OPTIMAL RECOVERY OF THE SOLUTION OF THE HEAT EQUATION FROM INACCURATE DATA

G. G. MAGARIL-IL'YAEV, K. YU. OSIPENKO

ABSTRACT. In the paper the problem of optimal recovery of the solution of the heat equation on the hall space at the instant of time from inaccurate observations of the solution at some other instants of time is considered. Explicit forms for an optimal recovery method and its error are given. The solution of a similar problem with a priory information about temperature distributions at some instants of time is also given. In all cases an optimal method uses information about at most two observations.

INTRODUCTION

The initial stimulus for this paper was the following question: if we have a possibility to observe the temperature of some body at the instants of time $t_1, \dots, t_n$ with known errors, then what is the best way to use this information to recover its temperature at some other instant of time?

We answer this question for the problem of temperature distribution in the space $\mathbb{R}^d$. More precisely, we state the problem of optimal recovery of the solution of the heat equation on $\mathbb{R}^d$ at some instant of time from inaccurate observations of this solution at other instants of time and give explicit forms of optimal recovery method and its error.

Usually in practice besides observations there is an a priory information about temperature distribution which is in the fact that at some instants of time there are known the bounds such that the temperature could not be out of them. In this paper the explicit solutions of this problem is also given.

The structure of the paper is the following. The first three sections are devoted to the solution of the optimal recovery problem of the heat equation from inaccurate observations. In the fourth section the similar

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problem is solving when an a priory information is giving. Historical and bibliographical comments are in the fifth section.

1. Statement of the problem

It is well known that the temperature distribution in $\mathbb{R}^d$ is described by the equation

$$\frac{\partial u}{\partial t} = \Delta u \tag{1}$$

(where Δ is the Laplace operator in $\mathbb{R}^d$ and $u(\cdot,\cdot)$ is a function on $[0,\infty)\times\mathbb{R}^d$) with the given initial temperature distribution

$$u(0,\cdot) = u_0(\cdot). \tag{2}$$

We assume that $u_0(\cdot) \in L_2(\mathbb{R}^d)$. The unique solution of problem (1)–(2) for $t > 0$ is the Poisson integral

$$u(t,x) = u(t,x;u_0(\cdot)) = \frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}^d} e^{-\frac{|x-\xi|^2}{4t}} u_0(\xi)\,d\xi, \tag{3}$$

where $x = (x_1,\ldots,x_d)$, $\xi = (\xi_1,\ldots,\xi_d)$, $|x-\xi|^2 = \sum_{i=1}^d (x_i-\xi_i)^2$, and moreover, $u(t,\cdot) \to u_0(\cdot)$ as $t \to 0$ in the $L_2(\mathbb{R}^d)$-metric.

We state the following problem. Let there be temperature distributions $u(t_1,\cdot),\ldots,u(t_n,\cdot)$ at the instants of time $0 \le t_1 < \ldots < t_n$ given approximately. More precisely, we know functions $y_i(\cdot) \in L_2(\mathbb{R}^d)$ such that

$$\|u(t_i,\cdot) - y_i(\cdot)\|_{L_2(\mathbb{R}^d)} \le \delta_i, \quad i = 1,\ldots,n,$$

where $\delta_i > 0$, $i = 1,\ldots,n$. For every set of such functions we want to find a function $L_2(\mathbb{R}^d)$ which approximate a real temperature distribution in $\mathbb{R}^d$ at a fixed instant of time τ in a best way in some sense.

We mean by this the following. Any map m from $(L_2(\mathbb{R}^d))^n = L_2(\mathbb{R}^d)\times\ldots\times L_2(\mathbb{R}^d)$ to $L_2(\mathbb{R}^d)$ we call a method of recovery (of the temperature in $\mathbb{R}^d$ at the instant of time τ from the given information). The error of this method is the value

$$e(\tau,\overline{\delta},m) = \sup_{\substack{u_0(\cdot),\ \overline{y}(\cdot)\in(L_2(\mathbb{R}^d))^n \\ \|u(t_i,\cdot)-y_i(\cdot)\|_{L_2(\mathbb{R}^d)}\le\delta_i,\ i=1,\ldots,n}} \|u(\tau,\cdot) - m(\overline{y}(\cdot))(\cdot)\|_{L_2(\mathbb{R}^d)},$$

where $\overline{y}(\cdot) = (y_1(\cdot),\ldots,y_n(\cdot))$ and $\overline{\delta} = (\delta_1,\ldots,\delta_n)$.

We are interested in the value

$$E(\tau,\overline{\delta}) = \inf_{m\colon (L_2(\mathbb{R}^d))^n \to L_2(\mathbb{R}^d)} e(\tau,\overline{\delta},m),$$

which we call the *error of optimal recovery* and in a method $\widehat{m}$, for which the lower bound is delivering, that is,

$$E(\tau, \overline{\delta}) = e(\tau, \overline{\delta}, \widehat{m}),$$

which is called an *optimal recovery method* (of the temperature in $\mathbb{R}^d$ at the instant of time τ from the given information).

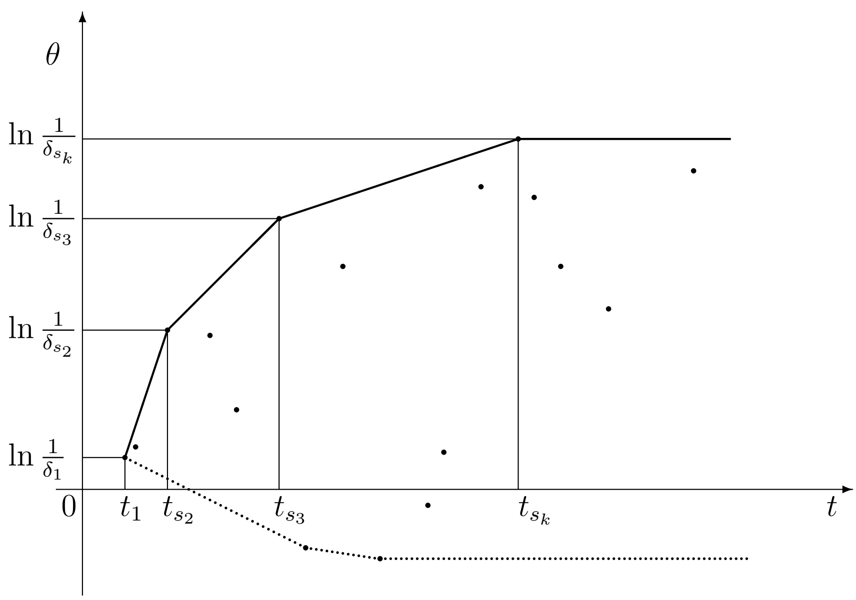
2. The statement of theorem

Before the statement of the theorem we make some constructions. On the two-dimensional plane (t, x) we construct a set

$$M = \operatorname{co}\{\, (t_j, \ln(1/\delta_j)),\ 1 \le j \le n \,\} + \{\, (t, 0) \mid\ t \ge 0 \,\},$$

where $\operatorname{co} A$ is the convex hall of A.

Define the function $\theta(\cdot)$ on $[0, \infty)$ by the equality $\theta(t) = \max\{\, x \mid (t, x) \in M \,\}$, where $\theta(t) = -\infty$, if $(t, x) \notin M$ for all x. It is clear that the function $\theta(\cdot)$ is a concave polygonal line on $[t_1, \infty)$. Denote by $t_{s_1} < \ldots < t_{s_k}$ its points of break (we consider the point t_1 as a point of break too, that is $t_{s_1} = t_1$), which are evidently the subset of the points $\{t_1, \ldots, t_n\}$ (see the figure where represented points have coordinates $(t_i, \ln(1/\delta_i))$ and the bold curve is the plot of $\theta(\cdot)$).



For every $t > 0$ formula (3) defines a continuous linear operator in $L_2(\mathbb{R}^d)$,[1] which we denote by P_t, and if P_0 denotes the identical operator, then $u(t, \cdot; u_0(\cdot)) = P_t u_0(\cdot)$ for all $t \ge 0$.

Theorem 1. *For all $\tau \ge 0$ the equlity*

$$E(\tau, \overline{\delta}) = e^{-\theta(\tau)}$$

holds.

(1) *If $t_1 > 0$ and $0 \le \tau < t_1$, then any method is optimal;*
(2) *if $\tau = t_{s_j}$, $1 \le j \le k$, then the method $\widehat{m}$ defined by the equality $\widehat{m}(\overline{y}(\cdot))(\cdot) = y_{s_j}(\cdot)$ is optimal;*
(3) *if $k \ge 2$ and $\tau \in (t_{s_j}, t_{s_{j+1}})$, $1 \le j \le k-1$, then the method $\widehat{m}$ defined by the equality*

$$\widehat{m}(\overline{y}(\cdot))(\cdot) = (K_{s_j} * y_{s_j})(\cdot) + (K_{s_{j+1}} * y_{s_{j+1}})(\cdot),$$

where $K_{s_j}(\cdot)$ $K_{s_{j+1}}(\cdot)$ are functions from $L_2(\mathbb{R}^d)$ with the Fourier transforms

$$FK_{s_j}(\xi) = \frac{(t_{s_{j+1}} - \tau)\delta^2_{s_{j+1}} e^{-|\xi|^2(\tau - t_{s_j})}}{(t_{s_{j+1}} - \tau)\delta^2_{s_{j+1}} + (\tau - t_{s_j})\delta^2_{s_j} e^{-2|\xi|^2(t_{s_{j+1}} - t_{s_j})}},$$

$$FK_{s_{j+1}}(\xi) = \frac{(\tau - t_{s_j})\delta^2_{s_j} e^{-|\xi|^2(\tau + t_{s_{j+1}} - 2t_{s_j})}}{(t_{s_{j+1}} - \tau)\delta^2_{s_{j+1}} + (\tau - t_{s_j})\delta^2_{s_j} e^{-2|\xi|^2(t_{s_{j+1}} - t_{s_j})}},$$

is optimal;
(4) *if $\tau > t_{s_k}$, then the method $\widehat{m}$ defined by the equality*

$$\widehat{m}(\overline{y}(\cdot))(\cdot) = P_{\tau - t_{s_k}} y_{s_k}(\cdot)$$

is optimal.

We give some remarks apropos to the formulated theorem.

1. If $t_1 > 0$ and $0 \le \tau < t_1$, then $\theta(\tau) = -\infty$ so that $E(\tau, \overline{\delta}) = +\infty$, that is, the past could not be recovered from inaccurate present. In this case any method may be considered as optimal.

2. Note that the optimal method is linear, it "smooths" observations (the convolution is an infinite differentiable function) and uses the information about at most two observations before and after the instant of time τ or only before τ (if $\tau > t_{s_k}$).

3. If $\tau = t_i$ and t_i is not a point of break of $\theta(\cdot)$, then the optimal recovery method makes possible to correct this observation.

4. The case $\tau > t_{s_k}$ means that the most precise observation of the temperature was before the instant of time τ. In this situation the

[1]It follows, for example, from Young's inequality since the Poisson integral is the convolution of bounded function with function from $L_2(\mathbb{R}^d)$.

optimal recovery method is the solution of the heat equation at the instant of time $\tau - t_{s_k}$ with the initial temperature distribution $y_{s_k}(\cdot)$.

3. Proof of Theorem 1

The proof consists of two parts: the lower bound of the optimal recovery error $E(\tau,\overline{\delta})$ and the upper bound of this value with presentation of optimal method.

1. *The lower bound of* $E(\tau,\overline{\delta})$. Recall that P_t is a continuous linear operator in $L_2(\mathbb{R}^d)$ which is defined by (3) for $t > 0$ and P_0 is the identical operator.

Let $\tau \geq 0$. Consider the problem

$$\|P_\tau u_0(\cdot)\|_{L_2(\mathbb{R}^d)} \to \max, \quad \|P_{t_j} u_0(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \delta_j, \quad j = 1,\dots,n, \quad u_0(\cdot) \in L_2(\mathbb{R}^d). \tag{4}$$

Denote its value (that is, the upper bound of $\|P_\tau u_0(\cdot)\|_{L_2(\mathbb{R}^d)}$ with the given constraints) by S and show that $E(\tau,\overline{\delta}) \geq S$.

Indeed, let $\overline{u}_0(\cdot)$ be an admissible function in (4) (that is, $\overline{u}_0(\cdot)$ satisfies all constraints of the problem). Then $-\overline{u}_0(\cdot)$ is also admissible in (4) and for any $m\colon (L_2(\mathbb{R}^d))^n \to L_2(\mathbb{R}^d)$ we have

$$\begin{aligned} 2\|P_\tau \overline{u}_0(\cdot)\|_{L_2(\mathbb{R}^d)} &= \|P_\tau \overline{u}_0(\cdot) - m(0)(\cdot) + m(0)(\cdot) - P_\tau(-\overline{u}_0(\cdot))\|_{L_2(\mathbb{R}^d)} \leq \\ &2 \sup_{\substack{u_0(\cdot)\in L_2(\mathbb{R}^d) \\ \|P_{t_j} u_0(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \delta_j,\ j=1,\dots,n,}} \|P_\tau u_0(\cdot) - m(0)(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \\ &\leq 2 \sup_{\substack{u_0(\cdot)\in L_2(\mathbb{R}^d),\ \overline{y}(\cdot)\in (L_2(\mathbb{R}^d))^n \\ \|P_{t_j} u_0(\cdot) - y_j(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \delta_j,\ j=1,\dots,n}} \|P_\tau u_0(\cdot) - m(\overline{y}(\cdot))(\cdot)\|_{L_2(\mathbb{R}^d)}. \end{aligned}$$

Passing to the lower bound over all methods m in the right hand side and to the upper bound over all admissible functions in (4) in the left hand side, we obtain that $E(\tau,\overline{\delta}) \geq S$.

The next step is the proof of the fact that $S = e^{-\theta(\tau)}$. Let $F\colon L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ be the Fourier transform. It is well known (see, for example, [1]) that for all $t \geq 0$ the equality

$$F(P_t u_0(\cdot))(\xi) = e^{-|\xi|^2 t} F u_0(\xi), \quad \xi \in \mathbb{R}^d,$$

holds, and therefore by Plancherel's theorem the squared value of problem (4) equals the value of the following problem

$$\text{(5)}\quad \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-2|\xi|^2\tau}|Fu_0(\xi)|^2\,d\xi \to \max,$$
$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-2|\xi|^2 t_j}|Fu_0(\xi)|^2\,d\xi \le \delta_j^2, \quad j=1,\dots,n, \quad u_0(\cdot)\in L_2(\mathbb{R}^d).$$

It can be shown that there is no existence in this problem, therefore we consider its "extension", namely we consider the following problem (formally replacing $(2\pi)^{-d}|Fu_0(\xi)|^2\,d\xi$ on a positive measure):

$$\text{(6)}\quad \int_{\mathbb{R}^d} e^{-2|\xi|^2\tau}\,d\mu(\xi) \to \max,$$
$$\int_{\mathbb{R}^d} e^{-2|\xi|^2 t_j}\,d\mu(\xi) \le \delta_j^2, \quad j=1,\dots,n, \quad d\mu(\cdot)\ge 0.$$

It is a convex problem. Its Lagrange function has the form

$$\mathcal{L}(d\mu(\cdot),\lambda) = \lambda_0\int_{\mathbb{R}^d} e^{-2|\xi|^2\tau}\,d\mu(\xi) + \sum_{j=1}^n \lambda_j\left(\int_{\mathbb{R}^d} e^{-2|\xi|^2 t_j}\,d\mu(\xi) - \delta_j^2\right),$$

where $\lambda = (\lambda_0,\lambda_1\dots,\lambda_n)$ is a set of Lagrange multipliers.

If we find an admissible measure $d\widehat{\mu}(\cdot)$ in (6) and Lagrange multipliers $\widehat{\lambda}_0 < 0$, $\widehat{\lambda}_j \ge 0$, $1\le j\le n$, such that

$$\text{(7)}\qquad \min_{d\mu(\cdot)\ge 0}\mathcal{L}(d\mu(\cdot),\widehat{\lambda}) = \mathcal{L}(d\widehat{\mu}(\cdot),\widehat{\lambda}),$$

where $\widehat{\lambda} = (\widehat{\lambda}_0,\widehat{\lambda}_1,\dots,\widehat{\lambda}_n)$ and

$$\text{(8)}\qquad \widehat{\lambda}_j\left(\int_{\mathbb{R}^d} e^{-2|\xi|^2 t_j}\,d\widehat{\mu}(\xi) - \delta_j^2\right) = 0, \quad j=1,\dots,n,$$

then $d\widehat{\mu}(\cdot)$ will be a solution of problem (6). Indeed, let $d\mu(\cdot)$ be an admissible measure in (6). Then using this fact (and taking into account that $\widehat{\lambda}_j \ge 0$, $1\le j\le n$), and then (7) with (8), we have

$$\widehat{\lambda}_0\int_{\mathbb{R}^d} e^{-2|\xi|^2\tau}\,d\mu(\xi) \ge \widehat{\lambda}_0\int_{\mathbb{R}^d} e^{-2|\xi|^2\tau}\,d\mu(\xi) + \\ + \sum_{j=1}^n \widehat{\lambda}_j\left(\int_{\mathbb{R}^d} e^{-2|\xi|^2 t_j}\,d\mu(\xi) - \delta_j^2\right) \ge \widehat{\lambda}_0\int_{\mathbb{R}^d} e^{-2|\xi|^2\tau}\,d\widehat{\mu}(\xi) + \\ + \sum_{j=1}^n \widehat{\lambda}_j\left(\int_{\mathbb{R}^d} e^{-2|\xi|^2 t_j}\,d\widehat{\mu}(\xi) - \delta_j^2\right) = \widehat{\lambda}_0\int_{\mathbb{R}^d} e^{-2|\xi|^2\tau}\,d\widehat{\mu}(\xi).$$

Dividing on $\widehat{\lambda}_0 < 0$ we obtain the required assertion.

From conditions (7) and (8) one can see what should be a measure $d\widehat{\mu}(\cdot)$ and Lagrange multipliers. Indeed, write the Lagrange function in the form

$$\mathcal{L}(d\mu(\cdot), \lambda) = \int_{\mathbb{R}^d} e^{-2|\xi|^2\tau} f(|\xi|^2)\, d\mu(\xi) - \sum_{j=1}^n \lambda_j \delta_j^2, \tag{9}$$

where

$$f(v) = \lambda_0 + \sum_{j=1}^n \lambda_j e^{-2v(t_j-\tau)}.$$

Hence we see that if $f(|\xi|^2) \geq 0$ for all $\xi \in \mathbb{R}^d$ and the support of measure $d\widehat{\mu}(\cdot)$ is at zeros of this function, then for all $d\mu(\cdot) \geq 0$ we have $\mathcal{L}(d\mu(\cdot), \lambda) \geq -\sum_{j=1}^n \lambda_j \delta_j^2 = \mathcal{L}(d\widehat{\mu}(\cdot), \lambda)$, that is, condition (7) holds. But for all non-negative $\lambda_1, \ldots, \lambda_n$ the function $f(\cdot)$ is convex on $\mathbb{R}$ and therefore if a point $v_0 \in \mathbb{R}$ such that $f(v_0) = f'(v_0) = 0$, then $f(v) \geq 0$ for all $v \in \mathbb{R}$. We will be guided by this observation.

Consider separately three cases: (a) $\tau \geq t_1$ and there is a break point of $\theta(\cdot)$ right of τ, (b) $\tau \geq t_1$ and there are not break points of $\theta(\cdot)$ right of τ, (c) $\tau < t_1$.

(a) Let $\tau \in [t_{s_j}, t_{s_{j+1}})$. Put $d\widehat{\mu}(\xi) = A\delta(\xi - \xi_0)$, where $\delta(\cdot - \xi_0)$ is the delta-function at the point ξ_0, and choose A with ξ_0 from the conditions

$$\int_{\mathbb{R}^d} e^{-2|\xi|^2 t_k}\, d\widehat{\mu}(\xi) = Ae^{-2|\xi_0|^2 t_k} = \delta_k^2, \quad k = s_j, s_{j+1}. \tag{10}$$

Hence it is easy to deduce that

$$A = \delta_{s_j}^{\frac{2t_{s_j+1}}{t_{s_{j+1}}-t_{s_j}}} \delta_{s_{j+1}}^{-\frac{2t_{s_j}}{t_{s_{j+1}}-t_{s_j}}}$$

and

$$|\xi_0|^2 = \frac{\ln(\delta_{s_j}/\delta_{s_{j+1}})}{t_{s_{j+1}} - t_{s_j}} = \frac{\ln 1/\delta_{s_{j+1}} - \ln 1/\delta_{s_j}}{t_{s_{j+1}} - t_{s_j}}.$$

Such point $\xi_0 \in \mathbb{R}^d$ exists since it follows from the construction of the polygonal line $\theta(\cdot)$ that the slope of the line which pass threw the points $(t_{s_j}, \ln 1/\delta_{s_j})$ and $(t_{s_{j+1}}, \ln 1/\delta_{s_{j+1}})$ is positive.

Put $\widehat{\lambda}_0 = -1$, $\widehat{\lambda}_k = 0$, $k \neq s_j, s_{j+1}$, and choose $\widehat{\lambda}_{s_j}$ and $\widehat{\lambda}_{s_{j+1}}$ such that $f(|\xi_0|^2) = f'(|\xi_0|^2) = 0$, that is, as the solution of the linear system

$$\lambda_{s_j} e^{-2|\xi_0|^2(t_{s_j}-\tau)} + \lambda_{s_{j+1}} e^{-2|\xi_0|^2(t_{s_{j+1}}-\tau)} = 1,$$
$$\lambda_{s_j}(t_{s_j} - \tau) e^{-2|\xi_0|^2(t_{s_j}-\tau)} + \lambda_{s_{j+1}}(t_{s_{j+1}} - \tau) e^{-2|\xi_0|^2(t_{s_{j+1}}-\tau)} = 0.$$

Hence

$$\widehat{\lambda}_{s_j}=\frac{t_{s_{j+1}}-\tau}{t_{s_{j+1}}-t_{s_j}}\left(\frac{\delta_{s_{j+1}}}{\delta_{s_j}}\right)^{\frac{2(\tau-t_{s_j})}{t_{s_{j+1}}-t_{s_j}}},$$

$$\widehat{\lambda}_{s_{j+1}}=\frac{\tau-t_{s_j}}{t_{s_{j+1}}-t_{s_j}}\left(\frac{\delta_{s_j}}{\delta_{s_{j+1}}}\right)^{\frac{2(t_{s_{j+1}}-\tau)}{t_{s_{j+1}}-t_{s_j}}}.$$

Thus, $f(|\xi|^2)\ge 0$ for all $\xi\in\mathbb{R}^d$ and the positive measure $d\widehat{\mu}(\cdot)$ is supported at the point ξ_0 where $f(|\xi_0|^2)=0$. Consequently, the condition (7) is fulfilled.

If $\tau\in(t_{s_j},t_{s_{j+1}})$, then evidently $\widehat{\lambda}_{s_j}>0$ and $\widehat{\lambda}_{s_{j+1}}>0$, and if $\tau=t_{s_j}$, then $\widehat{\lambda}_{s_j}=1$ and $\widehat{\lambda}_{s_{j+1}}=0$, so that in view of (10) the condition (8) is also fulfilled. It remains to check the admissibility of the measure $d\widehat{\mu}(\cdot)$ in problem (6).

It follows from the construction of the polygonal line $\theta(\cdot)$ that all points $(t_i,\ln 1/\delta_i)$, $i=1,\dots,n$, are not higher that its plot, and since this polygonal line is concave its plot is not higher than the line

$$p(t)=\frac{\ln 1/\delta_{s_{j+1}}-\ln 1/\delta_{s_j}}{t_{s_{j+1}}-t_{s_j}}(t-t_{s_j})+\ln\frac{1}{\delta_{s_j}}=\ln\delta_{s_j}^{-\frac{t_{s_{j+1}}-t}{t_{s_{j+1}}-t_{s_j}}}\delta_{s_{j+1}}^{-\frac{t-t_{s_j}}{t_{s_{j+1}}-t_{s_j}}},$$

connecting the points $(t_{s_j},\ln 1/\delta_{s_j})$ and $(t_{s_{j+1}},\ln 1/\delta_{s_{j+1}})$. Then (taking into account the expressions for A and $|\xi_0|^2$) we have

$$\int_{\mathbb{R}^d}e^{-2|\xi|^2t_i}\,d\widehat{\mu}(\xi)=Ae^{-2|\xi_0|^2t_i}=\delta_{s_j}^{2\frac{t_{s_{j+1}}-t_i}{t_{s_{j+1}}-t_{s_j}}}\delta_{s_{j+1}}^{2\frac{t_i-t_{s_j}}{t_{s_{j+1}}-t_{s_j}}}=$$
$$=e^{-2p(t_i)}\le e^{-2\ln\frac{1}{\delta_i}}=\delta_i^2,\quad i=1,\dots,n.$$

that is, $\widehat{\mu}(\cdot)$ is an admissible measure in problem (6) and, moreover, is a solution of it.

Substituting $\widehat{\mu}(\cdot)$ in the functional that should be maximize, we obtain the value of problem (6)

$$\int_{\mathbb{R}^d}e^{-2|\xi|^2\tau}\,d\widehat{\mu}(\xi)=Ae^{-2|\xi_0|^2\tau}=\delta_{s_j}^{2\frac{t_{s_{j+1}}-\tau}{t_{s_{j+1}}-t_{s_j}}}\delta_{s_{j+1}}^{2\frac{\tau-t_{s_j}}{t_{s_{j+1}}-t_{s_j}}}=e^{-2p(\tau)}=e^{-2\theta(\tau)}.$$

Approximating the delta-function by a sequence of delta-shaped functions in a standard way, we obtain that the value of problem (5) is the same. But then $e^{-\theta(\tau)}$ is the value of problem (4), that is, $S=e^{-\theta(\tau)}$.

(*b*) Let $\tau\ge t_{s_k}$ (in particular, $t_{s_k}=t_1$, if $\theta(\cdot)$ is a line). Put $\widehat{\lambda}_0=-1$, $\widehat{\lambda}_{s_k}=1$, $\widehat{\lambda}_{s_j}=0$, $j\ne k$, and $d\widehat{\mu}(\cdot)=\delta_{s_k}^2\delta(\cdot)$ ($\delta(\cdot)$ is the delta-function at the zero). Then evidently (8) is fulfilled. Since for all $\xi \in \mathbb{R}^d$ the inequality

$$f(|\xi|^2) = -1 + e^{-2|\xi|^2(t_{s_k}-\tau)} \geq 0$$

holds and $f(0) = 0$, (7) is fulfilled. The function $\theta(\cdot)$ identically equals $\ln(1/\delta_{s_k})$ in the interval $[t_{s_k}, \infty)$ and it is clear that $\ln(1/\delta_i) \leq \ln(1/\delta_{s_k})$, $1 \leq i \leq n$. Consequently,

$$\int_{\mathbb{R}^d} e^{-2|\xi|^2 t_i}\, d\widehat{\mu}(\xi) = \delta_{s_k}^2 = e^{-2\ln\frac{1}{\delta_{s_k}}} \leq e^{-2\ln\frac{1}{\delta_i}} = \delta_i^2, \quad i = 1, \ldots, n,$$

that is, the measure $d\widehat{\mu}(\xi)$ is admissible in problem (6) and therefore is a solution of it.

Te value of problem (6) is

$$\int_{\mathbb{R}^d} e^{-2|\xi|^2\tau}\, d\widehat{\mu}(\xi) = \delta_{s_k}^2 = e^{-2\ln\frac{1}{\delta_{s_k}}} = e^{-2\theta(\tau)}$$

and hence by the same arguments as in the previous case the value of problem (4) equals $e^{-\theta(\tau)}$.

(c) Let $\tau < t_1$. We show that in this case the value of problem (6) equals $+\infty$. Let $x_0 > 0$. Evidently there exists a line $x = at + b$, $a > 0$, which separate the point $(\tau, -x_0)$ and the set M, in particular,

$$-a\tau - x_0 \geq b \geq -at_i + \ln\frac{1}{\delta_i}, \quad 1 \leq i \leq n.$$

Denoting $A = e^{-2b}$ and choosing $\xi_0 \in \mathbb{R}^d$ such that $|\xi_0|^2 = a$, from these inequalities we obtain that $A\exp(-2|\xi_0|^2 t_i) \leq \delta_i^2$, $1 \leq i \leq n$, that is, the measure $d\mu(\cdot) = \delta(\cdot - \xi_0)$ is admissible in problem (6) and $A\exp(-2|\xi_0|^2\tau) \geq \exp(2x_0)$. In view of arbitrariness of x_0 the value of problem (6) equals $+\infty$. Hence as in the previous cases the value of problem (4) equals $+\infty$.

Thus it is proved that for all $\tau \geq 0$ the error of optimal recovery $E(\tau, \overline{\delta}) \geq e^{-\theta(\tau)}$.

2. *The upper bound of $E(\tau, \overline{\delta})$ and optimal method.* Let $\tau \geq t_1$ and $\widehat{\lambda}_j$, $1 \leq j \leq n$, be the Lagrange multipliers which were found for problem (6) for a given τ. The upper bound of $E(\tau, \overline{\delta})$ and finding of optimal method will be based on the following statement.

Lemma 1. *Let the function $\overline{y}(\cdot) = (y_1(\cdot), \ldots, y_n(\cdot)) \in (L_2(\mathbb{R}^d))^n$ be such that there exists a solution $\widehat{u}_0(\cdot) = \widehat{u}_0(\cdot, \overline{y}(\cdot))$ of the problem*

$$\sum_{j=1}^n \widehat{\lambda}_j \|P_{t_j} u_0(\cdot) - y_j(\cdot)\|^2_{L_2(\mathbb{R}^d)} \to \min, \quad u_0(\cdot) \in L_2(\mathbb{R}^d). \tag{11}$$

Then for all $\gamma_j > 0$, $1 \le j \le n$, the value of the problem

$$\|P_\tau u_0(\cdot) - P_\tau \widehat{u}_0(\cdot)\|_{L_2(\mathbb{R}^d)} \to \max, \quad \|P_{t_j} u_0(\cdot) - y_j(\cdot)\|_{L_2(\mathbb{R}^d)} \le \gamma_j, \quad 1 \le j \le n, \quad u_0(\cdot) \in L_2(\mathbb{R}^d) \tag{12}$$

is not greater than the value of the problem

$$\|P_\tau u_0(\cdot)\|_{L_2(\mathbb{R}^d)} \to \max, \quad \sum_{j=1}^n \widehat{\lambda}_j \|P_{t_j} u_0(\cdot)\|^2_{L_2(\mathbb{R}^d)} \le \sum_{j=1}^n \widehat{\lambda}_j \gamma_j^2, \quad u_0(\cdot) \in L_2(\mathbb{R}^d). \tag{13}$$

Proof. The minimizing functional in (11) is a smooth convex functional on $L_2(\mathbb{R}^d)$ and, consequently, vanishing of the derivative of this functional at the point $\widehat{u}_0(\cdot)$ is the necessary and sufficient condition for the function $\widehat{u}_0(\cdot)$ to be its minimum, that is, for all $u_0(\cdot) \in L_2(\mathbb{R}^d)$ the equality

$$\operatorname{Re} \sum_{j=1}^n \widehat{\lambda}_j \int_{L_2(\mathbb{R}^d)} (P_{t_j} \widehat{u}_0(x) - y_j(x)) \overline{P_{t_j} u_0(x)}\, dx = 0 \tag{14}$$

should be fulfilled.

Taking into account this fact it is easy to check that for all $u_0(\cdot) \in L_2(\mathbb{R}^d)$ the equality

$$\sum_{j=1}^n \widehat{\lambda}_j \|P_{t_j} u_0(\cdot) - y_j(\cdot)\|^2_{L_2(\mathbb{R}^d)} = \sum_{j=1}^n \widehat{\lambda}_j \|P_{t_j} u_0(\cdot) - P_{t_j} \widehat{u}_0(\cdot)\|^2_{L_2(\mathbb{R}^d)} + \sum_{j=1}^n \widehat{\lambda}_j \|P_{t_j} \widehat{u}_0(\cdot) - y_j(\cdot)\|^2_{L_2(\mathbb{R}^d)}$$

holds.

Let $u_0(\cdot)$ be an admissible function in (12). Then it follows from the last formula that

$$\sum_{j=1}^n \widehat{\lambda}_j \|P_{t_j}(u_0(\cdot) - \widehat{u}_0(\cdot))\|^2_{L_2(\mathbb{R}^d)} \le \sum_{j=1}^n \widehat{\lambda}_j \|P_{t_j} u_0(\cdot) - y_j(\cdot)\|^2_{L_2(\mathbb{R}^d)} \le \sum_{j=1}^n \widehat{\lambda}_j \gamma_j^2$$

and thus $u_0(\cdot) - \widehat{u}_0(\cdot)$ is an admissible function in (13). Moreover, the values of maximizing functionals in (12) and (13) coincide at elements $u_0(\cdot)$ and $u_0(\cdot) - \widehat{u}_0(\cdot)$. This yields the required result. $\square$

The scheme of using this lemma is the following. First, we prove that for $\gamma_j = \delta_j$, $1 \le j \le n$, the values of problems (4) and (13) coincide (that is, the value of problem (13) equals $e^{-\theta(\tau)}$). If we assume that for all $\overline{y}(\cdot) \in (L_2(\mathbb{R}^d))^n$ there exists a solution of problem (11), then the statement of Lemma means that the error $e(\tau, \overline{\delta}, \widehat{m})$ of the method $\widehat{m}\colon \overline{y}(\cdot) \mapsto P_\tau \widehat{u}_0(\cdot, \overline{y}(\cdot))$ does not exceed $e^{-\theta(\tau)}$ and, moreover, $E(\tau, \overline{\delta}) \le e^{-\theta(\tau)}$. Together with the proved lower bound hence $e(\tau, \overline{\delta}, \widehat{m}) = E(\tau, \overline{\delta})$ and so that $\widehat{m}$ is an optimal method.

However the solution of (11) exists not for all $\overline{y}(\cdot) \in (L_2(\mathbb{R}^d))^n$ and this fact will require some correction of the given arguments.

Thus we will prove the coincidences of the values of problems (4) and (13) for $\gamma_j = \delta_j$, $1 \le j \le n$. In just the same way as we passed from problem (4) to problem (6) (using Plancherel's theorem and then replacing $(2\pi)^{-d}|Fu_0(\xi)|^2\,d\xi$ by a positive measure), we pass from (13) to the problem

$$(15)\qquad \int_{\mathbb{R}^d} e^{-2|\xi|^2\tau}\,d\mu(\xi) \to \max,$$
$$\sum_{j=1}^n \widehat{\lambda}_j \int_{\mathbb{R}^d} e^{-2|\xi|^2 t_j}\,d\mu(\xi) \le \sum_{j=1}^n \widehat{\lambda}_j \delta_j^2, \quad d\mu(\cdot) \ge 0.$$

This is a convex problem. Its Lagrange function has the form

$$\mathcal{L}_1(d\mu(\cdot), \nu) = \nu_0 \int_{\mathbb{R}^d} e^{-2|\xi|^2\tau}\,d\mu(\xi) + \\ + \nu_1 \left(\sum_{j=1}^n \widehat{\lambda}_j \int_{\mathbb{R}^d} e^{-2|\xi|^2 t_j}\,d\mu(\xi) - \sum_{j=1}^n \widehat{\lambda}_j \delta_j^2 \right),$$

where $\nu = (\nu_0, \nu_1)$ are the set of Lagrange multipliers.

We show that the solution $d\widehat{\mu}(\cdot)$ of problem (6) is also the solution of this problem. For this reason (similarly to what was done for problem (6)) it is sufficient to check that the measure $d\widehat{\mu}(\cdot)$ is admissible in (15) and that for some $\widehat{\nu}_0 < 0$ and $\widehat{\nu}_1 \ge 0$ analogs of conditions (7) and (8) for this problem are fulfilled, namely,

$$\min_{d\mu(\cdot)\ge 0} \mathcal{L}_1(d\mu(\cdot), \widehat{\nu}) = \mathcal{L}_1(d\widehat{\mu}(\cdot), \widehat{\nu}),$$

where $\widehat{\nu} = (\widehat{\nu}_0, \widehat{\nu}_1)$ and

$$\widehat{\nu}_1 \left(\sum_{j=1}^n \widehat{\lambda}_j \int_{\mathbb{R}^d} e^{-2|\xi|^2 t_j}\,d\widehat{\mu}(\xi) - \sum_{j=1}^n \widehat{\lambda}_j \delta_j^2 \right) = 0.$$

It follows immediately from the admissibility of the measure $d\widehat{\mu}(\cdot)$ in problem (6) its admissibility in problem (15). Put $\widehat{\nu}_0 = -1$ and $\widehat{\nu}_1 = 1$. Then $\mathcal{L}_1(d\mu(\cdot), \widehat{\nu}) = \mathcal{L}(d\mu(\cdot), \widehat{\lambda})$ and, consequently, the first of the written relations is equivalent to (7) and therefore is fulfilled. The second relation immediately follows from (8). Thus, $d\widehat{\mu}(\cdot)$ is the solution of problem (15) and it means that its value coincides with the value of problem (6).

Further, as above, approximating the delta-function by delta-shaped functions we obtain that the squared of the value of problem (13) equals the value of (15) and it means that the values of problems (4) and (13) coincide.

Now we use the lemma. For this reason at first we find the value of problem (11) for the function $\overline{y}(\cdot) = (y_1(\cdot), \dots, y_n(\cdot)) \in (L_2(\mathbb{R}^d))^n$, for which functions $Fy_i(\cdot)$, $1 \le i \le n$, are compactly supported.

Let $\tau \in [t_{s_j}, t_{s_{j+1}})$. In this case, as it was proved, only Lagrange multipliers $\widehat{\lambda}_{s_j}$ and $\widehat{\lambda}_{s_{j+1}}$ may not be zeros (and simultaneously are not zeros) and therefore problem (11) has the form

$$\widehat{\lambda}_{s_j}\|P_{t_{s_j}}u_0(\cdot)-y_{s_j}(\cdot)\|^2_{L_2(\mathbb{R}^d)}+\widehat{\lambda}_{s_{j+1}}\|P_{t_{s_{j+1}}}u_0(\cdot)-y_{s_{j+1}}(\cdot)\|^2_{L_2(\mathbb{R}^d)} \to \min, \quad u_0(\cdot) \in L_2(\mathbb{R}^d).$$

If $\widehat{u}_0(\cdot) = \widehat{u}_0(\cdot, \overline{y}(\cdot))$ is the solution of this problem, then condition (14) is fulfilled, which according to Plancherel's theorem after the Fourie transform will be written in the form

$$\operatorname{Re}\sum_{k=j}^{j+1}\widehat{\lambda}_{s_k}\int_{L_2(\mathbb{R}^d)}(e^{-|\xi|^2 t_{s_k}}F\widehat{u}_0(\xi) - Fy_{s_k}(\xi))e^{-|\xi|^2 t_{s_k}}\overline{Fu_0(\xi)}\,d\xi = 0.$$

It is easy to verify that this relation will be fulfilled for all $u_0(\cdot) \in L_2(\mathbb{R}^d)$, if the function $\widehat{u}_0(\cdot) \in L_2(\mathbb{R}^d)$ such that its Fourier transform has the form

$$F\widehat{u}_0(\xi) = \frac{\widehat{\lambda}_{s_j}e^{-|\xi|^2 t_{s_j}}Fy_{s_j}(\xi) + \widehat{\lambda}_{s_{j+1}}e^{-|\xi|^2 t_{s_{j+1}}}Fy_{s_{j+1}}(\xi)}{\widehat{\lambda}_{s_j}e^{-2|\xi|^2 t_{s_j}} + \widehat{\lambda}_{s_{j+1}}e^{-2|\xi|^2 t_{s_{j+1}}}}. \tag{16}$$

But the expression in the right-hand side belongs $L_2(\mathbb{R}^d)$ since the functions $Fy_{s_j}(\cdot)$ and $Fy_{s_{j+1}}(\cdot)$ are compactly supported and therefore $\widehat{u}_0(\cdot) = \widehat{u}_0(\cdot, \overline{y}(\cdot)) \in L_2(\mathbb{R}^d)$. In view of sufficiency of condition (14) the function $\widehat{u}_0(\cdot)$ defined by formula (16) is the solution of problem (11).

Note that if $\tau = t_{s_j}$, then $\widehat{\lambda}_{s_j} = 1$, $\widehat{\lambda}_{s_{j+1}} = 0$, and the solution of equation (11) in this case is evident (and of course it follows from (16)) and has the form

$$F\widehat{u}_0(\xi) = e^{|\xi|^2 t_{s_j}}Fy_{s_j}(\xi). \tag{17}$$

It is well known that compactly supported functions are dense in $L_2(\mathbb{R}^d)$. Then by Plancherel's theorem it follows that functions which have compactly supported Fourier transforms are also dense in $L_2(\mathbb{R}^d)$.

Now let $\widetilde{u}_0(\cdot) \in L_2(\mathbb{R}^d)$ and $\overline{y}(\cdot) = (y_1(\cdot), \dots, y_n(\cdot)) \in (L_2(\mathbb{R}^d))^n$ be such that $\|P_{t_j}\widetilde{u}_0(\cdot) - y_j(\cdot)\|_{L_2(\mathbb{R}^d)} \le \delta_j$, $1 \le j \le n$. Further, let $\overline{y}_k(\cdot) = (y_{1k}(\cdot), \dots, y_{nk}(\cdot)) \in (L_2(\mathbb{R}^d))^n$, $k \in \mathbb{N}$, be a sequence with the property that functions $Fy_{jk}(\cdot)$ are compactly supported and $\|y_j(\cdot) - y_{jk}(\cdot)\|_{L_2(\mathbb{R}^d)} \le 1/k$, $1 \le j \le n$, $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$. As it was proved, for $\overline{y}_k(\cdot)$ there exists the solution $\widehat{u}_0(\cdot, \overline{y}_k(\cdot))$ of problem (11). Since $\|P_{t_j}\widetilde{u}_0(\cdot) - y_{jk}(\cdot)\|_{L_2(\mathbb{R}^d)} \le \|P_{t_j}\widetilde{u}_0(\cdot) - y_j(\cdot)\|_{L_2(\mathbb{R}^d)} + \|y_j(\cdot) - y_{jk}(\cdot)\|_{L_2(\mathbb{R}^d)} \le \delta_j + 1/k$, $1 \le j \le n$, the function $\widetilde{u}_0(\cdot)$ is admissible in problem (12) with $\gamma_j = \gamma_j(k) = \delta_j + 1/k$, $1 \le j \le n$. Due to the statement of lemma the value of this problem does not exceed the value of problem (13), which after replacing $u_0(\cdot) = a(k)v_0(\cdot)$, where $a(k) = \sqrt{\sum_{j=1}^n \widehat{\lambda}_j \gamma_j^2(k) / \sum_{j=1}^n \widehat{\lambda}_j \delta_j^2}$, takes the form

$$a(k)\|P_\tau v_0(\cdot)\|_{L_2(\mathbb{R}^d)} \to \max, \quad \sum_{j=1}^n \widehat{\lambda}_j \|P_{t_j} v_0(\cdot)\|^2_{L_2(\mathbb{R}^d)} \le \sum_{j=1}^n \widehat{\lambda}_j \delta_j^2,$$
$$v_0(\cdot) \in L_2(\mathbb{R}^d).$$

The value of this problem as it has proved coincides with the value of problem (4) multiplied by $a(k)$, that is, it equals $a(k)e^{-\theta(\tau)}$. In particular (in view of admissibility of $\widetilde{u}_0(\cdot)$ in (12)), we obtain that

$$\|P_\tau \widetilde{u}_0(\cdot) - P_\tau \widehat{u}_0(\cdot, \overline{y}_k(\cdot))\|_{L_2(\mathbb{R}^d)} \le a(k)e^{-\theta(\tau)}. \tag{18}$$

Let $\tau \in (t_{s_j}, t_{s_{j+1}})$. The Fourier transform of functions $K_{s_j}(\cdot)$ and $K_{s_{j+1}}(\cdot)$ from the statement of the theorem belong to the space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$. In this space the Fourier transform is an isomorphism and therefore the functions $K_{s_j}(\cdot)$ and $K_{s_{j+1}}(\cdot)$ belong to this space. In particular, they are bound and then according to Young's inequality the method $\widehat{m}$ from the statement of theorem is continuous linear operator from $(L_2(\mathbb{R}^d))^n$ to $L_2(\mathbb{R}^d)$.

It follows from the form of method $\widehat{m}$, expressions of functions $FK_{s_j}(\cdot)$ and $FK_{s_{j+1}}(\cdot)$, and formula (16) that

$$F\widehat{m}(\overline{y}_k(\cdot))(\xi) = FK_{s_j}(\xi)Fy_{s_jk}(\xi) + FK_{s_{j+1}}(\xi)Fy_{s_{j+1}k}(\xi) =$$
$$= e^{-|\xi|^2\tau}F\widehat{u}_0(\cdot, \overline{y}_k(\cdot))(\xi),$$

that is,

$$\widehat{m}(\overline{y}_k(\cdot))(\cdot) = P_\tau \widehat{u}_0(\cdot, \overline{y}_k(\cdot)). \tag{19}$$

If $\tau = t_{s_j}$, then it follows from the form of method $\widehat{m}$ that

$$F\widehat{m}(\overline{y}_k(\cdot))(\xi) = Fy_{s_jk}(\xi) = e^{-|\xi|^2\tau}F\widehat{u}_0(\cdot, \overline{y}_k(\cdot))(\xi),$$

that is, again formula (19) holds.

Returning to $\widetilde{u}_0(\cdot) \in L_2(\mathbb{R}^d)$ and $\overline{y}(\cdot) = (y_1(\cdot), \ldots, y_n(\cdot)) \in (L_2(\mathbb{R}^d))^n$ such that $\|P_{t_j}\widetilde{u}_0(\cdot) - y_j(\cdot)\|_{L_2(\mathbb{R}^d)} \le \delta_j$, $1 \le j \le n$, according to (19) and (18), we have

$$\begin{aligned}\|P_\tau\widetilde{u}_0(\cdot) - \widehat{m}(\overline{y}(\cdot))(\cdot)\|_{L_2(\mathbb{R}^d)} &\le \|P_\tau\widetilde{u}_0(\cdot) - P_\tau\widehat{u}_0(\cdot, \overline{y}_k(\cdot))\|_{L_2(\mathbb{R}^d)} + \\ &+ \|\widehat{m}(\overline{y}_k(\cdot))(\cdot) - \widehat{m}(\overline{y}(\cdot))(\cdot)\|_{L_2(\mathbb{R}^d)} \le \\ &\le a(k)e^{-\theta(\tau)} + \|\widehat{m}(\overline{y}_k(\cdot) - y(\cdot))(\cdot)\|_{L_2(\mathbb{R}^d)}.\end{aligned}$$

This is true for any $k \in \mathbb{N}$. Passing to the limit as $k \to \infty$ (taking into account that $a(k) \to 1$ and that the method $\widehat{m}$ is continuous), we obtain the inequality

$$\|P_\tau\widetilde{u}_0(\cdot) - \widehat{m}(\overline{y}(\cdot))(\cdot)\|_{L_2(\mathbb{R}^d)} \le e^{-\theta(\tau)}.$$

Passing here to the upper bound over all $\widetilde{u}_0(\cdot) \in L_2(\mathbb{R}^d)$ and $\overline{y}(\cdot) = (y_1(\cdot), \ldots, y_n(\cdot)) \in (L_2(\mathbb{R}^d))^n$ such that $\|P_{t_j}\widetilde{u}_0(\cdot) - y_j(\cdot)\|_{L_2(\mathbb{R}^d)} \le \delta_j$, $1 \le j \le n$, we obtain that $e(\tau, \overline{\delta}, \widehat{m}) \le e^{-\theta(\tau)}$. This and the proved lower bound yield that

$$e^{-\theta(\tau)} \le E(\tau, \overline{\delta}) \le e(\tau, \overline{\delta}, \widehat{m}) \le e^{-\theta(\tau)},$$

that is, $E(\tau, \overline{\delta}) = e^{-\theta(\tau)}$ and $\widehat{m}$ is an optimal method.

Thus for the case when $\tau \in [t_{s_j}, t_{s_{j+1}})$ the theorem is proved.

Let $\tau \ge t_{s_k}$. If $\tau = t_{s_k}$, then just the same arguments as for the case when $\tau = t_{s_j}$ give the required estimate and optimal method.

Let $\tau > t_{s_k}$. Here arguments are also similar to the previous ones but rather more simply, therefore we will be short. In the given case $\widehat{\lambda}_{s_k} = 1$ and all the rest Lagrange multipliers are vanishing therefore problem (11) takes the form

$$\|P_{t_{s_k}}u_0(\cdot) - y_{s_k}(\cdot)\|^2_{L_2(\mathbb{R}^d)} \to \min, \quad u_0(\cdot) \in L_2(\mathbb{R}^d).$$

If $\overline{y}(\cdot) = (y_1(\cdot), \ldots, y_n(\cdot))$ such that functions $Fy_j(\cdot)$, $1 \le j \le n$, are compactly supported, then the solution $\widehat{u}_0(\cdot) = \widehat{u}_0(\cdot, \overline{y}(\cdot))$ of the given problem exists and $F\widehat{u}_0(\xi) = e^{|\xi|^2 t_{s_k}}Fy_{s_k}(\xi)$.

Further, repeating word for word the previous arguments we arrive at the inequality (18).

The method $\widehat{m}$ from the statement of the theorem by definition is a continuous linear operator from $(L_2(\mathbb{R}^d))^n$ to $L_2(\mathbb{R}^d)$. If $\overline{y}(\cdot) = \overline{y}_k(\cdot)$, then it is clear that $F\widehat{m}(\overline{y}_k(\cdot))(\xi) = e^{-|\xi|^2(\tau - t_{s_k})}Fy_{s_k}(\xi) =$

$e^{-|\xi|^2\tau}F\widehat{u}_0(\cdot,\overline{y}_k(\cdot))(\xi)$, . . $\widehat{m}(\overline{y}_k(\cdot))(\cdot) = P_\tau\widehat{u}_0(\cdot,\overline{y}_k(\cdot))(\cdot)$. Further arguments are the same as in the previous case. The theorem is proved.

4. Optimal recovery with a priory information

We again consider problem (1)–(2) and instants of time $0 \le t_1 < \ldots < t_n$. Let A and B be subsets of $\{1,\ldots,n\}$ (one of which may be empty) such that $A\cap B = \emptyset$ and $A\cup B = \{1,\ldots,n\}$. We state the following problem. We know the following a priory information: the temperature could not fall outside some limits at instants of time t_i, $i \in A$, that is, we know that $\|u(t_i,\cdot)\|_{L_2(\mathbb{R}^d)} \le \delta_i$, where $\delta_i > 0$, $i \in A$.

Let $B \ne \emptyset$ and assume we know approximately temperature distributions $u(t_i,\cdot)$ at instants of time t_i, $i \in B$, that is, we know functions $y_i(\cdot) \in L_2(\mathbb{R}^d)$ such that $\|u(t_i,\cdot) - y_i(\cdot)\|_{L_2(\mathbb{R}^d)} \le \delta_i$, where $\delta_i > 0$. As above, with any set of such functions we want to associate a function from $L_2(\mathbb{R}^d)$ which approximate the real temperature distribution in $\mathbb{R}^d$ at a fixed instant of time τ in some sense in a best way.

If $B \ne \emptyset$ and $\operatorname{card} B = l$, then again any map m from $(L_2(\mathbb{R}^d))^l$ to $L_2(\mathbb{R}^d)$ is considered as a method of recovery. The quantity

$$e(\tau,A,B,\overline{\delta},m) = \sup_{\substack{u_0(\cdot),\ \overline{y}_B(\cdot)\in(L_2(\mathbb{R}^d))^l \\ \|u(t_i,\cdot)\|_{L_2(\mathbb{R}^d)}\le\delta_i,\ i\in A \\ \|u(t_i,\cdot)-y_i(\cdot)\|_{L_2(\mathbb{R}^d)}\le\delta_i,\ i\in B}} \|u(\tau,\cdot) - m(\overline{y}_B(\cdot))(\cdot)\|_{L_2(\mathbb{R}^d)},$$

where $\overline{y}_B(\cdot) = \{y_i(\cdot)\}_{i\in B}$ and $\overline{\delta} = (\delta_1,\ldots,\delta_n)$, is called the error of this method.

We are interested in the quantity

$$E(\tau,A,B,\overline{\delta}) = \inf_{m\colon (L_2(\mathbb{R}^d))^l\to L_2(\mathbb{R}^d)} e(\tau,\overline{\delta},m),$$

which also called the *error of optimal recovery* and in the method $\widehat{m}$ delivering the lower bound, that is, for which

$$E(\tau,A,B,\overline{\delta}) = e(\tau,A,B,\overline{\delta},\widehat{m}),$$

which is called an *optimal recovery method* (of the temperature in $\mathbb{R}^d$ at the insant of time τ from the giving information).

Note that if $A = \emptyset$, then we arrive at the previous setting. If $B = \emptyset$, then there are no observations and thus there are no sense to speak about any recovery method. But we can speak about estimate of temperature at the instant of time τ, that is, about finding of bounds which temperature certainly could not exceed for a given a priory information. It is natural to take the Chebyshev radius of the set $\{\, u(\tau,\cdot)(\cdot) \in L_2(\mathbb{R}^d) \mid \|u(t_i,\cdot)\|_{L_2(\mathbb{R}^d)} \le \delta_i,\ 1 \le i \le n \,\}$ as such estimate, that is, the minimal radius of balls containing the given set.

Since the set is centrally symmetric it is easy to verify that this quantity is

$$E(\tau,A,\emptyset,\overline{\delta})=\sup_{\substack{u_0(\cdot)\in L_2(\mathbb{R}^d)\\ \|u(t_i,\cdot)\|_{L_2(\mathbb{R}^d)}\le\delta_i,\ i=1,\dots,n}}\|u(\tau,\cdot)(\cdot)\|_{L_2(\mathbb{R}^d)}.$$

The mentioned setting, as it was noted, generalized the initial setting and we could consider precisely the problem with a priory information at the very beginning. But we wanted to remain the simplicity of the initial setting the more so the proof of the generalized result actually the same and we only show those changes which one should do in the previous arguments.

Theorem 2. *For all A and B and any $\tau\ge 0$ the equality*

$$E(\tau,A,B,\overline{\delta})=e^{-\theta(\tau)}$$

holds. Let $B\ne\emptyset$. Then

1. *if $t_1>0$, $0\le\tau<t_1$, then any method is optimal;*
2. *if $\tau=t_{s_j}$, $1\le j\le k$, and $s_j\in B$, then the method $\widehat{m}$, defined be equality $\widehat{m}(\overline{y}(\cdot))(\cdot)=y_{s_j}(\cdot)$ is optimal, and if $s_j\notin B$, then the zero mapping is optimal method;*
3. *if $k\ge 2$, $\tau\in(t_{s_j},t_{s_{j+1}})$, $1\le j\le k-1$, and $s_j,s_{j+1}\in B$, then the method $\widehat{m}$, defined in (3) in Theorem 1 is optimal; if $s_j\in B$, and $s_{j+1}\notin B$, then $\widehat{m}(\overline{y}(\cdot))(\cdot)=(K_{s_j}*y_{s_j})(\cdot)$ is optimal method ($K_{s_j}(\cdot)$ from Theorem 1); if $s_j\notin B$, and $s_{j+1}\in B$, then the method $\widehat{m}(\overline{y}(\cdot))(\cdot)=(K_{s_{j+1}}*y_{s_{j+1}})(\cdot)$ is optimal ($K_{s_{j+1}}(\cdot)$ from Thorem 1); finally, if $s_j,s_{j+1}\notin B$, then the zero mapping is optimal method;*
4. *if $\tau>t_{s_k}$ and $s_k\in B$, then the method $\widehat{m}$, defined in (4) in Thorem 1, is optimal, if $s_k\notin B$, then the zero mapping is optimal method.*

Proof. Let $B=\emptyset$. Then $E(\tau,A,\emptyset,\overline{\delta})$ coincides with the value of problem (4) (which, as it was proved, equals $e^{-\theta(\tau)}$ for all $\tau\ge 0$) so that $E(\tau,A,\emptyset,\overline{\delta})=e^{-\theta(\tau)}$.

Let $B\ne\emptyset$. Then repeating word for word the arguments from the beginning of the proof of Theorem 1, we obtain that $E(\tau,A,B,\overline{\delta})$ no less than the value of problem (4) and thus for any sets A and B and for all $\tau\ge 0$ the lower bound $E(\tau,A,B,\overline{\delta})\ge e^{-\theta(\tau)}$ holds.

We proceed to the proof of the upper bound and to presentation of appropriate optimal methods. Here we will be based on the following statement which formally generalize Lemma 1 but is proved in just the same way.

Lemma 2. *Let the function $\overline{y}_B(\cdot) = \{y_j(\cdot)\}_{j\in B}$ be such that there exists a solution $\widehat{u}_0(\cdot) = \widehat{u}_0(\cdot, \overline{y}_B(\cdot))$ of the problem*

$$\sum_{j\in A} \widehat{\lambda}_j \|P_{t_j} u_0(\cdot)\|^2_{L_2(\mathbb{R}^d)} + \sum_{j\in B} \widehat{\lambda}_j \|P_{t_j} u_0(\cdot) - y_j(\cdot)\|^2_{L_2(\mathbb{R}^d)} \to \min, \quad u_0(\cdot) \in L_2(\mathbb{R}^d).$$

Then for any $\gamma_j > 0$, $1 \le j \le n$, the value of the problem

$$\|P_\tau u_0(\cdot) - P_\tau \widehat{u}_0(\cdot)\|_{L_2(\mathbb{R}^d)} \to \max, \quad \|P_{t_j} u_0(\cdot)\|_{L_2(\mathbb{R}^d)} \le \gamma_j,\ j \in A,$$
$$\|P_{t_j} u_0(\cdot) - y_j(\cdot)\|_{L_2(\mathbb{R}^d)} \le \gamma_j,\ j \in B, \quad u_0(\cdot) \in L_2(\mathbb{R}^d),$$

is not greater than the value of the problem

$$\|P_\tau u_0(\cdot)\|_{L_2(\mathbb{R}^d)} \to \max, \quad \sum_{j=1}^n \widehat{\lambda}_j \|P_{t_j} u_0(\cdot)\|^2_{L_2(\mathbb{R}^d)} \le \sum_{j=1}^n \widehat{\lambda}_j \gamma_j^2, \quad u_0(\cdot) \in L_2(\mathbb{R}^d).$$

Let $\tau \in [t_{s_j}, t_{s_{j+1}})$. If $s_j, s_{j+1} \in B$, then just the same arguments as in Theorem 1 prove the optimality of appropriate methods.

Let $s_j \in B$ and $s_{j+1} \notin B$. In this case according to Lemma 2 the analog of problem (11) has the form

$$\widehat{\lambda}_{s_j} \|P_{t_{s_j}} u_0(\cdot) - y_{s_j}(\cdot)\|^2_{L_2(\mathbb{R}^d)} + \widehat{\lambda}_{s_{j+1}} \|P_{t_{s_{j+1}}} u_0(\cdot)\|^2_{L_2(\mathbb{R}^d)} \to \min, \quad u_0(\cdot) \in L_2(\mathbb{R}^d).$$

Again the same arguments as in Theorem 1 (with $y_{s_{j+1}}(\cdot) = 0$) lead to the proof of the optimality of appropriate methods.

If $s_j \notin B$ and $s_{j+1} \notin B$, then the analog of problem (11) has the form

$$\widehat{\lambda}_{s_j} \|P_{t_{s_j}} u_0(\cdot)\|^2_{L_2(\mathbb{R}^d)} + \widehat{\lambda}_{s_{j+1}} \|P_{t_{s_{j+1}}} u_0(\cdot)\|^2_{L_2(\mathbb{R}^d)} \to \min, \quad u_0(\cdot) \in L_2(\mathbb{R}^d),$$

and here the zero function evidently is a solution. The optimality of the zero method immediately follows from Lemma 2.

The rest cases are considered similarly. □

5. Comments

Optimal recovery problems solved in this paper are included in the following general scheme. Let X be a linear space, W a subset (class) in X, $Y_1, \dots, Y_r$, and Z normed spaces, $I_i\colon X \to Y_i$, $1 \le i \le r$, linear operators. We state the problem of optimal recovery of a linear operator $\Lambda\colon X\to Z$ on the class W from the following information about elements from this class: for every element $x\in W$ we know a vector $\overline{y}=(y_1,\dots,y_r)\in Y_1\times\ldots\times Y_r$ such that $\|I_ix-y_i\|_{Y_i}\le\delta_i$, $\delta_i\ge 0$, $1\le i\le r$.

By optimal recovery of Λ on W from the given information we mean the following. Any map m from $Y_1\times\ldots\times Y_r$ to Z is admitted as a recovery method (of Λ on W from the given information). The quantity

$$e(\Lambda,W,\overline{I},\overline{\delta},m)=\sup_{\substack{x\in W,\ \overline{y}\in Y_1\times\ldots\times Y_r\\ \|I_ix-y_i\|_{Y_i}\le\delta_i,\ i=1,\dots,r}}\|\Lambda x-m(\overline{y})\|_Z,$$

where $\overline{I}=(I_1,\dots,I_r)$ and $\overline{\delta}=(\delta_1,\dots,\delta_r)$, is called the error of this method.

The quantity

$$E(\Lambda,W,\overline{I},\overline{\delta})=\inf_{m\colon Y_1\times\ldots\times Y_r\to Z}e(\Lambda,W,\overline{I},\overline{\delta},m),$$

is called the *error of optimal recovery*, and a method $\widehat{m}$ delivering the lower bound, that is, for which

$$E(\Lambda,W,\overline{I},\overline{\delta})=e(\Lambda,W,\overline{I},\overline{\delta},\widehat{m}),$$

is called an *optimal method of recovery* (of Λ on W from the given information).

For example, in according with these notation in the problem with a priory information and when $B\ne\emptyset$ ($\operatorname{card}B=l$) we have: $r=l$, $X=Y_1=\ldots=Y_l=Z=L_2(\mathbb{R}^d)$,

$$W=\{\,u_0(\cdot)\in L_2(\mathbb{R}^d)\mid \|P_{t_i}u_0(\cdot)\|_{L_2(\mathbb{R}^d)}\le\delta_i,\ i\in A\,\}$$

(if $A=\emptyset$, the we put $W=X=L_2(\mathbb{R}^d)$), $I_i=P_{t_i}$, $i\in B$, $\Lambda=P_\tau$.

The stated approach to definition of optimal method (in an abstract problem) ideologically goes back to the papers of A. N. Kolmogorov of the 1930's devoted to finding of the best approximation tool for all functions from the given class at once. The setting mentioned here for the case when $r=1$, X and Y are finite-dimensional spaces, $Z=\mathbb{R}$ (the problem about the recovery of a linear functional) and $\delta_1=0$ (the information is given precisely) was considered for the first time by S. A. Smolyak [2]. He proved that if W is a convex centrally symmetric set, then among optimal methods there exists a linear one. Quite many papers (see [3]–[7]) were devoted to the extension of this fact to more general situations but in some sense the final result in this field, namely necessary and sufficient conditions of existing of linear optimal method, was obtained by authors [8]. Quite extensive literature is devoted to optimal recovery of linear functionals. The general approach to the solution of similar problems based on standard methods of extremum theory is explained in [9]. Many concrete results and further references may be found in the books [10]–[14].

The general result concerning the existence of linear method for operators (Z is a Hilbert space) was proved in [15] and there were also obtained concrete results about optimal recovery of linear operators. Further development of these subjects was given by authors [16]–[18] where other approaches were used based on general principles of extremum theory.

An application of optimal recovery of linear operators to problems of mathematical physics may be found in [19]–[23].

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Moscow State Institute of Radio Engineering, Electronics and Automation (Technology University)

MATI — Russian State Technological University