Optmal Recovery of Derivatives and Exact Inequalities in Hardy Spaces

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Denote by $\mathcal{H}_2^\beta$ the Hardy space of functions $f(\cdot)$ analytic in the strip $S_\beta = \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$ and satisfying the condition

$$\|f(\cdot)\|_{\mathcal{H}_2^\beta} = \left(\sup_{0\le\eta<\beta} \frac{1}{2}\int_{\mathbb{R}} (|f(t+i\eta)|^2 + |f(t-i\eta)|^2)\,dt\right)^{1/2} < \infty.$$

The Hardy-Sobolev space $\mathcal{H}_2^{r,\beta}$ is the space of functions $f(\cdot)$ analytic in the strip S_β for which $f^{(r)}(\cdot) \in \mathcal{H}_2^\beta$. The class $H_2^{r,\beta}$ is the set of functions $f(\cdot) \in \mathcal{H}_2^{r,\beta}$ for which $\|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta} \le 1$.

We consider the problem of optimal recovery of $f^{(k)}(\cdot)$ on $\mathbb{R}$ for functions $f(\cdot) \in H_2^{r,\beta} \cap L_2(\mathbb{R})$, $k \le r$, by the information about the trace of $f(\cdot)$ on $\mathbb{R}$ given with some error. Put

$$E_k(H_2^{r,\beta},\delta) = \inf_{\varphi\colon L_2(\mathbb{R})\to L_2(\mathbb{R})} \sup_{\substack{f(\cdot)\in H_2^{r,\beta}\cap L_2(\mathbb{R}),\ y(\cdot)\in L_2(\mathbb{R})\\ \|f(\cdot)-y(\cdot)\|_{L_2(\mathbb{R})}\le\delta}} \|f^{(k)}(\cdot) - \varphi(y)(\cdot)\|_{L_2(\mathbb{R})}.$$

Any method $\widehat{\varphi}$ for which this infimum is attained is called optimal.

Denote by $\mu_{r\beta}(x)$ the unique solution of the equation $t^r\sqrt{\operatorname{ch} 2\beta t} = x$ which belongs to the interval $[0, +\infty)$.

Theorem. *For all $r, k \in \mathbb{N}$, $k \le r$, and $\delta > 0$*

$$E_k(H_2^{r,\beta},\delta) = \sup_{\substack{f(\cdot)\in H_2^{r,\beta}\cap L_2(\mathbb{R})\\ \|f(\cdot)\|_{L_2(\mathbb{R})}\le\delta}} \|f^{(k)}(\cdot)\|_{L_2(\mathbb{R})} = \delta\mu_{r\beta}^k(\delta^{-1}).$$

Moreover, the method

$$\widehat{\varphi}(y)(\cdot) = (\mathcal{K}_{k,\delta}^{r,\beta} * y)(\cdot),$$

where

$$\mathcal{K}_{k,\delta}^{r,\beta}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} (it)^k \left(1 + \frac{k\delta^2 t^{2r}\operatorname{ch} 2\beta t}{r - k + \beta\mu_{r\beta}(\delta^{-1})\operatorname{th}(2\beta\mu_{r\beta}(\delta^{-1}))}\right)^{-1} e^{ixt}\,dt,$$

is optimal.

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