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ON OPTIMAL RECOVERY OF SOLUTIONS TO DIFFERENCE EQUATIONS FROM INACCURATE DATA

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We prove a theorem on the optimal recovery of powers of a normal operator. To illustrate the result, we prove assertion concerning the optimal recovery of the temperature of a body in the difference model of the heat equation and the optimal recovery of a solution in the difference model of a system of ordinary differential equations. Bibliography: 6 *titles.*

1 The Main Result

Let $T\colon \mathbb{C}^d \to \mathbb{C}^d$ be a linear operator. We assume that $x \in \mathbb{C}^d$ and $T^n x$ (where T^n denotes the nth power of an operator T) are known approximately, i.e., we know vectors $y_0, y_n \in \mathbb{C}^d$ such that $\|x - y_0\| \leqslant \delta_0$ and $\|T^n x - y_n\| \leqslant \delta_n$, where $\|\cdot\|$ is the Euclidean norm and $\delta_0, \delta_n > 0$. Based on this information, we wish to recover (in the best way if possible) the values $T^k x$, $0 < k < n$. By a *recover method* we mean any mapping $\varphi\colon \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$. The *error* of a method φ is defined by the formula

$$e(T^k, T^n, \delta_0, \delta_n, \varphi) = \sup \|T^k x - \varphi(y_0, y_n)\|,$$

where the supremum is taken over all $x, y_0, y_n \in \mathbb{C}^d$ such that $\|x - y_0\| \leqslant \delta_0$ and $\|T^n x - y_n\| \leqslant \delta_n$. We are interested in the *optimal recovery error* defined by

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$$E(T^k, T^n, \delta_0, \delta_n) = \inf_{\varphi\colon \mathbb{C}^d\times\mathbb{C}^d\to\mathbb{C}^d} e(T^k, T^n, \delta_0, \delta_n, \varphi). \tag{1}$$

A method on which the infimum is attained is called an *optimal recovery method.*

Let $T\colon \mathbb{C}^d \to \mathbb{C}^d$ be a nonzero normal operator, i.e., $TT^* = T^*T$. Then there exists an orthonormal basis of eigenvectors of the operator T. Let $\lambda_1, \dots, \lambda_d$ be the corresponding eigenvalues. Suppose that their moduli are arranged in ascending order:

$$|\lambda_1| = \ldots = |\lambda_{s_1}| < \ldots < |\lambda_{s_{r-1}+1}| = \ldots = |\lambda_{s_r}|.$$

The common value of the moduli of eigenvalues in the jth group is denoted by μ_j, $1 \leqslant j \leqslant r$. We divide the half-line $(0, \infty)$ into intervals

$$\Delta_0 = (0, \mu_1^n],\quad \Delta_1 = (\mu_1^n, \mu_2^n],\quad \ldots,\quad \Delta_{r-1} = (\mu_{r-1}^n, \mu_r^n],\quad \Delta_r = (\mu_r^n, \infty),$$

where the half-interval Δ_0 is absent if $\mu_1 = 0$. With each Δ_j we associate a pair of numbers u_j, v_j, $0 \leqslant j \leqslant r$ ($1 \leqslant j \leqslant r$ if $\mu_1 = 0$) by the rule

$$u_0 = 0, \quad u_j = \frac{\mu_j^{2k}\mu_{j+1}^{2n} - \mu_j^{2n}\mu_{j+1}^{2k}}{\mu_{j+1}^{2n} - \mu_j^{2n}},\ 1 \leqslant j \leqslant r-1, \quad u_r = \mu_r^{2k}$$

and

$$v_0 = \mu_1^{-2(n-k)}, \quad v_j = \frac{\mu_{j+1}^{2k} - \mu_j^{2k}}{\mu_{j+1}^{2n} - \mu_j^{2n}},\ 1 \leqslant j \leqslant r-1, \quad v_r = 0.$$

Theorem 1. *Let $T\colon \mathbb{C}^d \to \mathbb{C}^d$ be a nonzero normal operator, and let $\lambda_1, \dots, \lambda_d$ be eigenvalues of T in the orthonormal basis of the eigenvectors of T. If $\delta_n/\delta_0 \in \Delta_j$, $0 \leqslant j \leqslant r$, then*

$$E(T^k, T^n, \delta_0, \delta_n) = \sqrt{\delta_0^2 u_j + \delta_n^2 v_j},$$

and for any $\theta \in \mathbb{C}$ such that $|\theta| \leqslant 1$ and any linear operator $B\colon \mathbb{C}^d \to \mathbb{C}^d$ with the same basis of its eigenvectors corresponding to the eigenvalues

$$\beta_i = \frac{v_j \overline{\lambda}_i^n \lambda_i^k}{u_j + |\lambda_i|^{2n} v_j} + \theta \frac{\sqrt{u_j v_j}}{u_j + |\lambda_i|^{2n} v_j} \sqrt{-|\lambda_i|^{2k} + u_j + |\lambda_i|^{2n} v_j}, \quad 1 \leqslant i \leqslant d,$$

the linear operator $\widehat{\varphi}\colon \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$ acting by the rule

$$\widehat{\varphi}(\xi, \eta) = (T^k - BT^n)\xi + B\eta$$

is an optimal recovery method.

We begin with particular cases of this theorem.

Suppose that $\mu_1 > 0$ and $\delta_n/\delta_0 \in \Delta_0$. Then $u_0 = 0$ and thereby $\beta_i = \lambda_i^{-(n-k)}$, $1 \leqslant i \leqslant d$, i.e., $B = T^{-(n-k)}$. Consequently, the action of the optimal method $\widehat{\varphi}$ is as follows:

$$\widehat{\varphi}(\xi, \eta) = T^k(T^{-n}\eta).$$

The method uses only the measurement of η, namely, x is found from the equality $T^n x = \eta$ and then the kth power of T is taken.

Let $\delta_n/\delta_0 \in \Delta_r$. Then $v_0 = 0$ and, consequently, $\beta_i = 0$, $1 \leqslant i \leqslant d$, i.e., B is the zero operator. In this case, the method uses only the measurement of ξ:

$$\widehat{\varphi}(\xi, \eta) = T^k\xi.$$

Proof of Theorem 1. We estimate from below the optimal recovery error $E(T^k, T^n, \delta_0, \delta_n)$. We show that it is not less than the value of the following problem (i.e., the upper bound of the maximized functional)

$$\|T^k x\| \to \max, \quad \|x\| \leqslant \delta_0, \quad \|T^n x\| \leqslant \delta_n, \quad x \in \mathbb{C}^d. \tag{2}$$

Indeed, let x_0 be an admissible vector in (2). Then it is obvious that the vector $-x_0$ is also admissible and for any $\varphi\colon \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$

$$2\|T^k x_0\| = \|T^k x_0 - \varphi(0,0) - (T^k(-x_0) - \varphi(0,0))\| \leqslant \|T^k x_0 - \varphi(0,0)\| + \|T^k(-x_0) - \varphi(0,0)\|$$

$$\leqslant 2 \sup_{\substack{x \in \mathbb{C}^d \\ \|x\| \leqslant \delta_0,\ \|T^n x\| \leqslant \delta_n}} \|T^k x - \varphi(0,0)\| \leqslant 2 \sup_{\substack{x, y_0, y_n \in \mathbb{C}^d \\ \|x - y_0\| \leqslant \delta_0,\ \|T^n x - y_n\| \leqslant \delta_n}} \|T^k x - \varphi(y_0, y_n)\|.$$

Passing in (2) to the upper bound over all admissible functions on the left-hand side and to the lower bound over all methods φ on the right-hand side, we obtain the required assertion.

Let $e_1, \ldots, e_d$ be the orthonormal basis for $\mathbb{C}^d$ consisting of eignevectors of the operator T, and let $x = x_1 e_1 + \ldots + x_d e_d$. Then the squared value of (2) is equal to the value of the following problem:

$$\sum_{j=1}^d |\lambda_j|^{2k} |x_j|^2 \to \max, \quad \sum_{j=1}^d |x_j|^2 \leqslant \delta_0^2, \quad \sum_{j=1}^d |\lambda_j|^{2n} |x_j|^2 \leqslant \delta_n^2. \tag{3}$$

Let us estimate from below its value. We consider several cases separately.

Case 1: $\mu_1 > 0$ and $\delta_n/\delta_0 \in \Delta_0$. We define $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_d)$ by the rule $\widehat{x}_1 = \delta_n/\mu_1^n$ and $\widehat{x}_j = 0$, $2 \leqslant j \leqslant d$. Since $\delta_n/\delta_0 \leqslant \mu_1^n$, we have $\widehat{x}_1^2 \leqslant \delta_0^2$ and, consequently, $\widehat{x}$ is admissible in (3). Hence the value of the problem (3) is not less than

$$\sum_{j=1}^d |\lambda_j|^{2k} |\widehat{x}_j|^2 = \mu_1^{2k} \frac{\delta_n^2}{\mu_1^{2n}} = \delta_n^2 \mu_1^{-(n-k)} = \delta_0^2 u_0 + \delta_n^2 v_0.$$

Case 2: $\mu_1 > 0$ and $\delta_n/\delta_0 \in \Delta_j$, $1 \leqslant j \leqslant r-1$. Let k_1 and k_2 be such that $|\lambda_{k_1}| = \mu_j$, and let $|\lambda_{k_2}| = \mu_{j+1}$. We choose $\widehat{x}_{k_1}$ and $\widehat{x}_{k_2}$ from the equalities

$$\widehat{x}_{k_1}^2 + \widehat{x}_{k_2}^2 = \delta_0^2, \quad \mu_j^{2n} \widehat{x}_{k_1}^2 + \mu_{j+1}^{2n} \widehat{x}_{k_2}^2 = \delta_n^2,$$

i.e.,

$$\widehat{x}_{k_1} = \sqrt{\frac{\delta_0^2 \mu_{j+1}^{2n} - \delta_n^2}{\mu_{j+1}^{2n} - \mu_j^{2n}}}, \quad \widehat{x}_{k_2} = \sqrt{\frac{\delta_n^2 - \delta_0^2 \mu_j^{2n}}{\mu_{j+1}^{2n} - \mu_j^{2n}}}.$$

We set $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_d)$, where $\widehat{x}_{k_1}$ and $\widehat{x}_{k_2}$ are as above, whereas the remaining components vanish. Then it is easy to see that $\widehat{x}$ is admissible in (3) and, consequently, the value of the problem (3) is not less than

$$\sum_{j=1}^d |\lambda_j|^{2k} |\widehat{x}_j|^2 = \mu_j^{2k} \widehat{x}_{k_1}^2 + \mu_{j+1}^{2k} \widehat{x}_{k_2}^2 = \mu_j^{2k} \frac{\delta_0^2 \mu_{j+1}^{2n} - \delta_n^2}{\mu_{j+1}^{2n} - \mu_j^{2n}} + \mu_{j+1}^{2k} \frac{\delta_n^2 - \delta_0^2 \mu_j^{2n}}{\mu_{j+1}^{2n} - \mu_j^{2n}} = \delta_0^2 u_j + \delta_n^2 v_j.$$

Case 3: $\mu_1 > 0$ and $\delta_n/\delta_0 \in \Delta_r$. We set $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_d)$, where $\widehat{x}_d = \delta_0$ and $\widehat{x}_j = 0$, $1 \leqslant j \leqslant d-1$. Since $\mu_r^{2n}|\widehat{x}_d|^2 = \mu_r^{2n}\delta_0^2 < \delta_n^2$, the vector $\widehat{x}$ is admissible in the problem (3) and, consequently, the value of the problem (3) is not less than

$$\sum_{j=1}^{d} |\lambda_j|^{2k}|\widehat{x}_j|^2 = \mu_r^{2k}\delta_0^2 = \delta_0^2 u_r + \delta_n^2 v_r.$$

Case 4. If $\mu_1 = 0$, then, repeating the above reasoning, we conclude that for $\delta_n/\delta_0 \in \Delta_j$, $1 \leqslant j \leqslant r$, the value of the problem (3) is not less than $\delta_0^2 u_j + \delta_n^2 v_j$.

Therefore, the optimal recovery error is not less than the value of the problem (2). Hence for $\delta_n/\delta_0 \in \Delta_j$, $j = 0, 1, \ldots, r$, it is proved that

$$E(T^k, T^n, \delta_0, \delta_n) \geqslant \sqrt{\delta_0^2 u_j + \delta_n^2 v_j}. \tag{4}$$

We proceed by obtaining an upper estimate for $E(T^k, T^n, \delta_0, \delta_n)$ and constructing optimal recovery methods. We look for an optimal method among linear operators acting from $\mathbb{C}^d \times \mathbb{C}^d$ to $\mathbb{C}^d$, i.e., operators acting by the rule: $(\xi, \eta) \mapsto A\xi + B\eta$, where A and B are linear operators from $\mathbb{C}^d$ to $\mathbb{C}^d$. The optimality of such a method means that the error of this method, i.e., the value of the problem

$$\|T^k x - Ay_0 - By_n\| \to \max, \quad \|x - y_0\| \leqslant \delta_0, \quad \|T^n x - y_n\| \leqslant \delta_n, \quad x, y_0, y_n \in \mathbb{C}^d \tag{5}$$

is equal to $E(T^k, T^n, \delta_0, \delta_n)$.

Denote $\xi = x - y_0$ and $\eta = T^n x - y_n$. Then this problem can be written in the form

$$\|T^k x - Ax - BT^n x + A\xi + B\eta\| \to \max, \quad \|\xi\| \leqslant \delta_0, \quad \|\eta\| \leqslant \delta_n, \quad x, \xi, \eta \in \mathbb{C}^d.$$

We note that if $T^k - A - BT^n$ is a nonzero operator, then the value of this problem is infinite. Indeed, if there exists $x_0 \in \mathbb{C}^d$ such that $T^k x_0 - Ax_0 - BT^n x_0 \neq 0$, then we can make the maximized functional as large as desired by the choice of a constant $C > 0$ on admissible elements Cx_0, $\xi = 0$, and $\eta = 0$.

Further, we assume that $A = T^k - BT^n$ and the eigenvectors $e_1, \ldots, e_d$ of the operator T are simultaneously eigenvectors of the operator B. But, in this case, they are also eigenvectors of the operator A. If α_i and β_i, $1 \leqslant i \leqslant d$, are the corresponding eigenvalues of the operators A and B, then

$$\alpha_i = \lambda_i^k - \beta_i \lambda_i^n, \quad 1 \leqslant i \leqslant d. \tag{6}$$

Taking into account the above assumptions and the decompositions $\xi = \xi_1 e_1 + \ldots + \xi_d e_d$ and $\eta = \eta_1 e_1 + \ldots + \eta_d e_d$, we conclude that the squared value of the problem (5) is equal to the value of the problem

$$\sum_{i=1}^{d} |\xi_i \alpha_i + \eta_i \beta_i|^2 \to \max, \quad \sum_{i=1}^{d} |\xi_i|^2 \leqslant \delta_0^2, \quad \sum_{i=1}^{d} |\eta_i|^2 \leqslant \delta_n^2. \tag{7}$$

Suppose that $\delta_n/\delta_0 \in \Delta_j$; moreover, $\mu_1 > 0$ and $1 \leqslant j \leqslant r-1$ or $\mu_1 = 0$ and $2 \leqslant j \leqslant r-1$. Then u_j and v_j are positive. Let us estimate the terms under the sign of sum in the maximized functional (7) by the Cauchy–Bunyakovksy inequality

$$|\xi_i \alpha_i + \eta_i \beta_i|^2 \leqslant \left(\frac{|\alpha_i|^2}{u_j} + \frac{|\beta_i|^2}{v_j}\right)(|\xi_i|^2 u_j + |\eta_i|^2 v_j), \quad 1 \leqslant i \leqslant d. \tag{8}$$

If (cf. (6))

$$\frac{|\alpha_i|^2}{u_j} + \frac{|\beta_i|^2}{v_j} = \frac{|\lambda_i^k - \beta_i\lambda_i^n|^2}{u_j} + \frac{|\beta_i|^2}{v_j} \leqslant 1, \quad 1 \leqslant i \leqslant d, \tag{9}$$

then, adding the inequalities (8), we see that the value of the problem (7) does not exceed $\delta_0^2 u_j + \delta_n^2 v_j$; moreover it is equal to this expression in view of (4), and the method indicated in the theorem with the operator B, given by the data β_i, $i = 1, \dots, d$, is optimal.

We write the left-hand sides of the inequalities (9) the form

$$\frac{|\lambda_i^k - \beta_i\lambda_i^n|^2}{u_j} + \frac{|\beta_i|^2}{v_j} = \frac{u_j + |\lambda_i|^{2n} v_j}{u_j v_j}\left|\beta_i - \frac{v_j \overline{\lambda}_i^n \lambda_i^k}{u_j + |\lambda_i|^{2n} v_j}\right|^2 + \frac{|\lambda_i|^{2k}}{u_j + |\lambda_i|^{2n} v_j}.$$

The condition that these expressions do not exceed 1 is equivalent to the inequalities

$$\left|\beta_i - \frac{v_j \overline{\lambda}_i^n \lambda_i^k}{u_j + |\lambda_i|^{2n} v_j}\right| \leqslant \frac{\sqrt{u_j v_j}}{u_j + |\lambda_i|^{2n} v_j}\sqrt{-|\lambda_i|^{2k} + u_j + |\lambda_i|^{2n} v_j}, \tag{10}$$

$1 \leqslant i \leqslant d$, which, in turn, are equivalent to the expressions for β_i, $i = 1, \dots, d$, in the theorem (in the cases under consideration, where $\delta_n/\delta_0 \in \Delta_j$ and j is such that u_j and v_j are simultaneously different from zero). Moreover, the expressions under the root sign in (10) are nonnegative. Indeed, the points (μ_j^{2n}, μ_j^{2k}) and $(\mu_{j+1}^{2n}, \mu_{j+1}^{2k})$ lie on the concave curve $y = x^{k/n}$, and the line $y = u_j + v_j x$ passes through these points. Hence for all μ_i we have $\mu_i^{2k} \leqslant u_j + v_j \mu_i^{2n}$ and, consequently, $-|\lambda_i|^{2k} + u_j + |\lambda_i|^{2n} v_j \geqslant 0$. In particular, from (10) it follows that there exist numbers β_i, $i = 1, \dots, d$, satisfying (9).

Thus, if $\delta_n/\delta_0 \in \Delta_j$ and j are such that u_j and v_j are simultaneously different from zero, then the numbers β_i, $i = 1, \dots, d$ satisfying the assumptions of the theorem yield the required expression for the optimal recovery error and provide the optimality of the method indicated in the theorem.

We prove that, in the remaining cases, the expressions for β_i, $i = 1, \dots, d$, found from (10) possess the same property. Indeed, suppose that $\mu_1 > 0$ and $\delta_n/\delta_0 \in \Delta_0$. Then $u_0 = 0$ and $v_0 = \mu_1^{-2(n-k)}$ by definition. From (10) it follows that $\beta_i = \lambda_i^{-(n-k)}$ and, by (6), $\alpha_i = 0$, $i = 1, \dots, d$, and we find (taking into account that $\mu_1 \leqslant |\lambda_i|$, $i = 1, \dots, d$) the upper estimate for the value of the problem (7):

$$\sum_{i=1}^d |\xi_i\alpha_i + \eta_i\beta_i|^2 = \sum_{i=1}^d |\lambda_i|^{-2(n-k)}|\eta_i|^2 \leqslant \mu_1^{-2(n-k)} \sum_{i=1}^d |\eta_i|^2 \leqslant \delta_n^2 v_0 = \delta_0^2 u_0 + \delta_n^2 v_0.$$

Hence the theorem holds if $\mu_1 > 0$ and $\delta_n/\delta_0 \in \Delta_0$.

Let $\delta_n/\delta_0 \in \Delta_r$. By definition, $u_r = \mu_r^{2k}$ and $v_r = 0$. From (10) we find that $\beta_i = 0$ and because of (6) $\alpha_i = \lambda_i^k$, $i = 1, \dots, d$. Consequently (taking into account that $|\lambda_i| \leqslant \mu_r$, $i = 1, \dots, d$),

$$\sum_{i=1}^d |\xi_i\alpha_i + \eta_i\beta_i|^2 = \sum_{i=1}^d |\lambda_i|^{2k}|\xi_i|^2 \leqslant \mu_r^{2k} \sum_{i=1}^d |\xi_i|^2 \leqslant \delta_0^2 u_r = \delta_0^2 u_r + \delta_n^2 v_r,$$

i.e., in this case, the assertions of the theorem are true. $\square$

2 Optimal Recovery of Temperature from Inaccurate Data

We consider the heat equation on a circle given by the implicit difference scheme

$$\frac{u_{s+1,j} - u_{sj}}{\tau} = \frac{u_{s+1,j+1} - 2u_{s+1,j} + u_{s+1,j-1}}{h^2}. \tag{11}$$

Here, τ and h are positive numbers, $(s, j) \in \mathbb{Z}_+ \times \mathbb{Z}_m$, where $\mathbb{Z}_m$ is the group of residuals modulo $m \geqslant 1$ which will be realized as a collection of numbers $\{0, 1, \ldots, m-1\}$ with the addition modulo m, $u_{s,j}$ is the temperature of a body at time $s\tau$ at the point jh.

We denote by l_2^m the space of functions (vectors) $x = (x_0, x_1, \ldots, x_{m-1})$ on $\mathbb{Z}_m$ equipped with the norm

$$\|x\|_{l_2^m} = \left(\sum_{j=0}^{m-1} |x_j|^2\right)^{1/2}.$$

We assume that the temperature of the body was approximately measured at times 0 and $n\tau$, i.e., we are know approximately the vectors $u_0 = (u_{0,0}, \ldots, u_{0,m-1})$ and $u_n = (u_{n,0}, \ldots, u_{n,m-1})$ or, more exactly, $y_0 = (y_{0,0}, \ldots, y_{0,m-1})$ and $y_n = (y_{n,0}, \ldots, y_{n,m-1})$ such that

$$\|u_q - y_q\|_{l_2^m} \leqslant \delta_q, \quad q = 0, n,$$

where $\delta_q > 0$, $q = 0, n$. It is required to recover $u_k = (u_{k,0}, \ldots, u_{k,m-1})$, $0 < k < n$, on the basis of this information, i.e., to find the value of the temperature of the body at time $k\tau$. By a *recovery method* we mean any possible mapping $\varphi\colon l_2^m \times l_2^m \to l_2^m$. The *error of a method* φ is defined by the formula

$$e_{kn}(\delta_0, \delta_n, \varphi) = \sup \|u_k - \varphi(y_0, y_n)\|_{l_2^m},$$

where the supremum is taken over $u_0, y_0, y_n \in l_2^m$ such that $\|u_q - y_q\|_{l_2^m} \leqslant \delta_q$, $q = 0, n$. The *optimal recovery error* is defined by

$$E_{kn}(\delta_0, \delta_n) = \inf_{\varphi\colon l_2^m \times l_2^m \to l_2^m} e_{kn}(\delta_0, \delta_n, \varphi).$$

A method at which the infimum is attained is said to be *optimal.*

On $\mathbb{Z}_m$, we can introduce the Fourier transform, i.e., the linear mapping sending $\mathbb{Z}_m$ to itself and, given by the matrix

$$F = \frac{1}{\sqrt{m}} \left(e^{-\frac{2\pi i p}{m} j}\right)_{p,j=0}^{m-1}.$$

It is easy to see that this matrix is unitary.

We apply the Fourier transform with respect to j to both sides of Equation (11), taking into account that it sends the translation by ± 1 to the multiplication by $\exp(\pm 2\pi i p/m)$. After simple transformations, we find that $Fu_{s+1} = \Lambda F u_s$ for all $s \in \mathbb{Z}_+$, where Λ is a diagonal matrix with diagonal entries

$$\lambda_p = \left(1 + \frac{4\tau}{h^2} \sin^2 \frac{\pi p}{m}\right)^{-1}, \quad p = 0, 1, \ldots, m-1. \tag{12}$$

From the relations $Fu_{s+1} = \Lambda F u_s$ it follows that

$$u_{s+1} = T u_s, \quad s \in \mathbb{Z}_+,$$

where $T = F^{-1}\Lambda F$.

Thus, we obtain the above problem, and it is clear that

$$E_{kn}(\delta_0, \delta_n) = E(T^k, T^n, \delta_0, \delta_n).$$

We formulate the corresponding result which follows from Theorem 1. We set

$$\mu_j = \left(1 + \frac{4\tau}{h^2}\sin^2\frac{\pi}{m}(r-j)\right)^{-1}, \quad j = 1, \dots r,$$

where $r = [m/2] + 1$, and define Δ_j, v_j, u_j, $j = 1, \dots, r$ by the same formulas as in Theorem 1. Then Theorem 1 implies the following assertion.

Theorem 2. *If $\delta_n/\delta_0 \in \Delta_j$, $0 \leqslant j \leqslant r$, then*

$$E_{kn}(\delta_0, \delta_n) = \sqrt{\delta_0^2 u_j + \delta_n^2 v_j},$$

and for any $\theta \in \mathbb{C}$ such that $|\theta| \leqslant 1$ and any diagonal matrix B with diagonal entries

$$\beta_p = \frac{v_j \lambda_p^{n+k}}{u_j + \lambda_p^{2n} v_j} + \theta \frac{\sqrt{u_j v_j}}{u_j + \lambda_p^{2n} v_j}\sqrt{-\lambda_p^{2k} + u_j + \lambda_p^{2n} v_j}, \quad p = 0, 1 \dots m-1,$$

the method

$$\widehat{\varphi}(y_0, y_n) = (T^k - T^n \widetilde{B}) y_0 + \widetilde{B} y_n,$$

where $\widetilde{B} = F^{-1} B F$, is optimal.

Similar problems for continuous models were considered in [1]–[5]. A discrete model in the nonperiodic case was studied in [4].

3 Discrete Analog of a System of Linear Differential Equations

We consider the discrete model of displacements of a d-dimensional vector

$$\frac{x_{s+1} - x_s}{\tau} = A x_s, \quad s = 0, 1, \dots,$$

where $x_s \in \mathbb{C}^d$, $\tau > 0$, and A is a square matrix of order d with constant entries.

We assume that we know vectors $y_0, y_n \in \mathbb{C}^d$ such that $\|x_0 - y_0\| \leqslant \delta_0$ and $\|x_n - y_n\| \leqslant \delta_n$, where $\|\cdot\|$ is the Euclidean norm and $\delta_0, \delta_n > 0$. Possessing on this information, we wish to recover the values x_k, $0 < k < n$. By a *recover method* we mean any mapping $\varphi\colon \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$, and the *error* of a method φ is defined by the formula

$$e(k, n, \delta_0, \delta_n, \varphi) = \sup \|x_k - \varphi(y_0, y_n)\|,$$

where the supremum is taken over $x_0, y_0, y_n \in \mathbb{C}^d$ such that $\|x_0 - y_0\| \leqslant \delta_0$, $\|x_n - y_n\| \leqslant \delta_n$. We are interested in the quantity

$$E(k, n, \delta_0, \delta_n) = \inf_{\varphi\colon \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d} e(k, n, \delta_0, \delta_n, \varphi).$$

By the equalities $x_s = T^s x_0$ and $T = E + \tau A$, the problem is reduced to the problem (1). We assume that A is a normal matrix with eigenvalues $\mu_1, \ldots, \mu_d$. Then T is also a normal matrix with eigenvalues $\lambda_j = 1 + \tau\mu_j$, $j = 1, \ldots, d$. Applying Theorem 1, we obtain a solution to the problem under consideration in this case.

An analog of this problem for a continuous model was considered in [6].

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