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THE BEST APPROXIMATION OF A SET WHOSE ELEMENTS ARE KNOWN APPROXIMATELY

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Abstract. This paper is concerned with the problem of the best (in a precisely defined sense) approximation with given accuracy of periodic functions and functions on the real line from, respectively, a finite tuple of noisy Fourier coefficients or noisy Fourier transform on an arbitrary set of finite measure.

Introduction

We start with the general formulation of the best approximation problem with fixed accuracy to elements of a given class, provided that these elements are known approximately. Let X be a vector space, and W be a nonempty subset (class) of elements of X. Next, let Y be a normed linear space, $I\colon X \to Y$ be a linear operator, and $\delta \geq 0$. The elements from W are known approximately, i.e., regarding each element $x \in W$ one knows ("observes") an element $y \in Y$ such that $\|Ix - y\|_Y \leq \delta$ (if $\delta = 0$, then one knows Ix). So, the information about elements from W is contained in the triple (Y, I, δ).

To recover from a given information the values of some linear operator $T\colon X \to Z$ on a class W with given accuracy is to put forward a recovery method $\varphi\colon Y \to Z$ with the required approximation accuracy (error); the latter is defined as follows:

$$e(\delta, \varphi) = e(T, W, Y, I, \delta, \varphi) = \sup_{\substack{x \in W,\ y \in Y \\ \|Ix - y\|_Y \leq \delta}} \|Tx - \varphi(y)\|_Z.$$

Assume that pairs (Y, I) are taken from the set $\mathcal{I}$, and moreover, corresponding to each (Y, I) there is some nonnegative number $v(Y, I)$, referred to as the *amount of information used*. The question is: Which minimal amount of information is required to recover the values of a given operator T on the class W with error not exceeding a given number ε? More precisely, if $\Phi(Y)$ is the set of all mappings from Y to Z, then it is required, for given $\varepsilon > 0$ and $\delta \geq 0$, to find, first, the quantity

$$V(\varepsilon, \delta) = \inf\bigl\{v(Y, I) \mid \exists\, \bigl(\varphi, (Y, I)\bigr) \in \Phi(Y) \times \mathcal{I}\colon e(\delta, \varphi) \leq \varepsilon\bigr\},$$

and second, the tuples $\bigl(\hat{\varphi}, (\hat{Y}, \hat{I})\bigr)$ on which the infimum is attained. Such tuples will be called *optimal*.

If the set of those $\bigl(\varphi, (Y, I)\bigr) \in \Phi(Y) \times \mathcal{I}$ for which $e(\delta, \varphi) \leq \varepsilon$ is empty, then we put $V(\varepsilon, \delta) = +\infty$. This means that from the given information one is unable to recover the elements from W with given accuracy.

This statement of the problem has its source in the definition of the ε-entropy of a set—this is the quantity characterizing the nearly best ε-approximation of a set by a finite set of elements (see, e.g., [1]). Moreover, this statement is, in a certain sense, converse to the problem of optimal recovery of functions from a given class from their noisy spectrum. Similar problems have been studied by a number of authors. We note the works [2–7].

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1. Periodic Case

Given a natural number n, let $W_2^n(\mathbb{T})$ be the Sobolev class of 2π-periodic functions $x(\cdot)$ for which the $(n-1)$th derivative is absolutely continuous and $\|x^{(n)}(\cdot)\|_{L_2(\mathbb{T})} \le 1$, where

$$\|x(\cdot)\|_{L_2(\mathbb{T})} = \left(\frac{1}{2\pi}\int_{\mathbb{T}} |x(t)|^2\,dt\right)^{1/2}.$$

The Fourier coefficients of $x(\cdot) \in W_2^n(\mathbb{T})$ are given by

$$c_j = c_j\big(x(\cdot)\big) = \frac{1}{2\pi}\int_{\mathbb{T}} x(t)e^{-ijt}\,dt, \quad j \in \mathbb{Z}.$$

We pose the following problem. Assume that, for any finite tuple of integers, one is able to measure (exactly or approximately) the Fourier coefficients of each function $x(\cdot) \in W_2^n(\mathbb{T})$ with subscripts from a given tuple. Our aim is to use this information to recover in the metric of $L_2(\mathbb{T})$, with fixed accuracy, the original elements $x(\cdot)$ and their kth derivatives ($1 \le k \le n-1$), choosing from the set of tuples the one having the least number of elements.

We refine the statement of the problem to be in line with the general scheme outlined above. Let X be the space of 2π-periodic functions for which the $(n-1)$th derivative is absolutely continuous and the nth derivative belongs to the space $L_2(\mathbb{T})$, $W = W_2^n(\mathbb{T})$, and $Z = L_2(\mathbb{T})$. To each finite tuple α of integers we assign the pair $\big(l_\infty^{N(\alpha)}, I_\alpha\big)$, where $N(\alpha)$ is the number of elements in a tuple, $l_\infty^{N(\alpha)}$ is the space $\mathbb{C}^{N(\alpha)}$ of vectors $y = \big(y_1, \dots, y_{N(\alpha)}\big)$ with norm $\|y\|_{l_\infty^{N(\alpha)}} = \max\limits_{1\le i\le N(\alpha)} |y_i|$, and $I_\alpha\colon X \to l_\infty^{N(\alpha)}$ is the linear operator assigning to a function $x(\cdot)$ its Fourier coefficients with numbers from the tuple α. So, in our setting, $\mathcal{I}$ is the set of pairs $\big(l_\infty^{N(\alpha)}, I_\alpha\big)$ labeled by finite subsets of the set of integers, and for $\delta = 0$ the available information $\big(l_\infty^{N(\alpha)}, I_\alpha, \delta\big)$ about a function $x(\cdot) \in W_2^n(\mathbb{T})$ is contained in the Fourier coefficients of $x(\cdot)$ with numbers from α, while for $\delta > 0$, we have at our disposal $N(\alpha)$ numbers, each of which differs in absolute value from the corresponding Fourier coefficient at most by δ. We set $v\big(l_\infty^{N(\alpha)}, I_\alpha\big) = N(\alpha)$.

Given $0 \le k \le n-1$, let D^k be the kth order differential operator (D^0 is the identity operator) and let $\varphi\colon l_\infty^{N(\alpha)} \to L_2(\mathbb{T})$ be a recovery method. In accordance with the general scheme, its error is as follow:

$$e(\delta,\varphi) = e\big(D^k, W_2^n(\mathbb{T}), l_\infty^{N(\alpha)}, I_\alpha, \delta, \varphi\big) = \sup_{\substack{x(\cdot)\in W_2^n(\mathbb{T}),\ y\in l_\infty^{N(\alpha)} \\ \|I_\alpha x(\cdot)-y\|_{l_\infty^{N(\alpha)}}\le\delta}} \|D^k x(\cdot) - \varphi(y)(\cdot)\|_{L_2(\mathbb{T})}.$$

In this setting, the quantity we are interested in is

$$V(\varepsilon,\delta) = \inf\Big\{N(\alpha) \;\Big|\; \exists \Big(\varphi, \big(l_\infty^{N(\alpha)}, I_\alpha\big)\Big) \in \Phi\big(l_\infty^{N(\alpha)}\big) \times \mathcal{I}\colon e(\delta,\varphi) \le \varepsilon\Big\}.$$

Moreover, we are interested in $\hat\alpha$ and $\hat\varphi$ for which the infimum is attained on tuples $\Big(\hat\varphi, \big(l_\infty^{N(\hat\alpha)}, I_{\hat\alpha}\big)\Big)$. In this case, we say that $\hat\alpha$ is an optimal tuple and $\hat\varphi$ is an optimal method.

Given $\delta > 0$, we set

$$N_\delta = \max\Big\{N \in \mathbb{Z}_+ \;\Big|\; 2\delta^2 \sum_{j=0}^{N} j^{2n} < 1\Big\}$$

and $N_0 = +\infty$. Next, for each $m \in \mathbb{Z}_+$, we define

$$\varepsilon_m = \left(\frac{1}{(m+1)^{2(n-k)}} + 2\delta^2 \sum_{j=0}^{m} j^{2k}\left(1 - \left(\frac{j}{m+1}\right)^{2(n-k)}\right)\right)^{1/2}.$$

It is readily checked that $1 = \varepsilon_0 > \varepsilon_1 > \cdots > \varepsilon_{N_\delta}$ for $\delta > 0$.

Theorem. *Let n be a natural number, k be an integer number, $0 \le k \le n-1$, and let $\varepsilon > 0$. If $\varepsilon \ge 1$ and $k \ge 1$, then $V(\varepsilon,\delta) = 0$. If $\varepsilon \ge 1$ and $k = 0$, then $V(\varepsilon,\delta) = 1$ and $\alpha = \{0\}$ is an optimal set. Next, if $\varepsilon_m \le \varepsilon < \varepsilon_{m-1}$, $m = 1, 2, \dots, N_\delta$, then*

$$V(\varepsilon,\delta) = \begin{cases} 2m+1, & k = 0, \\ 2m, & k \ge 1. \end{cases}$$

The set

$$\alpha = \begin{cases} \{0, \pm 1, \dots, \pm m\}, & k = 0, \\ \{\pm 1, \dots, \pm m\}, & k \ge 1, \end{cases}$$

is optimal, and the method

$$\tilde{\varphi}(y)(t) = \sum_{|j| \le m} (ij)^k \left(1 - \left(\frac{j}{m+1} \right)^{2(n-k)} \right) y_j e^{ijt}$$

is optimal. If $\delta > 0$ and $\varepsilon < \varepsilon_{N_\delta}$, then $V(\varepsilon,\delta) = +\infty$ is optimal.

Proof. Let α be a tuple of integers, $\delta \ge 0$, and $\varphi\colon l_\infty^{N(\alpha)} \to L_2(\mathbb{T})$.

(1) We estimate $e\big(D^k, W_2^n(\mathbb{T}), l_\infty^{N(\alpha)}, I_\alpha, \delta, \varphi\big)$ from below. First, we claim that this quantity is not smaller than the value of the problem

$$\|x^{(k)}(\cdot)\|_{L_2(\mathbb{T})} \to \max, \quad \|I_\alpha x(\cdot)\|_{l_\infty^{N(\alpha)}} \le \delta, \quad \|x^{(n)}(\cdot)\|_{L_2(\mathbb{T})} \le 1, \tag{1}$$

i.e., it is not smaller than the supremum of the functional to be maximized under these constraints.

Indeed, let $x(\cdot)$ be an admissible function for problem (1) (i.e., $x(\cdot)$ satisfies the constraints of the problem). Then, clearly, the function $-x(\cdot)$ is also admissible, and so

$$\begin{aligned} 2\|x^{(k)}(\cdot)\|_{L_2(\mathbb{T})} &\le \|x^{(k)}(\cdot) - \varphi(0)(\cdot)\|_{L_2(\mathbb{T})} + \|-x^{(k)}(\cdot) - \varphi(0)(\cdot)\|_{L_2(\mathbb{T})} \\ &\le 2 \sup_{\substack{\|I_\alpha x(\cdot)\|_{l_\infty^{N(\alpha)}} \le \delta, \\ \|x^{(n)}(\cdot)\|_{L_2(\mathbb{T})} \le 1}} \|x^{(k)}(\cdot) - \varphi(0)(\cdot)\|_{L_2(\mathbb{T})} \le 2 \sup_{\substack{\|I_\alpha x(\cdot) - y\|_{l_\infty^{N(\alpha)}} \le \delta, \\ y \in l_\infty^{N(\alpha)},\ \|x^{(n)}(\cdot)\|_{L_2(\mathbb{T})} \le 1}} \big\|x^{(k)}(\cdot) - \varphi\big(y(\cdot)\big)(\cdot)\big\|_{L_2(\mathbb{T})}. \end{aligned}$$

The required result follows if we take the supremum on the left over all admissible functions for problem (1).

(2) Now let us estimate from below the value of problem (1). By Parseval's identity, the squared value of this problem equals the value of the problem

$$\sum_{j \in \mathbb{Z}} j^{2k} |c_j|^2 \to \max, \quad |c_j| \le \delta, \quad j \in \alpha, \quad \sum_{j \in \mathbb{Z}} j^{2n} |c_j|^2 \le 1. \tag{2}$$

Further, we assume that $k \ge 1$ (the case $k = 0$ is simple and is dealt with similarly). For each $s \in \mathbb{N}$ we denote $\Delta_s = \{\pm 1, \dots, \pm s\}$ and set

$$\hat{s} = \max\{s \in \mathbb{N} \mid \operatorname{card}(\alpha \cap \Delta_s) = 2s\},$$

assuming that $\hat{s} = 0$ if the set in curly bracketed is empty. Next, let

$$p_0 = \max\left\{ p \;\middle|\; 2\delta^2 \sum_{j=0}^{p} j^{2n} < 1,\ 0 \le p \le \hat{s} \right\}.$$

Consider the sequence c_j, $j \in \mathbb{Z}$, defined as follows: if $p_0 < \hat{s}$, then $c_j = \delta$ for $|j| \le p_0$,

$$c_{p_0+1} = c_{-(p_0+1)} = \frac{1}{\sqrt{2}} (p_0+1)^{-n} \sqrt{1 - 2\delta^2 \sum_{j=0}^{p_0} j^{2n}},$$

and $c_j = 0$ for the remaining j. If $p_0 = \hat{s}$, then it is clear that either p_0+1 or $-(p_0+1)$ does not lie in α. Let m be the number (of these two) that does not lie in α. We set

$$c_m = (p_0+1)^{-n}\sqrt{1 - 2\delta^2 \sum_{j=0}^{p_0} j^{2n}},$$

$c_j = \delta$ for $|j| \le p_0$, and $c_j = 0$ for the remaining j. The so-defined sequence is admissible for problem (2). Indeed, if $p_0 < \hat{s}$, then $|c_j| \le \delta$ for $|j| = p_0+1$, for otherwise

$$\frac{1}{\sqrt{2}}(p_0+1)^{-n}\sqrt{1 - 2\delta^2 \sum_{j=0}^{p_0} j^{2n}} > \delta \iff 2\delta^2 \sum_{j=0}^{p_0+1} j^{2n} < 1,$$

contradicting the definition of p_0. Moreover, $\sum\limits_{j\in\mathbb{Z}} j^{2n}|c_j|^2 = 1$, which is clear. This equality holds if and only if $p_0 = \hat{s}$.

Hence, the value of problem (2) is not smaller than the value that the maximized functional takes on this sequence; i.e., it is not smaller than

$$2\delta^2 \sum_{j=0}^{p_0} j^{2k} + (p_0+1)^{-2(n-k)}\left(1 - 2\delta^2 \sum_{j=0}^{p_0} j^{2n}\right) = \varepsilon_{p_0}^2.$$

Therefore, it follows from (1) that

$$e(\delta,\varphi) = e\big(D^k, W_2^n(\mathbb{T}), l_\infty^{N(\alpha)}, I_\alpha, \delta, \varphi\big) \ge \varepsilon_{p_0}. \tag{3}$$

(3) We claim that this estimate is attained for the method

$$\tilde{\varphi}(y)(t) = \sum_{|j|\le p_0} (ij)^k \omega_j y_j e^{ijt},$$

where

$$\omega_j = 1 - \left(\frac{j}{p_0+1}\right)^{2(n-k)}, \quad |j| \le p_0.$$

By definition, the squared error of this method is equal to the value of the extremal problem

$$\sum_{|j|\le p_0} j^{2k}|c_j - \omega_j y_j|^2 + \sum_{|j|>p_0} j^{2k}|c_j|^2 \to \max, \quad |c_j - y_j| \le \delta_j, \quad j \in \alpha, \quad \sum_{j\in\mathbb{Z}} j^{2n}|c_j|^2 \le 1. \tag{4}$$

Let us estimate its value from above. We set

$$\lambda = (p_0+1)^{-2(n-k)}, \quad \lambda_j = j^{2k}\omega_j, \quad |j| \le p_0.$$

If $0 < |j| \le p_0$, then, using the straightforward equality

$$j^{2k}\left(\frac{\omega_j^2}{\lambda_j} + \frac{(1-\omega_j)^2}{j^{2n}\lambda}\right)$$

and applying the Cauchy–Schwarz inequality, we see that

$$\begin{aligned} j^{2k}|c_j - \omega_j y_j|^2 &= j^{2k}|\omega_j(c_j - y_j) + c_j(1-\omega_j)|^2 \\ &\le j^{2k}\left(\frac{\omega_j^2}{\lambda_j} + \frac{(1-\omega_j)^2}{j^{2n}\lambda}\right)\big(|c_j - y_j|^2\lambda_j + |c_j|^2\lambda j^{2n}\big) = |c_j - y_j|^2\lambda_j + |c_j|^2\lambda j^{2n}. \end{aligned}$$

If $|j| > p_0$, then clearly $j^{2k} \le \lambda j^{2n}$. From these relations it follows that the maximized functional in (4) is estimated from above by the quantity

$$\sum_{|j|\le p_0} |c_j - y_j|^2\lambda_j + \lambda \sum_{|j|>p_0} j^{2n}|c_j|^2 \le \delta^2 \sum_{|j|\le p_0} \lambda_j + \lambda = \varepsilon_{p_0}^2.$$

Hence by (3) it follows that

$$e(\delta, \tilde{\varphi}) = \varepsilon_{p_0}. \tag{5}$$

(4) Let $\varepsilon_m \le \varepsilon < \varepsilon_{m-1}$ for some $m \ge 1$. Then, for the tuple $\alpha = \{\pm 1, \dots, \pm m\}$, it follows from (5) that

$$e(0, \tilde{\varphi}) = \varepsilon_m \le \varepsilon.$$

Assume that there exists an α and a method φ such that $N(\alpha) < 2m$ and $e(0, \varphi) \le \varepsilon$. Then $\hat{s} \le m$ and $p_0 \le m-1$. Hence, in view of (3),

$$\varepsilon < \varepsilon_{m-1} \le \varepsilon_{p_0} \le e(0, \varphi) \le \varepsilon.$$

This contradiction shows that $V(\varepsilon, 0) = 2m$. The case $\varepsilon \ge 1$ is dealt with similarly.

Let $\delta > 0$ and $\varepsilon < \varepsilon_{N_\delta}$. Since $p_0 \le N_\delta$ for all α, we have

$$e(\delta, \varphi) \ge \varepsilon_{p_0} \ge \varepsilon_{N_\delta} > \varepsilon$$

for any method. This means that in our setting $V(\varepsilon, \delta) = +\infty$. □

2. Aperiodic Case

We shall be concerned with functions on the real line. Let $W_2^n(\mathbb{R})$ be the Sobolev class of functions $x(\cdot)$ for which the $(n-1)$th derivative is locally absolutely continuous and $\|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})} \le 1$.

Consider the following problem. Assume that, for any measurable subset of $\mathbb{R}$ of finite Lebesgue measure, one may find (exactly or approximately) the Fourier transform of each function $x(\cdot) \in W_2^n(\mathbb{T})$ on this set. Our aim is to use this information to recover in the metric of $L_2(\mathbb{R})$, with fixed accuracy, the original elements $x(\cdot)$ and their kth derivative ($1 \le k \le n-1$) by choosing from these subsets the one that has the smallest measure.

We refine the statement of the problem to be in line with the general scheme. Let X be the space of functions on $\mathbb{R}$ for which the $(n-1)$th derivative is locally absolutely continuous and the nth derivative belongs to the space $L_2(\mathbb{R})$. Also let $W = W_2^n(\mathbb{R})$, $Z = L_2(\mathbb{R})$, and let $F\colon L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be the Fourier transform. To each set $A \subset \mathbb{R}$ of finite Lebesgue measure corresponds the pair $(L_2(A), I_A)$, where $I_A\colon X \to L_2(A)$ is the linear operator assigning to a function $x(\cdot)$ the restriction $Fx(\cdot)|_A$ of $Fx(\cdot)$ to A. So, in this setting $\mathcal{I}$ is the set of pairs $(L_2(A), I_A)$ labeled by subsets A of finite measure, and for $\delta = 0$ the available information $(L_2(A), I_A, \delta)$ about a function $x(\cdot) \in W_2^n(\mathbb{R})$ is that we know the Fourier transform of $x(\cdot)$ on A, while for $\delta > 0$ we know only a function $y(\cdot) \in L_2(A)$ such that $\|Fx(\cdot) - y(\cdot)\|_{L_2(A)} \le \delta$. We set $v(L_2(A), I_A) = \operatorname{mes} A$.

Let, as before, D^k be the kth order differential operator (D^0 be the identity operator), A be a set of finite measure, and $\varphi\colon L_2(A) \to L_2(\mathbb{R})$ be a recovery method. In accordance with the general scheme, its error is

$$e(\delta, \varphi) = e(D^k, W_2^n(\mathbb{R}), L_2(A), I_A, \delta, \varphi) = \sup_{\substack{x(\cdot) \in W_2^n(\mathbb{R}),\ y(\cdot) \in L_2(A) \\ \|Fx(\cdot) - y(\cdot)\|_{L_2(A)} \le \delta}} \|D^k x(\cdot) - \varphi(y)(\cdot)\|_{L_2(\mathbb{R})}$$

and

$$V(\varepsilon, \delta) = \inf\bigl\{\operatorname{mes} A \mid \exists \bigl(\varphi, (L_2(A), I_A)\bigr) \in \Phi\bigl(L_2(A)\bigr) \times \mathcal{I}\colon e(\delta, \varphi) \le \varepsilon\bigr\}.$$

If for $\hat{A}$ and $\hat{\varphi}$ the minimum is attained at the tuple $\bigl(\hat{\varphi}, (L_2(\hat{A}), I_{\hat{A}})\bigr)$, then we say that $\hat{A}$ is an optimal set and $\hat{\varphi}$ is an optimal method.

Theorem. *Let n be a natural number, k be an integer number, $0 \le k \le n-1$, and let $\varepsilon > 0$. Then*

$$V(\varepsilon, 0) = \varepsilon^{-1/(n-k)}.$$

The optimal set is the interval $\hat{A} = [-\sigma_\varepsilon, \sigma_\varepsilon]$, where $2\sigma_\varepsilon = V(\varepsilon, 0)$, and the optimal method is

$$\hat{\varphi}(Fx(\cdot)|_{\hat{A}})(t) = \frac{1}{2\pi} \int\limits_{\hat{A}} (i\xi)^k Fx(\xi) e^{i\xi t}\, d\xi.$$

Let $\delta > 0$. If

$$\varepsilon < \left(\frac{\delta^2}{2\pi}\right)^{(n-k)/(2n)},$$

then $V(\varepsilon, \delta) = +\infty$, and if

$$\varepsilon \geq \left(\frac{\delta^2}{2\pi}\right)^{(n-k)/(2n)},$$

then $V(\varepsilon, \delta) = 2\sigma_\varepsilon$, where σ_ε is the unique root of the equation

$$\frac{n-k}{n}\left(\frac{k}{n}\right)^{k/(n-k)}\frac{\delta^2}{2\pi}\sigma^{2k} + \frac{1}{\sigma^{2(n-k)}} = \varepsilon^2$$

(to include the case $k = 0$, we assume here that $0^0 = 1$ and that $\infty^0 = 1$ in the optimal method considered below). The optimal set is the interval $\hat{A} = [-\sigma_\varepsilon, \sigma_\varepsilon]$ and the optimal method is

$$\hat{\varphi}\big(y(\cdot)\big)(t) = \frac{1}{2\pi}\int\limits_{\hat{A}}(i\xi)^k\left(1 + \frac{n}{n-k}\left(\frac{n}{k}\right)^{k/(n-k)}\left(\frac{\xi}{\sigma_\varepsilon}\right)^{2n}\right)^{-1} y(\xi)e^{i\xi t}\,d\xi.$$

We note that from the noise-free information ($\delta = 0$) about the Fourier transform one may recover elements from the class $W_2^n(\mathbb{R})$ with any accuracy and that the best recovery method of the kth derivative ($0 \leq k \leq n-1$) is "natural": one needs to take the kth derivative (if $k \geq 1$) of the inverse Fourier transform on the interval $[-\sigma_\varepsilon, \sigma_\varepsilon]$.

If $\delta > 0$, then not for all ε can one recover functions and their kth derivatives with given accuracy. Moreover, for ε for which it is possible, an optimal method utilizes information only from the interval $[-\sigma_\varepsilon, \sigma_\varepsilon]$, but first it "smooths" this information.

Proof. Let A be a subset of finite measure of $\mathbb{R}$, $\delta \geq 0$, and $\varphi\colon L_2(A) \to L_2(\mathbb{R})$.

(1) First, we show that the quantity $e(D^k, W_2^n(\mathbb{R}), L_2(A), I_A, \delta, \varphi)$ is not smaller than the value of the problem

$$\big\|x^{(k)}(\cdot)\big\|_{L_2(\mathbb{R})} \to \max, \quad \|Fx(\cdot)\|_{L_2(A)} \leq \delta, \quad \big\|x^{(n)}(\cdot)\big\|_{L_2(\mathbb{R})} \leq 1 \tag{6}$$

(where $\|Fx(\cdot)\|_{L_2(A)} \leq \delta$ with $\delta = 0$ means that $Fx(\cdot)|_A = 0$). Indeed, let $x(\cdot)$ be an admissible function in (6) (i.e., $x(\cdot)$ satisfies the constraints of the problem). Then, clearly, the function $-x(\cdot)$ is also admissible, and so

$$\begin{aligned}
2\big\|x^{(k)}(\cdot)\big\|_{L_2(\mathbb{R})} &\leq \big\|x^{(k)}(\cdot) - \varphi(0)(\cdot)\big\|_{L_2(\mathbb{R})} + \big\|-x^{(k)}(\cdot) - \varphi(0)(\cdot)\big\|_{L_2(\mathbb{R})} \\
&\leq 2\sup_{\|Fx(\cdot)\|_{L_2(A)}\leq\delta,\ \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}\leq 1}\big\|x^{(k)}(\cdot) - \varphi(0)(\cdot)\big\|_{L_2(\mathbb{R})} \\
&\leq 2\sup_{\substack{\|Fx(\cdot)-y(\cdot)\|_{L_2(A)}\leq\delta,\\ y(\cdot)\in L_2(A),\ \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}\leq 1}}\big\|x^{(k)}(\cdot) - \varphi\big(y(\cdot)\big)(\cdot)\big\|_{L_2(\mathbb{R})}.
\end{aligned}$$

The required result follows if we take the supremum on the left over all admissible functions for problem (6).

(2) Setting

$$\hat{a} = \sup\{a \geq 0 \mid \operatorname{mes}\{A \cap [-a, a]\} = 2a\},$$

we claim that if $\hat{a} = 0$, then the value of problem (6) is infinite. Indeed, it follows from Plancherel's theorem that in the Fourier images the squared value of this problem agrees with the value of the problem

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}}\xi^{2k}|Fx(\xi)|^2\,d\xi \to \max, \quad \int\limits_A |Fx(\xi)|^2(\xi)\,d\xi \leq \delta^2, \quad \frac{1}{2\pi}\int\limits_{\mathbb{R}}\xi^{2n}|Fx(\xi)|^2\,d\xi \leq 1. \tag{7}$$

We have $\hat{a}=0$, and hence $\operatorname{mes}\{M_\sigma\cap[-\varepsilon,\varepsilon]\}<2\varepsilon$ for any $\varepsilon>0$. Consequently, the measure of the set $\Omega_\varepsilon=\{(\mathbb{R}\setminus M_\sigma)\cap[-\varepsilon,\varepsilon]\}$ is positive. Let a function $x_\varepsilon(\cdot)$ be such that

$$Fx_\varepsilon(\xi)=\begin{cases}\sqrt{2\pi}\Big(\int\limits_{\Omega_\varepsilon}\tau^{2n}\,d\tau\Big)^{-1/2}, & \xi\in\Omega_\varepsilon,\\ 0, & \xi\notin\Omega_\varepsilon.\end{cases}$$

This function is admissible for problem (7), and so

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}}\xi^{2k}|Fx_\varepsilon(\xi)|^2\,d\xi=\frac{\int\limits_{\Omega_\varepsilon}\xi^{2k}\,d\xi}{\int\limits_{\Omega_\varepsilon}\tau^{2n}\,d\tau}=\frac{\int\limits_{\Omega_\varepsilon}\xi^{2n}\xi^{-2(n-k)}\,d\xi}{\int\limits_{\Omega_\varepsilon}\tau^{2n}\,d\tau}\geq\varepsilon^{-2(n-k)}.$$

Hence, since ε is arbitrary, the value of the functional to be maximized in (7) may be made arbitrarily large.

(3) Let $\delta=0$. We claim that

$$e(0,\varphi)=e(D^k,W_2^n(\mathbb{R}),L_2(A),I_A,0,\varphi)\geq\sigma^{-(n-k)},\tag{8}$$

where $2\sigma=\operatorname{mes}A$.

Since $\delta=0$, the first constraint in problem (7) means that $Fx(\xi)=0$ for a.a. $\xi\in A$, and now problem (7) itself can be rewritten as

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}\setminus A}\xi^{2k}|Fx(\xi)|^2\,d\xi\to\max,\quad\frac{1}{2\pi}\int\limits_{\mathbb{R}\setminus A}\xi^{2n}|Fx(\xi)|^2\,d\xi\leq1.\tag{9}$$

Let us estimate its value from below. If $\hat{a}=0$, then by (2) this value is $+\infty$. Hence, it follows from (1) that $e(0,\varphi)=+\infty$ and (8) is clearly satisfied. Let $\hat{a}>0$. For any natural m, the set $[-\hat{a}-1/m,\hat{a}+1/m]\setminus[-\hat{a},\hat{a}]$ contains a subset E_m of positive measure not lying in A (for otherwise this would contradict the definition of $\hat{a}$). Consider the sequence of functions $x_m(\cdot)\in L_2(\mathbb{R})$ with Fourier transforms

$$Fx_m(\xi)=\begin{cases}\sqrt{\frac{2\pi}{\operatorname{mes}E_m}}\left(\hat{a}+\frac{1}{m}\right)^{-n}, & \xi\in E_m,\\ 0, & \xi\notin E_m.\end{cases}$$

That these functions are admissible for problem (9) is clear. Next,

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}\setminus A}\xi^{2k}|Fx_m(\xi)|^2\,d\xi\geq\frac{1}{\operatorname{mes}E_m}\left(\hat{a}+\frac{1}{m}\right)^{-2n}\hat{a}^{2k}\operatorname{mes}E_m\to\hat{a}^{-2(n-k)}\geq\sigma^{-2(n-k)},$$

since clearly $\hat{a}\leq\sigma$, and hence the value of problem (9) is not smaller than $\sigma^{-2(n-k)}$. Hence, by (1) and since the value of problem (9) is the squared value of problem (6), it follows that $e(0,\varphi)\geq\sigma^{-(n-k)}$, which is (8).

(4) We claim that inequality (8) becomes an equality with $A=[-\sigma,\sigma]$ and with the method from the statement of the theorem, but with σ taken for σ_ε. We denote this method by $\tilde{\varphi}$. Indeed, for any $x(\cdot)\in W_2^n(\mathbb{R})$, it follows from Plancherel's theorem that

$$\begin{aligned}\big\|x^{(k)}(\cdot)-\tilde{\varphi}\big(Fx(\cdot)|_{[-\sigma,\sigma]}\big)(\cdot)\big\|^2_{L_2(\mathbb{R})}&=\frac{1}{2\pi}\int\limits_{\mathbb{R}\setminus[-\sigma,\sigma]}\xi^{2k}|Fx(\xi)|^2\,d\xi\\&=\frac{1}{2\pi}\int\limits_{\mathbb{R}\setminus[-\sigma,\sigma]}\xi^{-2(n-k)}\xi^{2n}|Fx(\xi)|^2\,d\xi\leq\sigma^{-2(n-k)}\frac{1}{2\pi}\int\limits_{\mathbb{R}}\xi^{2n}|Fx(\xi)|^2\,d\xi\leq\sigma^{-2(n-k)}.\end{aligned}$$

This means that $e\big(D^k,W_2^n(\mathbb{R}),L_2([-\sigma,\sigma]),I_{[-\sigma,\sigma]},0,\tilde{\varphi}\big)\leq\sigma^{-(n-k)}$, and so inequality (8) is sharp.

(5) Now we are able to show that $V(\varepsilon, 0) = \varepsilon^{-1/(n-k)}$ for any $\varepsilon > 0$. Indeed, for a given $\varepsilon > 0$, we set $\sigma_\varepsilon = \varepsilon^{-1/(n-k)}$. By (4) the estimate

$$e(D^k, W_2^n(\mathbb{R}), L_2(A_\varepsilon), I_{A_\varepsilon}, 0, \tilde{\varphi}) \le \varepsilon$$

holds for the set $A_\varepsilon = [-\sigma_\varepsilon/2, \sigma_\varepsilon/2]$ and the corresponding method $\tilde{\varphi}$. Hence, $V(\varepsilon, 0) \le \varepsilon^{-1/(n-k)}$. Assume that this inequality is sharp; i.e., there exist a set A, $\operatorname{mes} A = 2\sigma$, and a method φ such that $\sigma < \sigma_\varepsilon$ and

$$e(D^k, W_2^n(\mathbb{R}), L_2(A), I_A, 0, \varphi) \le \varepsilon.$$

Hence, by (8)

$$\varepsilon = \sigma_\varepsilon^{-(n-k)} < \sigma^{-(n-k)} \le e(D^k, W_2^n(\mathbb{R}), L_2(A), I_A, 0, \varphi) \le \varepsilon.$$

This contradiction shows that $V(\varepsilon, 0) = \varepsilon^{-1/(n-k)}$.

That the set and the method from the theorem are optimal follows from the above analysis.

(6) Let $\delta > 0$. Henceforth we assume that $k \ge 1$ (the case $k = 0$ is dealt with similarly, but is technically simpler). Setting

$$\hat{\sigma} = \left(\frac{n}{k}\right)^{1/(2(n-k))} \left(\frac{\delta^2}{2\pi}\right)^{-1/(2n)},$$

we claim that the estimate

$$e(D^k, W_2^n(\mathbb{R}), L_2(A), I_A, \delta, \varphi) \ge \begin{cases} \sqrt{\frac{n-k}{n}\left(\frac{k}{n}\right)^{k/(n-k)} \frac{\delta^2}{2\pi}\sigma^{2k} + \frac{1}{\sigma^{2(n-k)}}}, & \sigma \le \hat{\sigma}, \\ \left(\frac{\delta^2}{2\pi}\right)^{(n-k)/(2n)}, & \sigma \ge \hat{\sigma}, \end{cases} \tag{10}$$

holds for any method φ and set A, $\operatorname{mes} A = 2\sigma$.

Note that the function (of σ) on the right is defined on the half-open interval $(0, \hat{\sigma}]$ and is monotonically decreasing on it. Moreover, at $\hat{\sigma}$ its minimal value is $(\delta^2/2\pi)^{(n-k)/(2n)}$. For any A and φ, the error may not be smaller than this quantity.

Let us estimate from below the value of problem (7). If $\hat{a} = 0$, then by (2) this value is $+\infty$, and now it follows from (1) that

$$e(D^k, W_2^n(\mathbb{R}), L_2(A), I_A, \delta, \varphi) = +\infty$$

and (10) is trivially satisfied.

Let $\hat{a} > 0$. Assume that $\sigma < \hat{\sigma}$. Given a natural number m, we set

$$C_m = 1 - \frac{1}{2\pi}\left(\left(\frac{k}{n}\right)^{1/(2(n-k))} \hat{a} + \frac{1}{2m}\right)^{2n}.$$

We have $\hat{a} \le \sigma < \hat{\sigma}$, and hence $C_m > 0$ for sufficiently large m. Next, $\gamma = (k/n)^{1/(2(n-k))} < 1$, and so the interval $\Delta_m = [\gamma\hat{a} - 1/2m, \gamma\hat{a} + 1/2m]$ lies in the interval $[-\hat{a}, \hat{a}]$ for sufficiently large m. Finally, let E_m be the set defined in (3). For this m, we consider the family of functions $x_m(\cdot)$, whose Fourier transform is as follows:

$$Fx_m(\xi) = \begin{cases} \sqrt{\frac{2\pi}{\operatorname{mes} E_m}}\left(\hat{a} + \frac{1}{m}\right)^{-n}\sqrt{C_m}, & \xi \in E_m, \\ \sqrt{m}\,\delta, & \xi \in \Delta_m, \\ 0 & \xi \notin E_m \cup \Delta_m. \end{cases}$$

It is easily checked that $x_m(\cdot)$ are admissible functions for problem (7). Next,

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}} \xi^{2k}|Fx_m(\xi)|^2\,d\xi = \frac{1}{2\pi}\int\limits_{E_m} \xi^{2k}|Fx_m(\xi)|^2\,d\xi + \frac{1}{2\pi}\int\limits_{\Delta_m} \xi^{2k}|Fx_m(\xi)|^2\,d\xi$$
$$\geq \left(\hat{a}+\frac{1}{m}\right)^{-2n} C_m\,\hat{a}^{2k} + \frac{\delta^2}{2\pi}\left(\frac{k}{n}\right)^{k/(n-k)}\hat{a}^{2k} \to \frac{n-k}{n}\left(\frac{k}{n}\right)^{k/(n-k)}\frac{\delta^2}{2\pi}\hat{a}^{2k} + \frac{1}{\hat{a}^{2(n-k)}}. \quad (11)$$

The expression on the right is monotonically decreasing (*qua* a function of $\hat{a}$) on the half-open interval $(0,\hat{\sigma}]$, and since $\hat{a}\leq\sigma<\hat{\sigma}$, the value of problem (7) is not smaller than the expression under the root sign on the right of (10). By (1) this proves estimate (10) for $\sigma<\hat{\sigma}$.

Let $\sigma\geq\hat{\sigma}$. If $\hat{a}<\hat{\sigma}$, then we arrive at formula (11) by the same argument as above. The minimal value of the expression on the right of (11) is $\big(\delta^2/(2\pi)\big)^{(n-k)/n}$. This proves (10) in the case in question.

Let $\hat{a}\geq\hat{\sigma}$. Since $\gamma_1=\big(\delta^2/(2\pi)\big)^{-1/(2n)}<\hat{\sigma}$, it follows that for sufficiently large m the interval $\Delta_m=[\gamma_1-1/m,\gamma_1]$ lies in the half-open interval $(0,\hat{\sigma}]$. For such m, we consider the sequence of functions $x_m(\cdot)$ with Fourier transforms

$$Fx_m(\xi)=\begin{cases}\sqrt{m}\,\delta, & \xi\in\Delta_m,\\ 0, & \xi\notin\Delta_m.\end{cases}$$

As before, it is easily checked that these are admissible functions for problem (7), and moreover,

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}} \xi^{2k}|Fx_m(\xi)|^2\,d\xi \geq \frac{\delta^2}{2\pi}\left(\left(\frac{\delta^2}{2\pi}\right)^{-1/(2n)}-\frac{1}{m}\right)^{2k} \to \left(\frac{\delta^2}{2\pi}\right)^{(n-k)/n},$$

proving estimate (10).

Now we claim that this estimate is attained on the set $A=[-\sigma,\sigma]$ and on the method from the statement of the theorem in which σ is taken for σ_ε. We denote this method by $\tilde{\varphi}$. Indeed, let $x(\cdot)\in W_2^n(\mathbb{R})$, $y(\cdot)\in L_2([-\sigma,\sigma])$, and $\|Fx(\cdot)-y(\cdot)\|_{L_2([-\sigma,\sigma])}\leq\delta$. Taking $\sigma_0=\min(\sigma,\hat{\sigma})$, we set

$$\lambda_1=\frac{n-k}{n}\left(\frac{k}{n}\right)^{k/(n-k)}\sigma_0^{2k}, \quad \lambda_2=\sigma_0^{-2(n-k)}.$$

Next, it is easily seen that

$$\frac{\lambda_1}{\lambda_1+\lambda_2\xi^{2n}}=\left(1+\frac{n}{n-k}\left(\frac{n}{k}\right)^{k/(n-k)}\left(\frac{\xi}{\sigma_0}\right)^{2n}\right)^{-1} \quad (12)$$

and $-\xi^{2k}+\lambda_1+\lambda_2\xi^{2n}\geq 0$ for any $\xi\in\mathbb{R}$.

By Plancherel's theorem it follows by (12) that

$$\big\|x^{(k)}(\cdot)-\tilde{\varphi}\big(y(\cdot)\big)(\cdot)\big\|^2_{L_2(\mathbb{R})}=\frac{1}{2\pi}\int\limits_{\mathbb{R}}\left|(i\xi)^kFx(\xi)-(i\xi)^k\frac{\lambda_1}{\lambda_1+\lambda_2\xi^{2n}}\chi_{\sigma_0}(\xi)y(\xi)\right|^2d\xi, \quad (13)$$

where $\chi_{\sigma_0}(\cdot)$ is the characteristic function of the interval $[-\sigma_0,\sigma_0]$.

Let $\xi\in[-\sigma_0,\sigma_0]$. Then, for such ξ, using simple transformations, applying the Cauchy–Schwarz inequality, and taking into account that the polynomial $\xi\mapsto-\xi^{2k}+\lambda_1+\lambda_2\xi^{2n}$ is nonnegative on $\mathbb{R}$, we have, for the integrand on the right of (13),

$$\begin{aligned}
&\left|(i\xi)^k Fx(\xi) - (i\xi)^k \frac{\lambda_1}{\lambda_1+\lambda_2\xi^{2n}} y(\xi)\right|^2 \\
&\quad = \xi^{2k}\left|\frac{\sqrt{\lambda_1}}{\lambda_1+\lambda_2\xi^{2n}}\sqrt{\lambda_1}\big(Fx(\xi)-y(\xi)\big) + \frac{\sqrt{\lambda_2}\xi^n}{\lambda_1+\lambda_2\xi^{2n}}\sqrt{\lambda_2}\xi^n Fx(\xi)\right|^2 \\
&\quad \le \xi^{2k}\left(\frac{\lambda_1}{(\lambda_1+\lambda_2\xi^{2n})^2} + \frac{\lambda_2\xi^{2n}}{(\lambda_1+\lambda_2\xi^{2n})^2}\right)\big(\lambda_1|Fx(\xi)-y(\xi)|^2 + \lambda_2\xi^{2n}|Fx(\xi)|^2\big) \\
&\quad = \frac{\xi^{2k}}{\lambda_1+\lambda_2\xi^{2n}}\big(\lambda_1|Fx(\xi)-y(\xi)|^2 + \lambda_2\xi^{2n}|Fx(\xi)|^2\big) \le \lambda_1|Fx(\xi)-y(\xi)|^2 + \lambda_2\xi^{2n}|Fx(\xi)|^2.
\end{aligned}$$

Integrating this inequality over the interval $[-\sigma_0, \sigma_0]$, we see that the integral on the right of (13) is estimated on this interval by the quantity

$$\lambda_1 \frac{1}{2\pi}\int\limits_{|\xi|\le\sigma_0} |Fx(\xi)-y(\xi)|^2\,d\xi + \lambda_2\frac{1}{2\pi}\int\limits_{|\xi|\le\sigma_0} \xi^{2n}|Fx(\xi)|^2\,d\xi.$$

If $|\xi| > \sigma_0$, then for such ξ the estimate for the expression on the right of (13) is given by

$$\begin{aligned}
\frac{1}{2\pi}\int\limits_{|\xi|>\sigma_0} \xi^{2k}|Fx(\xi)|^2\,d\xi &= \frac{1}{2\pi}\int\limits_{|\xi|>\sigma_0} \xi^{2n}\xi^{-2(n-k)}|Fx(\xi)|^2\,d\xi \\
&\le \sigma_0^{-2(n-k)}\frac{1}{2\pi}\int\limits_{|\xi|>\sigma_0} \xi^{2n}|Fx(\xi)|^2\,d\xi = \lambda_2\frac{1}{2\pi}\int\limits_{|\xi|>\sigma_0} \xi^{2n}|Fx(\xi)|^2\,d\xi.
\end{aligned}$$

Adding these inequalities, we have

$$\big\|x^{(k)}(\cdot) - \tilde{\varphi}\big(y(\cdot)\big)(\cdot)\big\|^2_{L_2(\mathbb{R})} \le \frac{\lambda_1\delta^2}{2\pi} + \lambda_2 = \frac{n-k}{n}\left(\frac{k}{n}\right)^{k/(n-k)}\frac{\delta^2}{2\pi}\sigma_0^{2k} + \frac{1}{\sigma_0^{2(n-k)}}$$

in view of the choice of $x(\cdot)$ and $y(\cdot)$ and the expression for λ_1 and λ_2.

If $\sigma_0 = \hat{\sigma}$, then the expression on the right assumes the required value. This proves that inequality (10) is sharp.

(7) Now we prove the second part of the theorem (with $\delta > 0$). Let $\varepsilon < \big(\delta^2/(2\pi)\big)^{(n-k)/2n}$. We claim that there do not exist a set A of finite measure and a method φ such that $e(D^k, W_2^n(\mathbb{R}), L_2(A), I_A, \delta, \varphi) \le \varepsilon$. Indeed, if such A and φ exist, then by (10) we would get

$$\varepsilon < \left(\frac{\delta^2}{2\pi}\right)^{(n-k)/2n} \le e(D^k, W_2^n(\mathbb{R}), L_2(A), I_A, \delta, \varphi) \le \varepsilon.$$

This contradiction proves that $V(\varepsilon, \delta) = +\infty$.

Let $\varepsilon \ge \big(\delta^2/(2\pi)\big)^{(n-k)/2n}$ and let σ_ε be from the statement of the theorem. By (6), the estimate $e(D^k, W_2^n(\mathbb{R}), L_2(A_\varepsilon), I_{A_\varepsilon}, \delta, \hat{\varphi}) \le \varepsilon$ is satisfied for the set $A_\varepsilon = [-\sigma_\varepsilon, \sigma_\varepsilon]$ and the method $\hat{\varphi}$, and hence $V(\varepsilon, \delta) \le 2\sigma_\varepsilon$. Assume that the inequality is sharp; i.e., there exist a set A, $\operatorname{mes} A = 2\sigma$, and φ such that $\sigma < \sigma_\varepsilon$ and $e(D^k, W_2^n(\mathbb{R}), L_2(A), I_A, \delta, \varphi) \le \varepsilon$. Now, if we denote by $f(\cdot)$ the function of σ on the right of (10) ($f(\cdot)$ is strictly monotone decreasing on $(0, \hat{\sigma}]$), then by the definition of σ_ε and estimate (10),

$$\varepsilon = f(\sigma_\varepsilon) < f(\sigma) \le e(D^k, W_2^n(\mathbb{R}), L_2(A), I_A, \delta, \varphi) \le \varepsilon.$$

This contradiction proves the required assertion.

The optimality of the set and the method from the statement of the theorem follow from the above arguments. $\square$

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