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{FirstPag

Discrete Analogs of Taikov's Inequality and Recovery of Sequences Given with an Error

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Abstract—We consider the problem of the recovery of the kth order divided difference from a sequence given with an error with bounded divided difference of nth order, $0 \le k < n$. The solution of this problem involves an extremal problem similar to that known in the continuous case as Taikov's inequality.

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1. INTRODUCTION {ssec1:v5

For functions $x(\,\cdot\,) \in L_2(\mathbb{R})$ whose $(n-1)$th derivative is locally absolutely continuous and

$$x^{(n)}(\,\cdot\,) \in L_2(\mathbb{R}) \qquad \text{for all} \quad 0 \le k \le n-1,$$

Taikov [1] obtained the sharp inequality

$$\|x^{(k)}(\,\cdot\,)\|_{L_\infty(\mathbb{R})} \le K \|x(\,\cdot\,)\|_{L_2(\mathbb{R})}^{(2n-2k-1)/2n} \|x^{(n)}(\,\cdot\,)\|_{L_2(\mathbb{R})}^{(2k+1)/2n},$$

where

$$K = \left((2k+1)\sin\pi\frac{2k+1}{2n}\right)^{-1/2}\left(\frac{2k+1}{2n-2k-1}\right)^{(2n-2k-1)/4n}$$

(this inequality is sharp, because it is impossible to replace the constant K by a smaller one). This problem is closely related to the following extremal problem:

$$x^{(k)}(0) \to \max, \qquad \|x(\,\cdot\,)\|_{L_2(\mathbb{R})} \le \delta, \quad \|x^{(n)}(\,\cdot\,)\|_{L_2(\mathbb{R})} \le 1. \tag{1.1}$$ {eq1.1:v5

By the generalized Smolyak theorem (see, for example, [2], [3]), the value of this problem is equal to the optimal recovery error of the kth derivative at zero from the function itself given with an error under the condition that this function belongs to the Sobolev class $W_2^n(\mathbb{R})$ consisting of functions $x(\,\cdot\,) \in L_2(\mathbb{R})$ whose $(n-1)$th derivative is locally absolutely continuous and $\|x^{(n)}(\,\cdot\,)\|_{L_2(\mathbb{R})} \le 1$. Problems of such type were studied in [4]. We consider a discrete analog of (1.1) and its relation with the problem of the recovery of a sequence given with an error.

Let $\mathcal{L}_{2,h}$, $h > 0$, denote the space of sequences $x = \{x_j\}_{j\in\mathbb{Z}}$, $x_j \in \mathbb{C}$ for which

$$\|x\|_{\mathcal{L}_{2,h}} = \left(h\sum_{j\in\mathbb{Z}} |x_j|^2\right)^{1/2} < \infty.$$

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Let

$$\Delta_h^1 x = \Delta_h x = \left\{\frac{x_{j+1}-x_j}{h}\right\}_{j\in\mathbb{Z}}, \qquad \Delta_h^k x = \Delta_h(\Delta_h^{k-1}x), \quad k = 2,3,\dots .$$

Let $l_{2,h}^n$ denote the class of sequences $x = \{x_j\}_{j\in\mathbb{Z}}$ for which

$$x \in \mathcal{L}_{2,h} \qquad \text{and} \qquad \|\Delta_h^n x\|_{\mathcal{L}_{2,h}} \le 1.$$

We consider the problem of the optimal recovery of $(\Delta_h^k x)_0$, $0 \le k < n$, from a sequence $x \in l_{2,h}^n$ given with an error. It is assumed that, for each $x \in l_{2,h}^n$, a sequence $\widetilde{x} \in \mathcal{L}_{2,h}$ such that $\|x-\widetilde{x}\|_{\mathcal{L}_{2,h}} \le \delta$ is known. As the recovery method, we consider all possible mappings $m\colon \mathcal{L}_{2,h} \to \mathbb{C}$. By the *error* of the method m we mean the quantity

$$e(k,n,h,\delta,m) = \sup_{\substack{x\in l_{2,h}^n,\, \widetilde{x}\in\mathcal{L}_{2,h} \\ \|x-\widetilde{x}\|_{\mathcal{L}_{2,h}}\le\delta}} |(\Delta_h^k x)_0 - m(\widetilde{x})|.$$

By the *optimal recovery error* we mean the quantity

$$E(k,n,h,\delta) = \inf_{m\colon \mathcal{L}_{2,h}\to\mathbb{C}} e(k,n,h,\delta,m).$$

The method for which the lower bound is attained is called the *optimal* method.

2. MAIN RESULTS {sec2:v515

Theorem 1. *For all $n \in \mathbb{N}$ and $0 \le k \le n-1$, the following relation holds*:

$$E(k,n,h,\delta) = K + \frac{\sin^{1/2}(\pi(2k+1)/2n)}{16\sqrt{2k+1}\,\sin(\pi(2k+3)/2n)}\left(\frac{(2k+1)\delta^2}{2n-2k-1}\right)^{(2n-2k-5)/4n} h^2 + o(h^2).$$

For a sufficiently small error, the optimal method is to take the kth divided difference of the approximate data. Then the optimal recovery error can be calculated exactly. {th2:v515

Theorem 2. *Suppose that $n \in \mathbb{N}$, $0 \le k \le n-1$, and the following condition holds*:

$$\delta \le \left(\frac{h}{2}\right)^n \left(\frac{(2k+2)(2k+4)\cdots(2k+2n)}{(2k+1)(2k+3)\cdots(2k+2n-1)}\right)^{1/2}. \tag{2.1}$$ {eq2.1:v5

Then

$$E(k,n,h,\delta) = \frac{2^k}{h^{k+1/2}}\left(\frac{(2k-1)!!}{(2k)!!}\right)^{1/2}\delta,$$

and the method $\widehat{m}(\widetilde{x}) = (\Delta_h^k \widetilde{x})_0$ is optimal.

For small n and k, the problem posed above can be solved for any δ and h. For $n = 1$ and $k = 0$, we have the following result. {th3:v515

Theorem 3. *The following equality holds*:

$$E(0,1,h,\delta) = \begin{cases} \dfrac{\delta}{\sqrt{h}}, & \delta \le \dfrac{h}{\sqrt{2}}, \\[2mm] \dfrac{1}{\sqrt{2}}(4\delta^2-h^2)^{1/4}, & \delta > \dfrac{h}{\sqrt{2}}. \end{cases}$$

For $\delta \le h/\sqrt{2}$, the method $\widehat{m}(\widetilde{x}) = \widetilde{x}_0$ is optimal, while, for $\delta > h/\sqrt{2}$, the method

$$\widehat{m}(\widetilde{x}) = \frac{h}{\sqrt{4\delta^2-h^2}} \sum_{j\in\mathbb{Z}} \left(\frac{2\delta^2 - h\sqrt{4\delta^2-h^2}}{2\delta^2-h^2}\right)^{|j|} \widetilde{x}_j$$

is optimal.

Thus, if the input data is given with a large error, for the best recovery of the sequence, we must use all the approximate values, smoothing them in a suitable way.

A similar result holds for $n = 2$ and $k = 0, 1$. In view of Theorem 2, we formulate the theorem only in the case $\delta > h^2/\sqrt{4k+6}$.

For $x = \{x_j\}_{j\in\mathbb{Z}} \in \mathcal{L}_{2,h}$ and $y = \{y_j\}_{j\in\mathbb{Z}} \in \mathcal{L}_{2,h}$, we set

$$(x, y)_{\mathcal{L}_{2,h}} = h\sum_{j\in\mathbb{Z}} x_j\overline{y}_j.$$

{th4:v515

Theorem 4. *Let $\delta > h^2/\sqrt{4k+6}$. Then, for $k = 0, 1$,*

$$E(k, 2, h, \delta) = \frac{(d_k+1)^{3/4-k/2}\sqrt{d_k}}{2^{5/4-k}\sqrt{(2k+1)d_k+2}}\,h^{3/2-k},$$

where d_k is the solution of the equation

$$\frac{(d_k+1)((3-2k)d_k^2+(2k-1)d_k+2)}{(2k+1)d_k+2} = \frac{16\delta^2}{h^4}. \tag{2.2}$$ {eq2.2:v5

The method

$$\widehat{m}(\widetilde{x}) = -\frac{4}{h\sqrt{d_k^2-1}}(\Delta_h^k\nu_k, \widetilde{x})_{\mathcal{L}_{2,h}},$$

where

$$(\nu_k)_j = \operatorname{Im}\frac{\mu_k^{|j|+1}}{1-\mu_k^2}, \qquad \mu_k = \left(1-\sqrt{\frac{2}{d_k+1}}\right)\left(1-i\sqrt{\frac{2}{d_k-1}}\right), \tag{2.3}$$ {eq2.3:v5

is optimal.

3. PROOFS

{ssec3:v5

We shall need the following auxiliary result.

{lem1:v51

Lemma 1. *Suppose that there exist $\widehat{\lambda}_1 \ge 0$, $\widehat{\lambda}_2 \ge 0$, and $\widehat{x} \in l_{2,h}^n$, such that, for all $x \in \mathcal{L}_{2,h}$, the following identity holds*:

$$(\Delta_h^k x)_0 = \widehat{\lambda}_1(x, \widehat{x})_{\mathcal{L}_{2,h}} + \widehat{\lambda}_2(\Delta_h^n x, \Delta_h^n\widehat{x})_{\mathcal{L}_{2,h}}, \tag{3.1}$$ {eq3.1:v5

and the element $\widehat{x}$ satisfies the conditions

$$\|\widehat{x}\|_{\mathcal{L}_{2,h}} = \delta, \quad \|\Delta_h^n\widehat{x}\|_{\mathcal{L}_{2,h}} = 1 \quad \textit{for } \widehat{\lambda}_2 > 0 \qquad \textit{or} \qquad \|\Delta_h^n\widehat{x}\|_{\mathcal{L}_{2,h}} \le 1 \quad \textit{for } \widehat{\lambda}_2 = 0.$$

Then the method

$$\widehat{m}(\widetilde{x}) = \widehat{\lambda}_1(\widetilde{x}, \widehat{x})_{\mathcal{L}_{2,h}}$$

is optimal and

$$E(k, n, h, \delta) = \widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2. \tag{3.2}$$ {eq3.2:v5

Proof. Let m be an arbitrary recovery method and $x \in l_{2,h}^n$ such that $\|x\|_{\mathcal{L}_{2,h}} \le \delta$. Then

$$2|(\Delta_h^k x)_0| = |(\Delta_h^k x)_0 - m(0) - ((\Delta_h^k(-x))_0 - m(0))| \le 2e(k, n, h, \delta, m).$$

Since the method is arbitrary, we have

$$E(k, n, h, \delta) \ge \sup_{\substack{x\in l_{2,h}^n\\ \|x\|_{\mathcal{L}_{2,h}}\le\delta}} |(\Delta_h^k x)_0| \ge (\Delta_h^k\widehat{x})_0 = \widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2.$$

On the other hand, using identity (3.1), we obtain

$$\begin{aligned}|(\Delta_h^k x)_0-\widehat{m}(\widetilde{x})| &= |(\Delta_h^k x)_0-\widehat{\lambda}_1(x,\widehat{x})_{\mathcal{L}_{2,h}}+\widehat{\lambda}_1(\widetilde{x}-x,\widehat{x})_{\mathcal{L}_{2,h}}| \\ &\le |\widehat{\lambda}_2(\Delta_h^n x,\Delta_h^n\widehat{x})_{\mathcal{L}_{2,h}}|+\widehat{\lambda}_1\delta^2\le\widehat{\lambda}_2+\widehat{\lambda}_1\delta^2.\end{aligned}$$

This implies relation (3.2) and also the optimality of the method $\widehat{m}$. The lemma is proved. □

With each sequence $\{x_j\}_{j\in\mathbb{Z}}\in\mathcal{L}_{2,h}$ we associate the function

$$Fx(t)=\sum_{j\in\mathbb{Z}}x_je^{ijt}\in L_2(\mathbb{T}),$$

where $\mathbb{T}$ is the closed interval $[-\pi,\pi]$ with identified endpoints and

$$\|x(\,\cdot\,)\|_{L_2(\mathbb{T})}=\biggl(\frac{1}{2\pi}\int_{\mathbb{T}}|x(t)|^2\,dt\biggr)^{1/2}.$$

Set

$$(x(\,\cdot\,),y(\,\cdot\,))_{L_2(\mathbb{T})}=\frac{1}{2\pi}\int_{\mathbb{T}}x(t)\overline{y(t)}\,dt.$$

Then it is readily verified that

$$(x,y)_{\mathcal{L}_{2,h}}=h(Fx(\,\cdot\,),Fy(\,\cdot\,))_{L_2(\mathbb{T})}.$$

Thus, identity (3.1) can be written as

$$\frac{1}{2\pi}\int_{\mathbb{T}}F(\Delta_h^kx)(t)\,dt=\frac{\widehat{\lambda}_1h}{2\pi}\int_{\mathbb{T}}Fx(t)\overline{F\widehat{x}(t)}\,dt+\frac{\widehat{\lambda}_2h}{2\pi}\int_{\mathbb{T}}F(\Delta_h^nx)(t)\overline{F(\Delta_h^n\widehat{x})(t)}\,dt. \qquad (3.3)$$

{eq3.3:v5

Since

$$F(\Delta_hx)(t)=\frac{1}{h}(e^{-it}-1)Fx(t),$$

we have

$$F(\Delta_h^kx)(t)=\frac{1}{h^k}(e^{-it}-1)^kFx(t).$$

Therefore, relation (3.3) can be rewritten as

$$\begin{aligned}&\frac{1}{2\pi h^k}\int_{\mathbb{T}}(e^{-it}-1)^kFx(t)\,dt\\ &\qquad=\frac{\widehat{\lambda}_1h}{2\pi}\int_{\mathbb{T}}Fx(t)\overline{F\widehat{x}(t)}\,dt+\frac{\widehat{\lambda}_2}{2\pi h^{2n-1}}\int_{\mathbb{T}}|e^{it}-1|^{2n}Fx(t)\overline{F\widehat{x}(t)}\,dt. \qquad (3.4)\end{aligned}$$

{eq3.4:v5

Hence

$$F\widehat{x}(t)=\frac{h^{2n-k-1}(e^{it}-1)^k}{\widehat{\lambda}_1h^{2n}+\widehat{\lambda}_2|e^{it}-1|^{2n}},$$

and $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ must be chosen from the conditions (we begin by searching for $\widehat{\lambda}_2>0$)

$$h\|F\widehat{x}(\,\cdot\,)\|^2_{L_2(\mathbb{T})}=\delta^2,\qquad h\|F(\Delta_h^n\widehat{x})(\,\cdot\,)\|^2_{L_2(\mathbb{T})}=1,$$

which are written as

$$\begin{aligned}\frac{h^{4n-2k-1}}{2\pi}\int_{\mathbb{T}}\frac{|e^{it}-1|^{2k}}{(\widehat{\lambda}_1h^{2n}+\widehat{\lambda}_2|e^{it}-1|^{2n})^2}\,dt&=\delta^2,\\ \frac{h^{2n-2k-1}}{2\pi}\int_{\mathbb{T}}\frac{|e^{it}-1|^{2(n+k)}}{(\widehat{\lambda}_1h^{2n}+\widehat{\lambda}_2|e^{it}-1|^{2n})^2}\,dt&=1.\end{aligned} \qquad (3.5)$$

{eq3.5:v5

Proof of Theorem 1. Making the replacement $u = \tan(t/2)$ in (3.5), we obtain

$$\begin{gathered}\frac{4^k h^{4n-2k-1}}{\pi}\int_{\mathbb{R}}\frac{u^{2k}(1+u^2)^{2n-k-1}}{(\widehat{\lambda}_1 h^{2n}(1+u^2)^n+\widehat{\lambda}_2 4^n u^{2n})^2}\,du=\delta^2,\\ \frac{4^{n+k} h^{2n-2k-1}}{\pi}\int_{\mathbb{R}}\frac{u^{2(n+k)}(1+u^2)^{n-k-1}}{(\widehat{\lambda}_1 h^{2n}(1+u^2)^n+\widehat{\lambda}_2 4^n u^{2n})^2}\,du=1.\end{gathered}\tag{3.6}$$ {eq3.6:v5

We us now pass to the variable $v = h^{-1}u$, obtaining the system

$$\begin{gathered}\frac{4^k}{\pi}\int_{\mathbb{R}}\frac{v^{2k}(1+h^2v^2)^{2n-k-1}}{(\widehat{\lambda}_1(1+h^2v^2)^n+\widehat{\lambda}_2 4^n v^{2n})^2}\,dv=\delta^2,\\ \frac{4^{n+k}}{\pi}\int_{\mathbb{R}}\frac{v^{2(n+k)}(1+h^2v^2)^{n-k-1}}{(\widehat{\lambda}_1(1+h^2v^2)^n+\widehat{\lambda}_2 4^n v^{2n})^2}\,dv=1.\end{gathered}$$

Set $b = 4^n\widehat{\lambda}_2/\widehat{\lambda}_1$. Then this system can be rewritten as

$$\begin{gathered}\frac{4^k}{\pi}\int_{\mathbb{R}}\frac{v^{2k}(1+h^2v^2)^{2n-k-1}}{((1+h^2v^2)^n+bv^{2n})^2}\,dv=\delta^2\widehat{\lambda}_1^2,\\ \frac{4^{n+k}}{\pi}\int_{\mathbb{R}}\frac{v^{2(n+k)}(1+h^2v^2)^{n-k-1}}{((1+h^2v^2)^n+bv^{2n})^2}\,dv=\widehat{\lambda}_1^2.\end{gathered}\tag{3.7}$$ {eq3.7:v5

Thus, for b, we obtain the equation

$$\Phi(\tau,b)=0,\tag{3.8}$$ {eq3.8:v5

where $\tau = h^2$, and

$$\Phi(\tau,b)=\int_{\mathbb{R}}\frac{v^{2k}(1+\tau v^2)^{2n-k-1}}{((1+\tau v^2)^n+bv^{2n})^2}\,dv-\delta^2 4^n\int_{\mathbb{R}}\frac{v^{2(n+k)}(1+\tau v^2)^{n-k-1}}{((1+\tau v^2)^n+bv^{2n})^2}\,dv.$$

Set $h = 0$. Then

$$\Phi(0,b)=\int_{\mathbb{R}}\frac{v^{2k}}{(1+bv^{2n})^2}\,dv-\delta^2 4^n\int_{\mathbb{R}}\frac{v^{2(n+k)}}{(1+bv^{2n})^2}\,dv.$$

Making the replacement $u^{2n} = bv^{2n}$, we obtain

$$\Phi(0,b)=b^{-(2k+1)/2n}\int_{\mathbb{R}}\frac{u^{2k}}{(1+u^{2n})^2}\,du-\delta^2 4^n b^{-(2n+2k+1)/2n}\int_{\mathbb{R}}\frac{u^{2(n+k)}}{(1+u^{2n})^2}\,du.$$

Let us use the equalities

$$\begin{gathered}\int_{\mathbb{R}}\frac{u^{2k}}{(1+u^{2n})^2}\,du=\frac{\pi}{2n^2}(2n-2k-1)\sin^{-1}\pi\frac{2k+1}{2n},\\ \int_{\mathbb{R}}\frac{u^{2(k+n)}}{(1+u^{2n})^2}\,du=\frac{\pi}{2n^2}(2k+1)\sin^{-1}\pi\frac{2k+1}{2n},\end{gathered}\tag{3.9}$$ {eq3.9:v5

which are easy to obtain by expressing the integrals on the left-hand sides in terms of the beta function $B(\,\cdot\,,\,\cdot\,)$ and applying the reduction formula

$$B(a,1-a)=\frac{\pi}{\sin\pi a}.$$

Then we have

$$\Phi(0,b)=b^{-(2k+1)/2n}\frac{\pi}{2n^2}\sin^{-1}\pi\frac{2k+1}{2n}(2n-2k-1-\delta^2 4^n b^{-1}(2k+1)).$$

Hence $\Phi(0,b_0)=0$, where

$$b_0=\delta^2 4^n\frac{2k+1}{2n-2k-1}.$$

In order to apply the implicit-function theorem, we calculate $\Phi'_\tau(0,b_0)$ and $\Phi'_b(0,b_0)$, obtaining

$$\Phi'_\tau(0,b) = (2n-k-1)\int_{\mathbb{R}} \frac{v^{2k+2}}{(1+bv^{2n})^2}\,dv - 2n\int_{\mathbb{R}} \frac{v^{2k+2}}{(1+bv^{2n})^3}\,dv - \delta^2 4^n(n-k-1)\int_{\mathbb{R}} \frac{v^{2n+2k+2}}{(1+bv^{2n})^2}\,dv + \delta^2 4^n 2n\int_{\mathbb{R}} \frac{v^{2n+2k+2}}{(1+bv^{2n})^3}\,dv.$$

Let us make the same replacement $u^{2n} = bv^{2n}$. Then

$$\Phi'_\tau(0,b) = (2n-k-1)b^{-(2k+3)/2n}\int_{\mathbb{R}} \frac{u^{2k+2}}{(1+u^{2n})^2}\,du - 2nb^{-(2k+3)/2n}\int_{\mathbb{R}} \frac{u^{2k+2}}{(1+u^{2n})^3}\,du - \delta^2 4^n(n-k-1)b^{-(2n+2k+3)(2n)}\int_{\mathbb{R}} \frac{u^{2n+2k+2}}{(1+u^{2n})^2}\,du + \delta^2 4^n 2nb^{-(2n+2k+3)/2n}\int_{\mathbb{R}} \frac{u^{2n+2k+2}}{(1+u^{2n})^3}\,du.$$

Just as in the case of (3.9), we can write

$$\int_{\mathbb{R}} \frac{u^{2k+2}}{(1+u^{2n})^3}\,du = \frac{\pi}{8n^3}(4n-2k-3)(2n-2k-3)\sin^{-1}\pi\frac{2k+3}{2n},$$

$$\int_{\mathbb{R}} \frac{u^{2n+2k+2}}{(1+u^{2n})^3}\,du = \frac{\pi}{8n^3}(2n-2k-3)(2k+3)\sin^{-1}\pi\frac{2k+3}{2n}.$$

Thus,

$$\Phi'_\tau(0,b) = \frac{\pi}{4n^2}b^{-(2k+3)/2n}(2n-2k-3)\sin^{-1}\pi\frac{2k+3}{2n} - \delta^2 4^n\frac{\pi}{4n^2}b^{-(2n+2k+3)/2n}(2k+3)\sin^{-1}\pi\frac{2k+3}{2n}.$$

Hence

$$\Phi'_\tau(0,b_0) = -\frac{\pi}{n(2k+1)}b_0^{-(2k+3)/2n}\sin^{-1}\pi\frac{2k+3}{2n}.$$

For $\Phi'_b(0,b)$, we have

$$\Phi'_b(0,b) = -2\int_{\mathbb{R}} \frac{v^{2n+2k}}{(1+bv^{2n})^3}\,dv + 2\delta^2 4^n\int_{\mathbb{R}} \frac{v^{4n+2k}}{(1+bv^{2n})^3}\,dv.$$

Replacing $u^{2n} = bv^{2n}$, we obtain

$$\begin{aligned}\Phi'_b(0,b) &= -2b^{-(2n+2k+1)/2n}\int_{\mathbb{R}} \frac{u^{2n+2k}}{(1+u^{2n})^3}\,du + 2\delta^2 4^n b^{-(4n+2k+1)/2n}\int_{\mathbb{R}} \frac{u^{4n+2k}}{(1+u^{2n})^3}\,du \\ &= -2b^{-(2n+2k+1)/2n}\frac{\pi(2k+1)}{8n^3} \\ &\quad\times\sin^{-1}\pi\frac{2k+1}{2n}(2n-2k-1-\delta^2 4^n b^{-1}(2n+2k+1)).\end{aligned}$$

Thus,

$$\Phi'_b(0,b_0) = \frac{\pi(2n-2k-1)}{2n^2}b_0^{-(2n+2k+1)/2n}\sin^{-1}\pi\frac{2k+1}{2n}.$$

By the implicit-function theorem, in a sufficiently small neighborhood of zero, there exists a function $b = b(\tau)$ satisfying Eq. (3.8) and

$$b'(0) = -\frac{\Phi'_\tau(0,b_0)}{\Phi'_b(0,b_0)} = \frac{2n\sin(\pi(2k+1)/2n)}{(2n-2k-1)(2k+1)\sin(\pi(2k+3)/2n)}\left(\frac{(2k+1)\delta^2 4^n}{2n-2k-1}\right)^{(n-1)/n}.$$

Hence

$$b(h) = \delta^2 4^n \frac{2k+1}{2n-2k-1} + \frac{2n\sin(\pi(2k+1)/2n)}{(2n-2k-1)(2k+1)\sin(\pi(2k+3)/2n)}\left(\frac{(2k+1)\delta^2 4^n}{2n-2k-1}\right)^{(n-1)/n} h^2 + o(h^2).$$

To find the asymptotics $\widehat{\lambda}_1$ consider the function

$$c(\tau) = \frac{4^{n+k}}{\pi}\int_{\mathbb{R}} \frac{v^{2(n+k)}(1+\tau v^2)^{n-k-1}}{((1+\tau v^2)^n + bv^{2n})^2}\,dv.$$

We have

$$\begin{aligned} c(0) &= \frac{4^{n+k}}{\pi}\int_{\mathbb{R}} \frac{v^{2(n+k)}}{(1+b_0 v^{2n})^2}\,dv = \frac{4^{n+k}}{\pi} b_0^{-(2n+2k+1)/2n}\int_{\mathbb{R}} \frac{u^{2(n+k)}}{(1+u^{2n})^2}\,du \\ &= 4^{n+k} b_0^{-(2n+2k+1)/2n}\frac{2k+1}{2n^2}\sin^{-1}\pi\frac{2k+1}{2n}. \end{aligned}$$

Further,

$$\begin{aligned} c'(0) &= \frac{4^{n+k}}{\pi}(n-k-1)\int_{\mathbb{R}} \frac{v^{2n+2k+2}}{(1+b_0 v^{2n})^2}\,dv \\ &\quad - \frac{4^{n+k}}{\pi} 2n \int_{\mathbb{R}} \frac{v^{2n+2k+2}}{(1+b_0 v^{2n})^3}\,dv - \frac{4^{n+k}}{\pi} 2b'(0)\int_{\mathbb{R}} \frac{v^{4n+2k}}{(1+b_0 v^{2n})^3}\,dv \\ &= 4^{n+k-1} b_0^{-(2n+2k+3)/2n}\frac{(2k+1)(2n-2k-5)}{n^2(2n-2k-1)}\sin^{-1}\pi\frac{2k+3}{2n}. \end{aligned}$$

Hence

$$\begin{aligned} \widehat{\lambda}_1 &= \sqrt{c(\tau)} = \sqrt{c(0)} + \frac{c'(0)}{2\sqrt{c(0)}}\tau + o(\tau) \\ &= 2^{n+k-1/2} b_0^{-(2n+2k+1)/4n}\frac{\sqrt{2k+1}}{n}\sin^{-1/2}\pi\frac{2k+1}{2n} \\ &\quad + 2^{n+k-5/2} b_0^{-(2n+2k+5)/4n}\frac{\sqrt{2k+1}\,(2n-2k-5)}{n(2n-2k-1)}\frac{\sin^{1/2}(\pi(2k+1)/2n)}{\sin(\pi(2k+3)/2n)} h^2 + o(h^2). \end{aligned}$$

Therefore, for the optimal recovery error we have

$$\begin{aligned} E(k,n,h,\delta) &= \widehat{\lambda}_1\delta^2 + \widehat{\lambda}_2 = \widehat{\lambda}_1\left(\delta^2 + \frac{b}{4^n}\right) \\ &= K\delta^{(2n-2k-1)/2n} \\ &\quad + \frac{\sin^{1/2}(\pi(2k+1)/2n)}{16\sqrt{2k+1}\sin(\pi(2k+3)/2n)}\left(\frac{(2k+1)\delta^2}{2n-2k-1}\right)^{(2n-2k-5)/4n} h^2 + o(h^2). \end{aligned}$$

The theorem is proved. □

Proof of Theorem 2. Now set $\widehat{\lambda}_2 = 0$. Then

$$F\widehat{x}(t) = \frac{(e^{it}-1)^k}{\widehat{\lambda}_1 h^{k+1}},$$

and $\widehat{\lambda}_1$ must be chosen from the conditions

$$h\|F\widehat{x}(\,\cdot\,)\|^2_{L_2(\mathbb{T})} = \delta^2, \qquad h\|F(\Delta_h^n\widehat{x})(\,\cdot\,)\|^2_{L_2(\mathbb{T})} \le 1,$$

which can be written as

$$\frac{1}{2\pi\widehat{\lambda}_1^2 h^{2k+1}}\int_{\mathbb{T}}|e^{it}-1|^{2k}\,dt=\delta^2, \qquad \frac{1}{2\pi\widehat{\lambda}_1^2 h^{2n+2k+1}}\int_{\mathbb{T}}|e^{it}-1|^{2(n+k)}\,dt\le 1. \tag{3.10}$$ {eq3.10:v

Using the well-known formula

$$\int_0^{\pi/2}\sin^{2k}\tau\,d\tau=\frac{\pi}{2}\frac{(2k-1)!!}{(2k)!!},$$

we obtain

$$\int_{\mathbb{T}}|e^{it}-1|^{2k}\,dt=4^k\int_{\mathbb{T}}\sin^{2k}\frac{t}{2}\,dt=2\pi 4^k\frac{(2k-1)!!}{(2k)!!}.$$

Thus, conditions (3.10) are written as

$$\frac{4^k}{\widehat{\lambda}_1^2 h^{2k+1}}\frac{(2k-1)!!}{(2k)!!}=\delta^2, \qquad \frac{4^{n+k}}{\widehat{\lambda}_1^2 h^{2n+2k+1}}\frac{(2n+2k-1)!!}{(2n+2k)!!}\le 1.$$

Hence

$$\widehat{\lambda}_1=\frac{2^k}{\delta h^{k+1/2}}\biggl(\frac{(2k-1)!!}{(2k)!!}\biggr)^{1/2},$$

and δ must satisfy condition (2.1). The theorem is proved. □

Proof of Theorem 3. For $\delta\le h/\sqrt{2}$, the assertion of the theorem follows from Theorem 2. We assume that $\delta>h/\sqrt{2}$. In the case under consideration, system (3.7) takes the form

$$\frac{1}{\pi}\int_{\mathbb{R}}\frac{1+h^2v^2}{(1+(h^2+b)v^2)^2}\,dv=\delta^2\widehat{\lambda}_1^2, \qquad \frac{4}{\pi}\int_{\mathbb{R}}\frac{v^2}{(1+(h^2+b)v^2)^2}\,dv=\widehat{\lambda}_1^2. \tag{3.11}$$ {eq3.11:v

Replacing $v=(h^2+b)^{-1/2}u$ we obtain (see (3.9))

$$\int_{\mathbb{R}}\frac{1}{(1+(h^2+b)v^2)^2}\,dv=(h^2+b)^{-1/2}\int_{\mathbb{R}}\frac{1}{(1+u^2)^2}\,du=\frac{\pi}{2}(h^2+b)^{-1/2}.$$

Similarly,

$$\int_{\mathbb{R}}\frac{v^2}{(1+(h^2+b)v^2)^2}\,dv=(h^2+b)^{-3/2}\int_{\mathbb{R}}\frac{u^2}{(1+u^2)^2}\,du=\frac{\pi}{2}(h^2+b)^{-3/2}.$$

Thus, using system (3.11), we find that

$$\frac{1}{2}(h^2+b)^{-1/2}+\frac{1}{2}h^2(h^2+b)^{-3/2}=\delta^2\widehat{\lambda}_1^2, \qquad 2(h^2+b)^{-3/2}=\widehat{\lambda}_1^2.$$

Hence

$$b=4\delta^2-2h^2, \qquad \widehat{\lambda}_1=\frac{\sqrt{2}}{(4\delta^2-h^2)^{3/4}}, \qquad \widehat{\lambda}_2=\frac{2\delta^2-h^2}{\sqrt{2}\,(4\delta^2-h^2)^{3/4}}.$$

Therefore,

$$F\widehat{x}(t)=\frac{h}{\widehat{\lambda}_1h^2+2\widehat{\lambda}_2(1-\cos t)}=\frac{h(4\delta^2-h^2)^{3/4}}{\sqrt{2}\,(2\delta^2-h^2)}\,\frac{1}{a-\cos t},$$

where $a=2\delta^2/(2\delta^2-h^2)$. To find the Fourier coefficients of the function $F\widehat{x}(\,\cdot\,)$, we use the easily verified equality

$$\sum_{j\in\mathbb{Z}}\mu^{|j|}e^{ijt}=\frac{(1-\mu^2)/2\mu}{(1+\mu^2)/2\mu-\cos t}, \tag{3.12}$$ {eq3.12:v

which holds for all μ, $0 < |\mu| < 1$. Setting

$$\mu = a - \sqrt{a^2 - 1} = \frac{2\delta^2 - h\sqrt{4\delta^2 - h^2}}{2\delta^2 - h^2},$$

we obtain

$$\frac{1}{a - \cos t} = \frac{2\mu}{1 - \mu^2} \sum_{j \in \mathbb{Z}} \mu^{|j|} e^{ijt} = \frac{1}{\sqrt{a^2 - 1}} \sum_{j \in \mathbb{Z}} \mu^{|j|} e^{ijt} = \frac{2\delta^2 - h^2}{h\sqrt{4\delta^2 - h^2}} \sum_{j \in \mathbb{Z}} \mu^{|j|} e^{ijt}.$$

Thus,

$$F\widehat{x}(t) = \frac{(4\delta^2 - h^2)^{1/4}}{\sqrt{2}} \sum_{j \in \mathbb{Z}} \mu^{|j|} e^{ijt}.$$

Hence

$$\widehat{x}_j = \frac{(4\delta^2 - h^2)^{1/4}}{\sqrt{2}} \mu^{|j|}.$$

The theorem is proved. □

Proof of Theorem 4. Here it is more convenient to deal with system (3.6), which, in the case under consideration, has the form

$$\frac{4^k h^{7-2k}}{\pi} \int_{\mathbb{R}} \frac{u^{2k}(1+u^2)^{3-k}}{(\widehat{\lambda}_1 h^4 (1+u^2)^2 + 16\widehat{\lambda}_2 u^4)^2}\, du = \delta^2,$$
$$\frac{4^{2+k} h^{3-2k}}{\pi} \int_{\mathbb{R}} \frac{u^{4+2k}(1+u^2)^{1-k}}{(\widehat{\lambda}_1 h^4 (1+u^2)^2 + 16\widehat{\lambda}_2 u^4)^2}\, du = 1.$$

This system can be rewritten as

$$\begin{aligned} \frac{4^k}{\pi} \int_{\mathbb{R}} \frac{u^{2k}(1+u^2)^{3-k}}{((1+u^2)^2 + a^2u^4)^2}\, du &= h^{1+2k}\delta^2 \widehat{\lambda}_1^2, \\ \frac{4^{2+k}}{\pi} \int_{\mathbb{R}} \frac{u^{4+2k}(1+u^2)^{1-k}}{((1+u^2)^2 + a^2u^4)^2}\, du &= h^{5+2k} \widehat{\lambda}_1^2, \end{aligned} \qquad (3.13)$$ {eq3.13:v

where $a = 4h^{-2}\sqrt{\lambda_2/\lambda_1}$. Set $c = p + iq$, where

$$p = \sqrt{\frac{\sqrt{1+a^2} - 1}{2(1+a^2)}}, \qquad q = \sqrt{\frac{\sqrt{1+a^2} + 1}{2(1+a^2)}}. \qquad (3.14)$$ {eq3.14:v

Then $c^2 = -(1+ia)^{-1}$ and

$$((1+u^2)^2 + a^2u^4)^2 = (1+a^2)^2 (u-c)^2 (u+c)^2 (u-\overline{c})^2 (u+\overline{c})^2.$$

Let us calculate each of the integrals using the residues (the multipliers 4^k in both integrals will be taken care of later). In the upper half-plane, there are two poles of the denominator of multiplicity 2, c and $-\overline{c}$. Therefore,

$$\begin{aligned} I_{1k} &= \frac{1}{\pi(1+a^2)^2} \int_{\mathbb{R}} \frac{u^{2k}(1+u^2)^{3-k}}{(u-c)^2(u+c)^2(u-\overline{c})^2(u+\overline{c})^2}\, du \\ &= \frac{ic^{2k}(1+c^2)^{3-k}}{2(1+a^2)^2 c^2 (c-\overline{c})^2 (c+\overline{c})^2} \left(\frac{2k-1}{c} + \frac{(6-2k)c}{1+c^2} - \frac{2}{c-\overline{c}} - \frac{2}{c+\overline{c}} \right) \\ &\quad + \frac{i\overline{c}^{2k}(1+\overline{c}^2)^{3-k}}{2(1+a^2)^2 \overline{c}^2 (c-\overline{c})^2 (c+\overline{c})^2} \left(-\frac{2k-1}{\overline{c}} - \frac{(6-2k)\overline{c}}{1+\overline{c}^2} - \frac{2}{c-\overline{c}} + \frac{2}{c+\overline{c}} \right). \end{aligned}$$

Since

$$(c-\overline{c})^2(c+\overline{c})^2 = -\frac{4a^2}{(1+a^2)^2}\,, \tag{3.15}$$

{eq3.15:v

we have

$$\begin{aligned} I_{1k} = &-\frac{ic^{2k}(1+c^2)^{3-k}}{8a^2c^2}\left(\frac{(6-2k)c}{1+c^2}+\frac{2k-1}{c}-\frac{4c}{c^2-\overline{c}^2}\right) \\ &-\frac{i\overline{c}^{2k}(1+\overline{c}^2)^{3-k}}{8a^2\overline{c}^2}\left(-\frac{(6-2k)\overline{c}}{1+\overline{c}^2}-\frac{2k-1}{\overline{c}}-\frac{4\overline{c}}{c^2-\overline{c}^2}\right). \end{aligned}$$

Taking into account the equalities $1+c^2=-iac^2$, and $1+\overline{c}^2=ia\overline{c}^2$, we obtain

$$I_{1k} = \frac{3-k}{4a^k i^{k+1}}(\overline{c}^3-(-1)^k c^3) + \frac{(2k-1)a^{1-k}}{8i^k}((-1)^k c^3+\overline{c}^3) + \frac{a^{1-k}(\overline{c}^5-(-1)^k c^5)}{2i^k(c^2-\overline{c}^2)}\,.$$

Thus,

$$\begin{aligned} I_{10} &= \frac{3}{2}q(q^2-3p^2) - \frac{a}{4p}(6p^4-13p^2q^2+q^4), \\ I_{11} &= -\frac{p(p^2-3q^2)}{a} - \frac{p^4-7p^2q^2+4q^4}{4q}\,. \end{aligned}$$

Substituting the expressions p and q from (3.14), we find that

$$I_{10} = \frac{\sqrt{\sqrt{1+a^2}+1}}{2\sqrt{2}(1+a^2)^{3/2}}(3a^2-\sqrt{1+a^2}+5), \qquad I_{11} = \frac{a^2+\sqrt{1+a^2}+3}{4\sqrt{2}\,(1+a^2)^{3/2}\sqrt{\sqrt{1+a^2}+1}}\,.$$

Let us now calculate the second integral:

$$\begin{aligned} I_{2k} &= \frac{16}{\pi(1+a^2)^2}\int_{\mathbb{R}} \frac{u^{4+2k}(1+u^2)^{1-k}}{(u-c)^2(u+c)^2(u-\overline{c})^2(u+\overline{c})^2}\,du \\ &= \frac{8ic^{2+2k}(1+c^2)^{1-k}}{(1+a^2)^2(c-\overline{c})^2(c+\overline{c})^2}\left(\frac{2(1-k)c}{1+c^2}+\frac{3+2k}{c}-\frac{2}{c-\overline{c}}-\frac{2}{c+\overline{c}}\right) \\ &\quad + \frac{8i\overline{c}^{2+2k}(1+\overline{c}^2)^{1-k}}{(1+a^2)^2(c-\overline{c})^2(c+\overline{c})^2}\left(-\frac{2(1-k)\overline{c}}{1+\overline{c}^2}-\frac{3+2k}{\overline{c}}-\frac{2}{c-\overline{c}}+\frac{2}{c+\overline{c}}\right). \end{aligned}$$

Using relation (3.15), we obtain

$$\begin{aligned} I_{2k} = &-\frac{2i}{a^2}c^{2+2k}(1+c^2)^{1-k}\left(\frac{2(1-k)c}{1+c^2}+\frac{3+2k}{c}-\frac{4c}{c^2-\overline{c}^2}\right) \\ &-\frac{2i}{a^2}\overline{c}^{2+2k}(1+\overline{c}^2)^{1-k}\left(-\frac{2(1-k)\overline{c}}{1+\overline{c}^2}-\frac{3+2k}{\overline{c}}-\frac{4\overline{c}}{\overline{c}^2-c^2}\right). \end{aligned}$$

Taking into account the equalities $1+c^2=-iac^2$, and $1+\overline{c}^2=ia\overline{c}^2$, we obtain

$$I_{2k} = \frac{4-4k}{i^{k+1}a^{2+k}}((-1)^k c^3-\overline{c}^3) - \frac{6+4k}{i^k a^{1+k}}((-1)^k c^3+\overline{c}^3) + \frac{8((-1)^k c^5-\overline{c}^5)}{i^k a^{1+k}(c^2-\overline{c}^2)}\,. \tag{3.16}$$

{eq3.16:v

Hence

$$\begin{aligned} I_{20} &= \frac{8q}{a^2}(3p^2-q^2) + \frac{4}{ap}(2p^4-p^2q^2+q^4), \\ I_{21} &= \frac{20q}{a^2}(3p^2-q^2) + \frac{4}{a^2q}(p^4-10p^2q^2+5q^4). \end{aligned}$$

The substitution of p and q from (3.14) yields

$$I_{20} = \frac{2^{3/2}(2+\sqrt{1+a^2}\,)}{\sqrt{\sqrt{1+a^2}+1}\,(1+a^2)^{3/2}}, \qquad I_{21} = \frac{2^{3/2}(2+3\sqrt{1+a^2}\,)}{(\sqrt{1+a^2}+1)^{3/2}(1+a^2)^{3/2}}.$$

First, consider the case $k=0$. Then system (3.13) is of the form

$$\frac{\sqrt{\sqrt{1+a^2}+1}}{2(2(1+a^2))^{3/2}}(3a^2-\sqrt{1+a^2}+5) = h\delta^2\widehat{\lambda}_1^2,$$

$$\frac{8(2+\sqrt{1+a^2}\,)}{\sqrt{\sqrt{1+a^2}+1}\,(2(1+a^2))^{3/2}} = h^5\widehat{\lambda}_1^2.$$

Setting $d=\sqrt{a^2+1}$ we see that d satisfies Eq. (2.2), where $k=0$:

$$\frac{(d+1)(3d^2-d+2)}{d+2} = \frac{16\delta^2}{h^4}.$$

It is readily verified that, for $d>1$, this equation has a unique solution for all $\delta > h^2/\sqrt{6}$ (it can be found explicitly by using the well-known formulas for the roots of the cubic equation). For $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$, we obtain

$$\widehat{\lambda}_1 = \frac{2^{3/4}(2+d)^{1/2}}{(d+1)^{1/4}d^{3/2}h^{5/2}}, \qquad \widehat{\lambda}_2 = \frac{a^2h^4\widehat{\lambda}_1}{16} = \frac{(d^2-1)h^4\widehat{\lambda}_1}{16}.$$

Therefore, for $\delta > h^2/\sqrt{6}$, we have

$$E(0,2,h,\delta) = \widehat{\lambda}_1\delta^2+\widehat{\lambda}_2 = \widehat{\lambda}_1\left(\delta^2+\frac{(d^2-1)h^4}{16}\right) = \frac{(d+1)^{3/4}\sqrt{d}}{2^{5/4}\sqrt{d+2}}h^{3/2}.$$

Let us now construct the optimal recovery method x_0. We have

$$F\widehat{x}(t) = \frac{h^3}{\widehat{\lambda}_1h^4+4\widehat{\lambda}_2(1-\cos t)^2} = \frac{h^3}{4\widehat{\lambda}_2}\frac{1}{\lambda^2+(1-\cos t)^2}$$
$$= \frac{ih^3}{8\lambda\widehat{\lambda}_2}\left(\frac{1}{1+i\lambda-\cos t}-\frac{1}{1-i\lambda-\cos t}\right),$$

where

$$\lambda = \frac{h^2}{2}\sqrt{\frac{\widehat{\lambda}_1}{\widehat{\lambda}_2}} = \frac{2}{\sqrt{d^2-1}}.$$

We choose $|\mu|<1$ from the condition

$$\frac{1+\mu^2}{2\mu} = 1+i\lambda,$$

obtaining

$$\mu = \left(1-\sqrt{\frac{2}{d+1}}\right)\left(1-i\sqrt{\frac{2}{d-1}}\right).$$

Using (3.12), we find that

$$\frac{1}{1+i\lambda-\cos t}-\frac{1}{1-i\lambda-\cos t} = \sum_{j\in\mathbb{Z}}\left(\frac{2\mu}{1-\mu^2}\mu^{|j|}-\frac{2\overline{\mu}}{1-\overline{\mu}^2}\,\overline{\mu}^{|j|}\right)e^{ijt}.$$

Therefore,

$$\widehat{x}_j = \frac{ih^3}{8\lambda\widehat{\lambda}_2}\left(\frac{2\mu}{1-\mu^2}\mu^{|j|}-\frac{2\overline{\mu}}{1-\overline{\mu}^2}\,\overline{\mu}^{|j|}\right) = -\frac{4}{h\sqrt{d^2-1}\,\widehat{\lambda}_1}\,\mathrm{Im}\,\frac{\mu^{|j|+1}}{1-\mu^2}.$$

The case $k=1$ is treated in a similar way. The theorem is proved. ☐

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