned}
e(L, H_{2,\beta}, I) &= \sup_{f \in H_{2,\beta}} \left| \int_{\mathbb{T}} f(t)B(t)p(t)\,dt \right| \\
&= \sup_{f \in H_{2,\beta}} \left| \int_{\mathbb{T}} \frac{1}{4\pi} \int_{\Gamma} f(\xi)\overline{g_t(\xi)}\,d\xi B(t)p(t)\,dt \right| \\
= \sup_{f \in H_{2,\beta}} &\left| \frac{1}{4\pi} \int_{\Gamma} f(\xi) \int_{\mathbb{T}} \overline{g_t(\xi)} B(t)p(t)\,dt\,d\xi \right| = \sup_{\|f\|_{\mathcal{H}_{2,\beta}} \leq 1} (f, G)_{\mathcal{H}_{2,\beta}},
\end{aligned}$$

where

$$G(\xi) = \frac{2K}{\pi} \int_{\mathbb{T}} \mathrm{dn}\left(\frac{K}{\pi}(t - \xi)\right) B(t)p(t)\,dt.$$

Hence it follows that

$$g_{B,2}(z)=\frac{G(z)}{\|G\|_{\mathcal{H}_{2,\beta}}}.$$

Theorem 4. *Let $\nu_1,\dots,\nu_n$ be even numbers and $1\le p\le\infty$. Then the quadrature formula*

$$\int_{\mathbb{T}} f(t)p(t)\,dt\approx\sum_{j=1}^{n}\sum_{\nu=0}^{\nu_j-1}a_{j\nu}f^{(\nu)}(x_j), \tag{19}$$

where

$$a_{j\nu}=\int_{\mathbb{T}}c_{j\nu}(t)p(t)\,dt,$$

$$\begin{aligned}c_{j\nu}(t)=\frac{K}{\pi}&\frac{B(t)g_{B,p}(t)}{\nu!(\nu_j-\nu-1)!}\\&\times\lim_{z\to x_j}\left(\frac{(z-x_j)^{\nu_j}}{B(z)g_{B,p}(z)}\operatorname{ctn}\left(\frac{K}{\pi}(t-z),k\right)\right)^{(\nu_j-\nu-1)},\end{aligned}$$

$$\operatorname{ctn}(z,k)=\frac{\operatorname{cn}(z,k)\operatorname{dn}(z,k)}{\operatorname{sn}(z,k)},$$

is an optimal method of integration on the class $H_{p,\beta}$.

Proof. Consider the integral

$$Jf:=\frac{K}{\pi}B(t)g_{B,p}(t)\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{f(z)}{B(z)g_{B,p}(z)}\operatorname{ctn}\left(\frac{K}{\pi}(z-t),k\right)dz, \tag{20}$$

where Γ_ε is the boundary of the rectangle $-\varepsilon\le\operatorname{Re}z\le2\pi-\varepsilon$, $|\operatorname{Im}z|\le\beta$, and ε is chosen from the requirement that the points $x_1,\dots,x_n$ lie inside this rectangle. In view of the fact that $g_{B,p}(z)$ does not vanish in the strip S_β by the residue theorem we have

$$Jf=f(t)-\sum_{j=1}^{n}\sum_{\nu=0}^{\nu_j-1}c_{j\nu}(t)f^{(\nu)}(x_j).$$

It follows from the properties of elliptic functions (see, for example, [20]) that

$$\begin{aligned}\operatorname{ctn}\left(\frac{K}{\pi}(t\pm i\beta),k\right)&=\operatorname{ctn}\left(\frac{K}{\pi}t\pm i\frac{K'}{2},k\right)\\&=\pm i(1+k)\frac{1-k\operatorname{sn}^2\left(\frac{K}{\pi}t,k\right)}{1+k\operatorname{sn}^2\left(\frac{K}{\pi}t,k\right)}=\pm i\frac{\Lambda}{K}\operatorname{dn}\left(\frac{\Lambda}{\pi}t,\lambda\right),\end{aligned}$$

where $\lambda = 2\sqrt{k}/(1+k)$ and Λ is the complete elliptic integral of the first kind for the modulus λ (in other words, λ is defined by the condition $\Lambda'/\Lambda = K'/(2K)$). Thus the integral (20) may be written in the form

$$Jf := B(t)g_{B,p}(t)\frac{1}{4\pi}\int_\Gamma \frac{f(z)}{B(z)g_{B,p}(z)}K_\beta(\operatorname{Re} z - t)\,dz,$$

where $\Gamma = [i\beta, 2\pi + i\beta] \cup [-i\beta, 2\pi - i\beta]$. Let $1 \le p < \infty$. Then for the error of the quadrature formula (19) we have

$$\begin{aligned} R_f &:= \left| \int_{\mathbb{T}} f(t)p(t)\,dt - \sum_{j=1}^{n}\sum_{\nu=0}^{\nu_j-1} a_{j\nu}f^{(\nu)}(x_j)\right| \\ &\le \int_{\mathbb{T}} B(t)g_{B,p}(t)p(t)\frac{1}{4\pi}\int_\Gamma \frac{|f(z)|}{|g_{B,p}(z)|}K_\beta(\operatorname{Re} z - t)\,dz\,dt \\ &\quad = \frac{1}{4\pi}\int_\Gamma \frac{|f(z)|}{|g_{B,p}(z)|}\int_{\mathbb{T}} B(t)g_{B,p}(t)K_\beta(\operatorname{Re} z - t)p(t)\,dt\,dz. \end{aligned}$$

Using (18) we obtain

$$R_f \le e(L, H_{p,\beta}, I)\frac{1}{4\pi}\int_\Gamma |f(z)||g_{B,p}(z)|^{p-1}\,dz.$$

By the Hölder inequality

$$\begin{aligned} R_f \le e(L, H_{p,\beta}, I)\left(\frac{1}{4\pi}\int_\Gamma |f(z)|^p\,dz\right)^{1/p}\left(\frac{1}{4\pi}\int_\Gamma |g_{B,p}(z)|^p\,dz\right)^{(p-1)/p} \\ \le e(L, H_{p,\beta}, I). \end{aligned}$$

If $p = \infty$, then $g_{B,p}(z) \equiv 1$ and

$$|Jf| \le B(t)\frac{1}{4\pi}\int_\Gamma |f(z)|K_\beta(\operatorname{Re} z - t)\,dz \le B(t),$$

because

$$\frac{1}{4\pi}\int_\Gamma K_\beta(\operatorname{Re} z - t)\,dz \equiv 1.$$

Consequently,

$$R_f \le \int_{\mathbb{T}} B(t)p(t)\,dt = e(L, H_{\infty,\beta}, I).$$

□

4. Recovery of functions from H_p^r by the Taylor information

Consider the problem of optimal recovery of the value $f(\xi)$, $\xi \in D$, on the class H_p^r by the values of the information operator

$$If = (f(0), f'(0), \dots, f^{(n+r-1)}(0)).$$

We denote by $e(\xi, H_p^r, I)$ the error of optimal recovery method in this case.

It is easily seen that if $f \in H_p^r$ и $If = 0$, then $f^{(r)}(z) = z^n\varphi(z)$ where $\varphi \in H_p$. Consequently, $f(z) = T_r(t^n\varphi(t))(z)$ where the operator T_r is defined by (7). It is obvious that for all $\varphi \in H_p$, $f(z) = T_r(t^n\varphi(t))(z) \in H_p^r$, moreover, $If = 0$. Thus, taking into account the duality formula (2),

$$e(\xi, H_p^r, I) = \sup_{\substack{f\in H_p^r \\ If=0}} |f(\xi)| = \sup_{\varphi\in H_p} \left| \int_0^\xi \frac{(\xi-t)^{r-1}}{(r-1)!} t^n \varphi(t)\, dt \right|. \tag{21}$$

Let $\xi \in (0,1)$. Then it follows from [11, p. 176] that there exists the unique function $\varphi_\xi \in H_p$ such that $\varphi_\xi(t) > 0$ for $t \in (-1,1)$ and

$$e(\xi, H_p^r, I) = \int_0^\xi \frac{(\xi-t)^{r-1}}{(r-1)!} t^n \varphi_\xi(t)\, dt. \tag{22}$$

Theorem 5. *For all $\xi \in D$ and $1 \le p \le \infty$ the method*

$$f(\xi) \approx \sum_{j=0}^{n+r-1} a_j \frac{\xi^j}{j!} f^{(j)}(0), \tag{23}$$

where $a_0 = \ldots = a_{r-1} = 1$,

$$a_{n+r-1} = \frac{(n+r-1)!}{(n-1)!\varphi_{|\xi|}(0)} h_{n+r-1},$$

$$a_k = \frac{k!}{(k-r)!\varphi_{|\xi|}(0)} \left(h_k - \sum_{j=k+1}^{n+r-1} a_j \frac{(j-r)!}{j!(j-k)!} |\xi|^{j-k} \varphi_{|\xi|}^{(j-k)}(0) \right), \tag{24}$$
$$k = n+r-2, \ldots, r,$$

$$h_k = \int_0^1 \frac{(1-\tau)^{r-1}}{(r-1)!} \tau^{k-r} \left(1 - (|\xi|\tau)^{2(n+r-k)}\right) \varphi_{|\xi|}(|\xi|\tau)\, d\tau,$$
$$k = r, \ldots, n+r-1,$$

is an optimal method of recovery on the class H_p^r.

Proof. Denote by $H_p^{r,\mathbb{R}}$ the class of all functions from H_p^r real on the interval $(-1,1)$. First, we shall show that the method (23) is optimal on the class $H_p^{r,\mathbb{R}}$ for $\xi \in (0,1)$. Since $\varphi_\xi \in H_p^{\mathbb{R}}$ the equality (22) is also valid for the class $H_p^{r,\mathbb{R}}$, that is the function

$$f_0(z) := \int_0^z \frac{(z-t)^{r-1}}{(r-1)!} t^n \varphi_\xi(t)\, dt \tag{25}$$

is extremal in the problem of optimal recovery of the value $f(\xi)$ on the class $H_p^{r,\mathbb{R}}$ by the Taylor information If.

Set $\omega_0(z) := 1$,

$$\omega_j(z) := \frac{z\omega_{j-1}(z) + \varepsilon_{n+r-j}}{1 + \varepsilon_{n+r-j} z\omega_{j-1}(z)}, \quad j = 1, \ldots, n.$$

For all $\varepsilon_r, \dots, \varepsilon_{n+r-1} \in (-1,1)$, $\omega_n\varphi_\xi \in H_p^{\mathbb{R}}$. For the points $P = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n+r-1}) \in \mathbb{R}^{n+r}$ consider the function

$$f_P(z) := \sum_{j=0}^{r-1} \varepsilon_j z^j + T_r(\omega_n\varphi_\xi)(z).$$

For all $P \in (-1,1)^{n+r}$, $f_P \in H_p^r$, and for $P = 0$ this function coincides with the extremal function (25). It follows from Theorem 1 that the coefficients a_j of optimal recovery method on the class $H_p^{r,\mathbb{R}}$ are found from the system

$$\sum_{j=0}^{n+r-1} a_j \frac{\xi^j}{j!} \frac{\partial f_P^{(j)}(0)}{\partial \varepsilon_k}\Big|_{P=0} = \frac{\partial f_P(\xi)}{\partial \varepsilon_k}\Big|_{P=0}, \quad k = 0, 1, \dots, n+r-1.$$

For $0 \le k \le r-1$ we have

$$\frac{\partial f_P^{(j)}(0)}{\partial \varepsilon_k}\Big|_{P=0} = \begin{cases} 0, & k \ne j, \\ j!, & k = j, \end{cases} \quad \frac{\partial f_P(\xi)}{\partial \varepsilon_k}\Big|_{P=0} = \xi^k,$$

and for $r \le k \le n+r-1$

$$\frac{\partial f_P^{(j)}(0)}{\partial \varepsilon_k}\Big|_{P=0} = \begin{cases} 0, & 0 \le j \le k-1, \\ C_{j-r}^{k-r}(k-r)!\varphi_\xi^{(j-k)}(0), & k \le j \le n+r-1, \end{cases}$$

$$\frac{\partial f_P(\xi)}{\partial \varepsilon_k}\Big|_{P=0} = (T_r g_k)(\xi),$$

where

$$g_k(z) = z^{k-r}(1 - z^{2(n+r-k)})\varphi_\xi(z).$$

Hence $a_0 = \ldots = a_{r-1} = 1$ and for finding other coefficients we obtain the system

$$\sum_{j=k}^{n+r-1} a_j \frac{\xi^j}{j!} C_{j-r}^{k-r}(k-r)!\varphi_\xi^{(j-k)}(0) = (T_r g_k)(\xi), \quad k = r, \dots, n+r-1.$$

Thus,

$$a_{n+r-1} = \frac{(n+r-1)!}{(n-1)!\varphi_\xi(0)} \frac{(T_r g_{n+r-1})(\xi)}{\xi^{n+r-1}},$$

$$a_k = \frac{k!}{(k-r)!\varphi_\xi(0)} \left(\frac{(T_r g_k)(\xi)}{\xi^k} - \sum_{j=k+1}^{n+r-1} a_j \frac{(j-r)!}{j!(j-k)!} \xi^{j-k} \varphi_\xi^{(j-k)}(0) \right),$$
$$k = n+r-2, \dots, r.$$

Making the substitution $t = \xi\tau$ we get that $(T_r g_k)(\xi) = \xi^k h_k$ and consequently the equalities (24) hold.

The optimality of the constructed method on the class H_p^r is proved by the method similar to the one used in the proof of Theorem 2.

Now let ξ be an arbitrary point of the disk D. If $\xi = |\xi|e^{i\theta}$ and $f \in H^r_p$, then the function $F(z) = f(ze^{i\theta})$ belongs to H^r_p, $F(|\xi|) = f(\xi)$, and

$$IF = (f(0), e^{i\theta} f'(0), \dots, e^{i(n+r-1)\theta} f^{(n+r-1)}(0)).$$

Applying the obtaining method to the function F at the point $|\xi|$, we have

$$\left| f(\xi) - \sum_{j=0}^{n+r-1} a_j \frac{\xi^j}{j!} f^{(j)}(0) \right| \le e(|\xi|, H^r_p, I).$$

Using the first equality of (21) it is easy to verify that

$$e(|\xi|, H^r_p, I) = e(\xi, H^r_p, I).$$

Thus the constructed method is optimal for all $\xi \in D$. □

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