EXTREMAL PROBLEMS OF THE HADAMARD THREE-CIRCLE THEOREM TYPES AND OPTIMAL RECOVERY OF OPERATORS

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The well-known Hadamard three-circle theorem states that if \( f(z) \) is a holomorphic function on the annulus \( r_1 \leq |z| \leq r_2 \) and

\[
M(r) = \max_{|z|=r} |f(z)|,
\]

then

\[
M(r) \leq M(r_1) \frac{\log r_2/r}{\log r_2/r_1} M(r_2) \frac{\log r_2/r_1}{\log r_2/r_1}
\]

for any three concentric circles of radii \( r_1 < r < r_2 \).

In 1913 E. Landau considered a very similar problem. He took derivatives instead of circles. He proved that for all functions \( x(\cdot) \in L_\infty(\mathbb{R}_+) \) with the first derivative locally absolutely continuous on \( \mathbb{R}_+ \) and \( x''(\cdot) \in L_\infty(\mathbb{R}_+) \) the following exact inequality

\[
\|x'(\cdot)\|_{L_\infty(\mathbb{R}_+)} \leq 2 \|x^{(1/2)}(\cdot)\|_{L_\infty(\mathbb{R}_+)} \|x''(\cdot)\|_{L_\infty(\mathbb{R}_+)}^{1/2}
\]

holds. Then in 1914 Hadamard solved the analogous problem for \( \mathbb{R} \).

In 1934 Hardy, Littlewood, and Pólya proved that for all integers \( 0 < k < r \) the exact inequality

\[
\|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \leq \|x^{(1/2)}(\cdot)\|_{L_2(\mathbb{R})}^{1-k/2} \|x^{(r)}(\cdot)\|_{L_2(\mathbb{R})}^{k/2}
\]

holds for all functions \( x(\cdot) \in L_2(\mathbb{R}) \) for which the \((r-1)\)-st derivative is locally absolute continuous on \( \mathbb{R} \) and \( x^{(r)}(\cdot) \in L_2(\mathbb{R}) \).

The exact inequality (1) may be easily obtained passing to the Fourier transforms from the following extremal problem

\[
\int_{\mathbb{R}} \xi^{2k}|F x(\xi)|^2 d\xi \rightarrow \max, \quad \int_{\mathbb{R}} |F x(\xi)|^2 d\xi \leq \delta_1^2, \quad \int_{\mathbb{R}} \xi^{2n}|F x(\xi)|^2 d\xi \leq \delta_2^2,
\]

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where \( Fx(\cdot) \) is the Fourier transform of \( x(\cdot) \). The value of this extremal problem coincides with the value of the extremal problem

\[
\int_{\mathbb{R}} \xi^{2k} |Fx(\xi)|^2 d\xi \rightarrow \max, \quad \int_{|\xi| \leq \sigma_1} |Fx(\xi)|^2 d\xi \leq \delta_1^2,
\]

\[
\int_{|\xi| \geq \sigma_2} \xi^{2n} |Fx(\xi)|^2 d\xi \leq \delta_2^2
\]

for some \( \sigma_1 \geq \sigma_2 \). Using this fact we obtain a collection of optimal recovery methods of \( x^{(k)}(\cdot) \) from inaccurate information about the Fourier transform of \( x(\cdot) \).

REFERENCES


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