

Optimal recovery of a solution of a system of linear differential equations from initial information given with a random error

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Abstract. The problem of the optimal recovery of a solution of a system of linear differential equations from initial information containing a random error is considered. Optimal methods are searched for among all possible (not necessarily linear) recovery methods. Depending on the given variance of random errors, the optimal recovery methods constructed in the paper, which turn out to be linear, use only part of the available information.

Bibliography: 17 titles.

Keywords: optimal recovery, systems of linear differential equations, information with random errors.

§ 1. Introduction

In the general statement of the recovery problem one must find the values of a fixed functional or operator acting on some classes of functions from incomplete information about these values. Classes are usually defined in terms of the smoothness or analyticity properties of functions forming them. Usually, local or individual information consists of some characteristic of a function which are available to us (the values of the function at some point, its moments, Fourier or Taylor coefficients, Fourier transform and so on). This information can be prescribed with an error, deterministic or random.

To this date, optimal methods have been found for various recovery problems in a considerable number of papers. Problems with deterministic errors were considered, for instance, in [1]–[9].

Problems with random errors were investigated in [10]–[16]. In [11] recovery methods were estimated on the basis of linear functionals, and in [12] an estimate for a nonlinear recovery method was found in terms of estimates for linear methods.

The problem of estimating the error of a method of recovery from a random variable with normal distribution was considered in [13], where inequalities for the minimax nonlinear risk were obtained.

For a solution of a system of linear homogeneous equations the recovery problem was considered in [17], but there the error in the data was deterministic.

AMS 2020 *Mathematics Subject Classification*. Primary 34A45, 41A65, 42A05; Secondary 41A46.

This paper is concerned with the construction of optimal recovery methods for a solution of a system of linear homogeneous equations from initial information known with a random error. We reproduce the statement of the problem from [14] and use a number of ideas from there to prove our general result in the final-dimensional case. We consider various forms of the prescription of information: the problem is solved under the assumptions that the initial point lies in an ellipsoid and its coordinates at the initial moment of time are known with a random error. We must recover the solution at time $\tau > 0$. We also consider the problem when a solution is known with a random error at some moment of time $t = T_1$, and we must recover it at another moment of time τ , $0 < \tau < T_1$.

The general result is also applied to the problem of the recovery of the k th derivative of a trigonometric polynomial from its coefficients, which are known with a random error.

In these problems we do not limit ourselves to random variables with normal distribution. We consider arbitrary distributions of a random vector with fixed expectation and a fixed estimate for variance. As in problems with deterministic error, here we discover such phenomena as the linearity of the optimal method and the opportunity to use only part of the information available for measurements.

§ 2. Statement of the optimal recovery problem for a solution of a system of linear differential equations

Consider the Cauchy problem for a system of linear differential equations

$$\begin{cases} \frac{dx}{dt} = Ax, \\ x(0) = x_0, \end{cases} \quad (2.1)$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, and

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad a_{ij} \in \mathbb{R}.$$

Assume that the matrix A is selfadjoint, and let

$$\mu_1, \mu_2, \dots, \mu_n$$

be its eigenvalues. Let $\{e_j\}_{j=1}^n$ denote the orthonormal basis of eigenvectors corresponding to the eigenvalues μ_j , $j = 1, \dots, n$. Let

$$x_0 = \sum_{j=1}^n x_j e_j.$$

Then the solution of (2.1) can be expressed by

$$x(t) = \sum_{j=1}^n e^{\mu_j t} x_j e_j.$$

Assume that we know the coordinate of the initial point x_0 with a random error. Moreover, we know some ellipsoid containing the point x_0 . We must recover the solution at time $\tau > 0$.

We turn to the precise statement of the problem. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ set

$$W = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n \nu_j x_j^2 \leq 1 \right\}, \quad Tx = (e^{\mu_1 \tau} x_1, \dots, e^{\mu_n \tau} x_n) \quad \text{and} \quad Ix = (x_1, \dots, x_n).$$

Fix $\delta > 0$, and for each $x \in W$ consider the set of random vectors

$$Y_\delta(x) = \{ y = (y_1, \dots, y_n) : \mathbf{E}(y) = Ix, \mathbf{D}(y_j) \leq \delta^2, j = 1, \dots, n \}.$$

Let l_2^n denote the space of vectors $x = (x_1, \dots, x_n)$ with the norm

$$\|x\|_{l_2^n} = \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2}.$$

A recovery method assigns to a random vector $y \in Y_\delta(x)$ an element of l_2^n , which is regarded as an approximation of Tx . The error of a recovery method $\varphi: \mathbb{R}^n \rightarrow l_2^n$ is the quantity

$$e(T, W, I, \delta, \varphi) = \left(\sup_{x \in W, y \in Y_\delta(x)} \mathbf{E}(\|Tx - \varphi(y)\|_{l_2^n}^2) \right)^{1/2}$$

(we consider only measurable maps φ). The problem consists in finding the optimal recovery error

$$E(T, W, I, \delta) = \inf_{\varphi: \mathbb{R}^n \rightarrow l_2^n} e(T, W, I, \delta, \varphi)$$

and a method delivering the infimum, which is called an optimal method.

To solve this problem we consider a more general problem, solve it and apply the result obtained to the original problem.

§ 3. General result

Let X be a linear space, Z be a normed linear space and $T: X \rightarrow Z$ be a linear operator. We must recover the values of T on a certain set (class) $W \subset X$ from values of a linear operator $I: X \rightarrow \mathbb{R}^n$, which are given with a random error. For each $x \in W$ and $\delta > 0$ we consider the set of random vectors

$$Y_\delta(x) = \{ y = (y_1, \dots, y_n) : \mathbf{E}(y) = Ix, \mathbf{D}(y_j) \leq \delta^2, j = 1, \dots, n \}$$

and, similarly to § 2, define the error of a recovery method $\varphi: \mathbb{R}^n \rightarrow Z$ by

$$e(T, W, I, \delta, \varphi) = \left(\sup_{x \in W, y \in Y_\delta(x)} \mathbf{E}(\|Tx - \varphi(y)\|_Z^2) \right)^{1/2}$$

(only measurable methods φ are considered). The problem consists in finding an optimal recovery method (if it exists) and the optimal recovery error

$$E(T, W, I, \delta) = \inf_{\varphi: \mathbb{R}^n \rightarrow Z} e(T, W, I, \delta, \varphi). \quad (3.1)$$

Set

$$W = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n \nu_j |x_j|^2 \leq 1 \right\},$$

where $\nu_j > 0$, $j = 1, \dots, n$. Consider the linear operators $T: \mathbb{R}^n \rightarrow l_2^n$ and $I: \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$Tx = (\mu_1 x_1, \dots, \mu_n x_n) \quad \text{and} \quad Ix = (x_1, \dots, x_n),$$

$|\mu_j| > 0$, $j = 1, \dots, n$.

We introduce the notation

$$\gamma_j = \frac{\sqrt{\nu_j}}{|\mu_j|}, \quad j = 1, \dots, n, \quad \text{and} \quad \xi_j = \left(\sum_{k=1}^j \nu_k \left(\frac{\gamma_j}{\gamma_k} - 1 \right) \right)^{1/2}, \quad j = 1, \dots, n.$$

Let $\gamma_1 \leq \dots \leq \gamma_n$. Then it is easy to see that $0 = \xi_1 \leq \dots \leq \xi_n$.

Theorem 1. *Let $1/\delta \in (\xi_s, \xi_{s+1}]$ for some $1 \leq s \leq n-1$ or $1/\delta \in (\xi_n, +\infty)$ (and then set $s = n$). Then*

$$E(T, W, I, \delta) = \delta \left(\sum_{k=1}^s |\mu_k|^2 \left(1 - \frac{\gamma_k(1 - c_1)}{\gamma_1} \right) \right)^{1/2},$$

where

$$c_1 = 1 - \frac{\delta^2 \gamma_1 \sum_{k=1}^s (\nu_k / \gamma_k)}{1 + \delta^2 \sum_{k=1}^s \nu_k}, \quad (3.2)$$

and the method

$$\varphi(y) = \sum_{k=1}^s \left(1 - \frac{\gamma_k(1 - c_1)}{\gamma_1} \right) \mu_k y_k e_k,$$

where $\{e_k\}$ is the standard basis in l_2^n , is optimal.

Proof. 1. *A lower bound.* Fix some $\tau = (\tau_1, \dots, \tau_n) \in W$ such that

$$\tau_1 \geq \dots \geq \tau_n > 0.$$

Consider the set

$$B = \{x \in l_2^n : x_j = \pm \tau_j, \quad j = 1, \dots, n\}.$$

It is obvious that $B \subset W$. Set

$$p_j = \frac{\delta^2}{\delta^2 + \tau_j^2}, \quad j = 1, \dots, n.$$

By the monotonicity condition on the τ_j we have

$$0 < p_1 \leq \dots \leq p_n < 1.$$

An arbitrary $x \in B$ is expressed in the form

$$x = \sum_{j=1}^n s_j(x) \tau_j e_j,$$

where $s_j(x) \in \{-1, 1\}$. We define the distribution $\eta(x)$ for $x \in B$ by

$$\eta(x) = \begin{cases} 0 & \text{with probability } p_1, \\ \frac{s_1(x) \tau_1}{1 - p_1} e_1 & \text{with probability } p_2 - p_1, \\ \sum_{j=1}^2 \frac{s_j(x) \tau_j}{1 - p_j} e_j & \text{with probability } p_3 - p_2, \\ \dots & \dots \\ \sum_{j=1}^{n-1} \frac{s_j(x) \tau_j}{1 - p_j} e_j & \text{with probability } p_n - p_{n-1}, \\ \sum_{j=1}^n \frac{s_j(x) \tau_j}{1 - p_j} e_j & \text{with probability } 1 - p_n. \end{cases}$$

Thus, the components of $\eta(x) = (\eta_1(x), \dots, \eta_n(x))$ has the following distributions:

$$\eta_j(x) = \begin{cases} 0 & \text{with probability } p_j, \\ \frac{s_j(x) \tau_j}{1 - p_j} & \text{with probability } 1 - p_j, \end{cases} \quad j = 0, 1, \dots, n.$$

It is easy to see that $\mathbf{E}(\eta_j(x)) = x_j$, $j = 1, \dots, n$. In addition,

$$\mathbf{D}(\eta_j(x)) = (1 - p_j) \frac{\tau_j^2}{(1 - p_j)^2} - \tau_j^2 = \delta^2, \quad j = 1, \dots, n.$$

Therefore, $\eta(x) \in Y_\delta(x)$ for $x \in B$.

Let φ be a recovery method. Since the cardinality of B is 2^n , we obtain

$$\begin{aligned} e^2(T, W, I, \delta, \varphi) &\geq \sup_{x \in B} \mathbf{E} \|Tx - \varphi(\eta(x))\|_{l_2^n}^2 \\ &= \sup_{x \in B} \left(\sum_{j=1}^{n+1} (p_j - p_{j-1}) \left\| Tx - \varphi \left(\sum_{k=1}^{j-1} \frac{s_k(x) \tau_k}{1 - p_k} e_k \right) \right\|_{l_2^n}^2 \right) \\ &\geq \frac{1}{2^n} \sum_{x \in B} \left(\sum_{j=1}^{n+1} (p_j - p_{j-1}) \left\| Tx - \varphi \left(\sum_{k=1}^{j-1} \frac{s_k(x) \tau_k}{1 - p_k} e_k \right) \right\|_{l_2^n}^2 \right) \\ &= \frac{1}{2^n} \sum_{j=1}^{n+1} (p_j - p_{j-1}) \sum_{x \in B} \left\| Tx - \varphi \left(\sum_{k=1}^{j-1} \frac{s_k(x) \tau_k}{1 - p_k} e_k \right) \right\|_{l_2^n}^2; \end{aligned} \quad (3.3)$$

here $p_0 = 0$ and $p_{n+1} = 1$. Set

$$B_{s_1, \dots, s_{j-1}} = \{x \in B: s_1(x) = s_1, \dots, s_{j-1}(x) = s_{j-1}\},$$

$j = 1, \dots, n+1$ (for $j = 1$ this set coincides with B). Then

$$\begin{aligned} & \frac{p_j - p_{j-1}}{2^n} \sum_{x \in B} \left\| Tx - \varphi \left(\sum_{k=1}^{j-1} \frac{s_k(x) \tau_k}{1 - p_k} e_k \right) \right\|_{l_2^n}^2 \\ &= \frac{p_j - p_{j-1}}{2^n} \sum_{s_1, \dots, s_{j-1}} \sum_{x \in B_{s_1, \dots, s_{j-1}}} \left\| Tx - \varphi \left(\sum_{k=1}^{j-1} \frac{s_k \tau_k}{1 - p_k} e_k \right) \right\|_{l_2^n}^2. \end{aligned}$$

If $x \in B_{s_1, \dots, s_{j-1}}$, then

$$x = \sum_{k=1}^{j-1} s_k \tau_k e_k + z(x), \quad \text{where } z(x) = \sum_{k=j}^n s_k(x) \tau_k e_k.$$

Moreover, together with each element

$$\sum_{k=1}^{j-1} s_k \tau_k e_k + z(x) \in B_{s_1, \dots, s_{j-1}}$$

the set $B_{s_1, \dots, s_{j-1}}$ contains also the element

$$\sum_{k=1}^{j-1} s_k \tau_k e_k - z(x).$$

Thus,

$$\begin{aligned} & \frac{p_j - p_{j-1}}{2^n} \sum_{s_1, \dots, s_{j-1}} \sum_{x \in B_{s_1, \dots, s_{j-1}}} \left\| Tx - \varphi \left(\sum_{k=1}^{j-1} \frac{s_k \tau_k}{1 - p_k} e_k \right) \right\|_{l_2^n}^2 \\ &= \frac{p_j - p_{j-1}}{2^n} \sum_{s_1, \dots, s_{j-1}} \sum_{x \in B_{s_1, \dots, s_{j-1}}} \left\| T \left(\sum_{k=1}^{j-1} s_k \tau_k e_k + z(x) \right) - \varphi \left(\sum_{k=1}^{j-1} \frac{s_k \tau_k}{1 - p_k} e_k \right) \right\|_{l_2^n}^2 \\ &= \frac{p_j - p_{j-1}}{2^n} \sum_{s_1, \dots, s_{j-1}} \sum_{x \in B_{s_1, \dots, s_{j-1}}} \left\| T \left(\sum_{k=1}^{j-1} s_k \tau_k e_k \right) + Tz(x) - \varphi \left(\sum_{k=1}^{j-1} \frac{s_k \tau_k}{1 - p_k} e_k \right) \right\|_{l_2^n}^2 \\ &= \frac{p_j - p_{j-1}}{2^{n+1}} \sum_{s_1, \dots, s_{j-1}} \sum_{x \in B_{s_1, \dots, s_{j-1}}} \left(\left\| T \left(\sum_{k=1}^{j-1} s_k \tau_k e_k \right) + Tz(x) \right. \right. \\ & \quad \left. \left. - \varphi \left(\sum_{k=1}^{j-1} \frac{s_k \tau_k}{1 - p_k} e_k \right) \right\|_{l_2^n}^2 + \left\| T \left(\sum_{k=1}^{j-1} s_k \tau_k e_k \right) - Tz(x) - \varphi \left(\sum_{k=1}^{j-1} \frac{s_k \tau_k}{1 - p_k} e_k \right) \right\|_{l_2^n}^2 \right) \\ &\geq \frac{p_j - p_{j-1}}{2^n} \sum_{s_1, \dots, s_{j-1}} \sum_{x \in B_{s_1, \dots, s_{j-1}}} \|Tz(x)\|_{l_2^n}^2 = \frac{p_j - p_{j-1}}{2^n} \sum_{x \in B} \|Tz(x)\|_{l_2^n}^2 \\ &= (p_j - p_{j-1}) \sum_{k=j}^n |\mu_k|^2 \tau_k^2. \end{aligned}$$

Substituting this into (3.3) we obtain

$$\begin{aligned} e^2(T, W, I, \delta, \varphi) &\geq \sum_{j=1}^{n+1} (p_j - p_{j-1}) \sum_{k=j}^n |\mu_k|^2 \tau_k^2 \\ &= \sum_{j=1}^n \left(p_j \sum_{k=j}^n |\mu_k|^2 \tau_k^2 - p_j \sum_{k=j+1}^n |\mu_k|^2 \tau_k^2 \right) = \sum_{j=1}^n p_j |\mu_j|^2 \tau_j^2 = \sum_{j=1}^n \frac{\delta^2}{\delta^2 + \tau_j^2} |\mu_j|^2 \tau_j^2. \end{aligned}$$

Since the method φ can be arbitrary, we have the estimate

$$E^2(T, W, I, \delta) \geq \sup_{\substack{\tau \in W \\ \tau_1 \geq \dots \geq \tau_n > 0}} \sum_{j=1}^n \frac{\delta^2}{\delta^2 + \tau_j^2} |\mu_j|^2 \tau_j^2. \quad (3.4)$$

Consider a vector $\tau = (\tau_1, \dots, \tau_k, 0, \dots, 0) \in W$ such that $\tau_1 \geq \dots \geq \tau_k > 0$, $1 \leq k < n$. For sufficiently small $\varepsilon > 0$ set $\tau_\varepsilon = (\tau_1(\varepsilon), \dots, \tau_n(\varepsilon))$, where

$$\begin{aligned} \tau_j(\varepsilon) &= \begin{cases} \sqrt{\tau_j^2 - \varepsilon}, & 1 \leq j \leq k, \\ C\sqrt{\varepsilon}, & k+1 \leq j \leq n, \end{cases} \\ C &= \left(\frac{\sum_{j=1}^k \nu_j}{\sum_{j=k+1}^n \nu_j} \right)^{1/2}. \end{aligned}$$

Then

$$\sum_{j=1}^n \nu_j \tau_j^2(\varepsilon) = \sum_{j=1}^k \nu_j \tau_j^2 - \varepsilon \sum_{j=1}^k \nu_j + C^2 \varepsilon \sum_{j=k+1}^n \nu_j = \sum_{j=1}^k \nu_j \tau_j^2 \leq 1.$$

Thus, $\tau_\varepsilon \in W$. For $\varepsilon < \tau_k^2/(1 + C^2)$ we have

$$\sqrt{\tau_k^2 - \varepsilon} > C\sqrt{\varepsilon}.$$

Hence for such ε

$$\tau_1(\varepsilon) \geq \dots \geq \tau_n(\varepsilon) > 0.$$

It follows from (3.4) that

$$E^2(T, W, I, \delta) \geq \sum_{j=1}^n \frac{\delta^2}{\delta^2 + \tau_j^2(\varepsilon)} |\mu_j|^2 \tau_j^2(\varepsilon).$$

Taking the limit as $\varepsilon \rightarrow 0$ we obtain

$$E^2(T, W, I, \delta) \geq \sum_{j=1}^k \frac{\delta^2}{\delta^2 + \tau_j^2} |\mu_j|^2 \tau_j^2.$$

Thus,

$$E^2(T, W, I, \delta) \geq \sup_{\substack{\tau \in W \\ \tau_1 \geq \dots \geq \tau_n \geq 0}} \sum_{j=1}^n \frac{\delta^2}{\delta^2 + \tau_j^2} |\mu_j|^2 \tau_j^2. \quad (3.5)$$

2. *An upper bound.* We find the error of methods of the form

$$\varphi(y) = \sum_{j=1}^n \alpha_j y_j e_j.$$

Set $z(x) = y(x) - Ix$. Then $\mathbf{E}(z(x)) = 0$ and $\mathbf{D}(z_j(x)) \leq \delta^2$, $j = 1, \dots, n$. We have

$$\begin{aligned} e^2(T, W, I, \delta, \varphi) &= \sup_{\substack{x \in W \\ y(x) \in Y_\delta(x)}} \mathbf{E}(\|Tx - \varphi(y(x))\|_{l_2^n}^2) \\ &= \sup_{\substack{x \in W \\ y(x) \in Y_\delta(x)}} \mathbf{E}(\|Tx - \varphi(Ix) - \varphi(z(x))\|_{l_2^n}^2) \\ &= \sup_{\substack{x \in W \\ y(x) \in Y_\delta(x)}} (\|Tx - \varphi(Ix)\|_{l_2^n}^2 + \mathbf{E}(\|\varphi(z(x))\|_{l_2^n}^2) - 2\mathbf{E}(\varphi(z(x)), Tx - \varphi(Ix))); \end{aligned}$$

here (\cdot, \cdot) is the inner product in l_2^n . It follows from the form of φ that

$$\begin{aligned} \mathbf{E}(\varphi(z(x)), Tx - \varphi(Ix)) &= \mathbf{E}\left(\sum_{j=1}^n \alpha_j z_j(x) e_j, Tx - \varphi(Ix)\right) \\ &= \sum_{j=1}^n (e_j, Tx - \varphi(Ix)) \alpha_j \mathbf{E}(z_j(x)) = 0. \end{aligned}$$

Since

$$\mathbf{E}(\|\varphi(z(x))\|_{l_2^n}^2) = \mathbf{E}\left(\sum_{j=1}^n |\alpha_j|^2 |z_j(x)|^2\right) = \sum_{j=1}^n |\alpha_j|^2 \mathbf{D}(z_j(x)),$$

we have

$$\begin{aligned} e^2(T, W, I, \delta, \varphi) &= \sup_{\substack{x \in W \\ y(x) \in Y_\delta(x)}} \left(\|Tx - \varphi(Ix)\|_{l_2^n}^2 + \sum_{j=1}^n |\alpha_j|^2 \mathbf{D}(z_j(x)) \right) \\ &= \sup_{x \in W} \|Tx - \varphi(Ix)\|_{l_2^n}^2 + \delta^2 \sum_{j=1}^n |\alpha_j|^2. \end{aligned}$$

Consider the extremal problem

$$\|Tx - \varphi(Ix)\|_{l_2^n}^2 \rightarrow \max, \quad x \in W.$$

We can write it as

$$\sum_{j=1}^n |\mu_j - \alpha_j|^2 |x_j|^2 \rightarrow \max, \quad \sum_{j=1}^n \nu_j |x_j|^2 \leq 1.$$

From the inequality

$$\begin{aligned} \sum_{j=1}^n |\mu_j - \alpha_j|^2 |x_j|^2 &= \sum_{j=1}^n \frac{|\mu_j - \alpha_j|^2}{\nu_j} \nu_j |x_j|^2 \\ &\leq \max \left\{ \frac{|\mu_1 - \alpha_1|^2}{\nu_1}, \dots, \frac{|\mu_n - \alpha_n|^2}{\nu_n} \right\} \sum_{j=1}^n \nu_j |x_j|^2 \end{aligned}$$

we deduce that

$$\sup_{x \in W} \|Tx - \varphi(Ix)\|_{l_n^2}^2 \leq \max \left\{ \frac{|\mu_1 - \alpha_1|^2}{\nu_1}, \dots, \frac{|\mu_n - \alpha_n|^2}{\nu_n} \right\}.$$

Thus,

$$e^2(T, W, I, \delta, \varphi) \leq \max \left\{ \frac{|\mu_1 - \alpha_1|^2}{\nu_1}, \dots, \frac{|\mu_n - \alpha_n|^2}{\nu_n} \right\} + \delta^2 \sum_{j=1}^n |\alpha_j|^2.$$

Set

$$c_j = \frac{\alpha_j}{\mu_j}, \quad j = 1, \dots, n.$$

Then the error of the method φ satisfies the inequality

$$e^2(T, W, I, \delta, \varphi) \leq \max \left\{ \frac{|1 - c_1|^2}{\gamma_1^2}, \dots, \frac{|1 - c_n|^2}{\gamma_n^2} \right\} + \delta^2 \sum_{j=1}^n |\mu_j|^2 |c_j|^2.$$

2.1. Let $1/\delta \in (\xi_s, \xi_{s+1}]$ for some $1 \leq s \leq n-1$. Then it is easy to show that

$$\frac{1}{\gamma_{s+1}} \leq \frac{\delta^2 \sum_{k=1}^s (\nu_k / \gamma_k)}{1 + \delta^2 \sum_{k=1}^s \nu_k} < \frac{1}{\gamma_s}.$$

Defining c_1 by (3.2) we obtain

$$\frac{1}{\gamma_{s+1}} \leq \frac{1 - c_1}{\gamma_1} < \frac{1}{\gamma_s}.$$

Let

$$c_k = 1 - \gamma_k \frac{1 - c_1}{\gamma_1}, \quad k = 2, \dots, s, \quad c_k = 0, \quad k = s+1, \dots, n.$$

Then we have

$$\frac{(1 - c_k)^2}{\gamma_k^2} = \frac{(1 - c_1)^2}{\gamma_1^2}, \quad k = 2, \dots, s.$$

For $k \geq s+1$

$$\frac{(1 - c_k)^2}{\gamma_k^2} = \frac{1}{\gamma_k^2} \leq \frac{1}{\gamma_{s+1}^2} \leq \frac{(1 - c_1)^2}{\gamma_1^2}.$$

Therefore,

$$\max \left\{ \frac{|1 - c_1|^2}{\gamma_1^2}, \dots, \frac{|1 - c_n|^2}{\gamma_n^2} \right\} = \frac{(1 - c_1)^2}{\gamma_1^2}. \quad (3.6)$$

Hence

$$\begin{aligned}
e^2(T, W, I, \delta, \varphi) &\leq \frac{(1 - c_1)^2}{\gamma_1^2} + \delta^2 \sum_{k=1}^s |\mu_k|^2 c_k^2 \\
&= \frac{(1 - c_1)^2}{\gamma_1^2} + \delta^2 \sum_{k=1}^s |\mu_k|^2 ((1 - c_k)^2 - (1 - c_k) + c_k) \\
&= \frac{(1 - c_1)^2}{\gamma_1^2} + \delta^2 \sum_{k=1}^s |\mu_k|^2 \frac{\gamma_k^2}{\gamma_1^2} (1 - c_1)^2 \\
&\quad - \delta^2 \sum_{k=1}^s |\mu_k|^2 \frac{\gamma_k}{\gamma_1} (1 - c_1) + \delta^2 \sum_{k=1}^s |\mu_k|^2 c_k \\
&= \delta^2 \sum_{k=1}^s |\mu_k|^2 c_k + \frac{1 - c_1}{\gamma_1^2} \left(1 - c_1 + \delta^2 \sum_{k=1}^s \nu_k (1 - c_1) - \delta^2 \gamma_1 \sum_{k=1}^s \frac{\nu_k}{\gamma_k} \right) \\
&= \delta^2 \sum_{k=1}^s |\mu_k|^2 c_k + \frac{1 - c_1}{\gamma_1^2} \left((1 - c_1) \left(1 + \delta^2 \sum_{k=1}^s \nu_k \right) - \delta^2 \gamma_1 \sum_{k=1}^s \frac{\nu_k}{\gamma_k} \right) \\
&= \delta^2 \sum_{k=1}^s |\mu_k|^2 c_k. \tag{3.7}
\end{aligned}$$

Consider a vector $\hat{\tau} \in l_2^n$ such that

$$\hat{\tau}_k^2 = \delta^2 \left(\frac{\gamma_1}{(1 - c_1)\gamma_k} - 1 \right), \quad k = 1, \dots, s, \quad \hat{\tau}_k = 0, \quad k = s + 1, \dots, n.$$

Then we have

$$\begin{aligned}
\sum_{k=1}^n \nu_k \hat{\tau}_k^2 &= \delta^2 \sum_{k=1}^s \nu_k \left(\frac{\gamma_1}{(1 - c_1)\gamma_k} - 1 \right) = \frac{\delta^2 \gamma_1}{1 - c_1} \sum_{k=1}^s \frac{\nu_k}{\gamma_k} - \delta^2 \sum_{k=1}^s \nu_k \\
&= \left(1 + \delta^2 \sum_{k=1}^s \nu_k \right) - \delta^2 \sum_{k=1}^s \nu_k = 1.
\end{aligned}$$

Thus, $\hat{\tau} \in W$. Substituting $\hat{\tau}$ into (3.5) we obtain

$$\begin{aligned}
E^2(T, W, I, \delta) &\geq \sum_{k=1}^s \frac{\delta^2 |\mu_k|^2 \hat{\tau}_k^2}{\delta^2 + \hat{\tau}_k^2} = \sum_{k=1}^s \frac{\delta^4 |\mu_k|^2 (\gamma_1 / ((1 - c_1)\gamma_k) - 1)}{\delta^2 \gamma_1 / ((1 - c_1)\gamma_k)} \\
&= \delta^2 \sum_{k=1}^s |\mu_k|^2 \left(1 - \frac{(1 - c_1)\gamma_k}{\gamma_1} \right) = \delta^2 \sum_{k=1}^s |\mu_k|^2 c_k \geq e^2(T, W, I, \delta, \varphi).
\end{aligned}$$

Hence φ is an optimal method.

2.2. Now let $1/\delta > \xi_n$. Then

$$\frac{\delta^2 \sum_{k=1}^n (\nu_k / \gamma_k)}{1 + \delta^2 \sum_{k=1}^n \nu_k} < \frac{1}{\gamma_n}.$$

Set

$$c_1 = 1 - \gamma_1 \frac{e \delta^2 \sum_{k=1}^n (\nu_k / \gamma_k)}{1 + \delta^2 \sum_{k=1}^n \nu_k}.$$

Then

$$\frac{1 - c_1}{\gamma_1} < \frac{1}{\gamma_n}.$$

Let

$$c_k = 1 - \gamma_k \frac{1 - c_1}{\gamma_1}, \quad k = 2, \dots, n.$$

Then we have

$$\frac{(1 - c_k)^2}{\gamma_k^2} = \frac{(1 - c_1)^2}{\gamma_1^2}, \quad k = 2, \dots, n.$$

Thus, as in the previous case, we have equality (3.6). Repeating the calculations in (3.7) for $s = n$, we obtain

$$e^2(T, W, I, \delta, \varphi) \leq \delta^2 \sum_{k=1}^n |\mu_k|^2 c_k.$$

Consider the vector $\hat{\tau} \in l_2^n$ of the form

$$\hat{\tau}_k = \delta^2 \left(\frac{\gamma_1}{(1 - c_1)\gamma_k} - 1 \right), \quad k = 1, \dots, n.$$

Then

$$\begin{aligned} \sum_{k=1}^n \nu_k \hat{\tau}_k^2 &= \delta^2 \sum_{k=1}^n \nu_k \left(\frac{\gamma_1}{(1 - c_1)\gamma_k} - 1 \right) = \frac{\delta^2 \gamma_1}{1 - c_1} \sum_{k=1}^n \frac{\nu_k}{\gamma_k} - \delta^2 \sum_{k=1}^n \nu_k \\ &= \left(1 + \delta^2 \sum_{k=1}^n \nu_k \right) - \delta^2 \sum_{k=1}^n \nu_k = 1. \end{aligned}$$

Thus, $\hat{\tau} \in W$. Plugging $\hat{\tau}$ into (3.5) we have

$$\begin{aligned} E^2(T, W, I, \delta) &\geq \sum_{k=1}^n \frac{\delta^2 |\mu_k|^2 \hat{\tau}_k^2}{\delta^2 + \hat{\tau}_k^2} = \sum_{k=1}^n \frac{\delta^4 |\mu_k|^2 (\gamma_1 / ((1 - c_1)\gamma_k) - 1)}{\delta^2 \gamma_1 / ((1 - c_1)\gamma_k)} \\ &= \delta^2 \sum_{k=1}^n |\mu_k|^2 \left(1 - \frac{(1 - c_1)\gamma_k}{\gamma_1} \right) = \delta^2 \sum_{k=1}^n |\mu_k|^2 c_k \geq e^2(T, W, I, \delta, \varphi). \end{aligned}$$

Hence φ is an optimal method.

Theorem 1 is proved.

§ 4. Optimal recovery of a solution of a system of linear differential equations

Here we present the solution of the recovery problem for an initial point in an ellipsoid and then consider the special case of a ball. After that we consider the problem of recovery from the coefficients at time T and for the terminal point in an ellipsoid. Next we discuss the special case of a ball.

4.1. Recovering solutions of a linear differential equations from information with a random error for the initial moment of time. Consider the Cauchy problem for a system of homogeneous linear differential equations

$$\begin{cases} \frac{dx}{dt} = Ax, \\ x(0) = x_0, \end{cases} \quad (4.1)$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, and $A = (a_{ij}) \in \mathbb{R}$.

Let A be a selfadjoint matrix and

$$\mu_1 > \mu_2 > \cdots > \mu_n$$

be its eigenvalues. Let $\{e_j\}_{j=1}^n$ denote the orthonormal basis of eigenvectors corresponding to the eigenvalues μ_j , $j = 1, \dots, n$.

Let

$$x_0 = \sum_{j=1}^n x_j e_j.$$

Then the solution of (4.1) can be expressed in the form

$$x(t) = \sum_{j=1}^n e^{\mu_j t} x_j e_j.$$

Assume that we know the coordinates of the initial point x_0 with a random error. We also know an ellipsoid containing x_0 . We must recover the solution at time $\tau > 0$.

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ set

$$W = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n \nu_j x_j^2 \leq 1 \right\}, \quad Tx = (e^{\mu_1 \tau} x_1, \dots, e^{\mu_n \tau} x_n) \quad \text{and} \quad Ix = (x_1, \dots, x_n).$$

As in the general statement, a recovery method assigns to a random vector $y \in Y_\delta(x)$ an element of \mathbb{R}^n viewed as an approximation of Tx . Thus, the present recovery problem reduces to the one considered above. We use Theorem 1.

Set

$$\gamma_j = \frac{\sqrt{\nu_j}}{e^{\mu_j \tau}}, \quad j = 1, \dots, n, \quad \text{and} \quad \xi_j = \left(\sum_{k=1}^j \nu_k \left(\frac{\gamma_j}{\gamma_k} - 1 \right) \right)^{1/2}, \quad j = 1, \dots, n.$$

Theorem 2. Let $1/\delta \in (\xi_s, \xi_{s+1}]$ for some $1 \leq s \leq n-1$, or let $1/\delta \in (\xi_n, +\infty)$ (and then set $s = n$). Then

$$E(T, W, I, \delta) = \delta \left(\sum_{k=1}^s e^{2\mu_k \tau} \left(1 - \frac{\gamma_k(1-c_1)}{\gamma_1} \right) \right)^{1/2},$$

where

$$c_1 = 1 - \frac{\delta^2 \gamma_1 \sum_{k=1}^s (\nu_k / \gamma_k)}{1 + \delta^2 \sum_{k=1}^s \nu_k}, \quad (4.2)$$

and the method

$$\varphi(y) = \sum_{k=1}^s \left(1 - \frac{\gamma_k(1-c_1)}{\gamma_1} \right) e^{\mu_k \tau} y_k e_k,$$

is optimal.

Let A be a selfadjoint matrix,

$$\lambda_1 > \lambda_2 > \dots > \lambda_m$$

be its eigenvalues and r_k be the multiplicity of the eigenvalue λ_k , $k = 1, \dots, m$. Let $\{e_{kj}\}_{j=1}^{r_k}$ denote an orthonormal system of vectors corresponding to the eigenvalue λ_k . Then

$$e_{11}, \dots, e_{1r_1}, \dots, e_{m1}, \dots, e_{mr_m}$$

is an orthonormal basis of \mathbb{R}^n .

Let

$$x_0 = \sum_{k=1}^m \sum_{j=1}^{r_k} c_{kj} e_{kj}.$$

Then the solution of (4.1) has the expression

$$x(t) = \sum_{k=1}^m e^{\lambda_k t} \sum_{j=1}^{r_k} c_{kj} e_{kj}.$$

Now assume that at the initial moment of time the point x_0 lies in a ball of radius R :

$$\sum_{k=1}^m \sum_{j=1}^{r_k} x_{kj}^2 \leq R^2.$$

Then the problem of the recovery of the solution at time $\tau > 0$ reduces to the previous problem for

$$W = \left\{ x \in l_2^n : \sum_{k=1}^m \sum_{j=1}^{r_k} R^{-2} x_{kj}^2 \leq 1 \right\},$$

$$Tx = (e^{\lambda_1 \tau} x_{11}, \dots, e^{\lambda_1 \tau} x_{1r_1}, \dots, e^{\lambda_m \tau} x_{m1}, \dots, e^{\lambda_m \tau} x_{mr_m})$$

and

$$Ix = (x_{11}, \dots, x_{1r_1}, \dots, x_{m1}, \dots, x_{mr_m}).$$

Set

$$\xi_k = R^{-1} \left(\sum_{j=1}^k r_j (e^{(\lambda_j - \lambda_k)\tau} - 1) \right)^{1/2}, \quad k = 1, \dots, m.$$

Theorem 3. Let $1/\delta \in (\xi_s, \xi_{s+1}]$ for some $1 \leq s \leq m-1$, or let $1/\delta \in (\xi_m, +\infty)$ (and then set $s = m$). Then

$$E(T, W, I, \delta) = \delta \left(\sum_{k=1}^s e^{2\lambda_k \tau} r_k (1 - e^{(\lambda_1 - \lambda_k)\tau} (1 - c_1)) \right)^{1/2},$$

where

$$c_1 = 1 - \frac{\delta^2 R^{-2} e^{-\lambda_1 \tau} \sum_{k=1}^s r_k e^{\lambda_k \tau}}{1 + \delta^2 R^{-2} \sum_{k=1}^s r_k}, \quad (4.3)$$

and the method

$$\varphi(y) = \sum_{k=1}^s (e^{\lambda_k \tau} - e^{\lambda_1 \tau} (1 - c_1)) \sum_{j=1}^{\tau_k} y_{kj} e_{kj},$$

is optimal.

4.2. Recovering solutions of linear differential equations from information with a random error for time T_1 . Consider the Cauchy problem for a system of homogeneous linear differential equations

$$\begin{cases} \frac{dx}{dt} = Ax, \\ x(0) = x_0, \end{cases} \quad (4.4)$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, and $A = (a_{ij})$, $a_{ij} \in \mathbb{R}$.

Let A be a selfadjoint matrix and

$$\lambda_1 > \lambda_2 > \dots > \lambda_n$$

be its eigenvalues. Let $\{e_j\}_{j=1}^n$ denote the orthonormal basis of eigenvectors corresponding to the eigenvalues λ_j , $j = 1, \dots, n$.

Let

$$x_0 = \sum_{j=1}^n x_j e_j.$$

Then the solution of (4.4) has the expression

$$x(t) = \sum_{j=1}^n e^{\lambda_j t} x_j e_j.$$

Moreover, assume that at the initial moment of time x_0 lies in an ellipsoid

$$B = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n b_j x_j^2 \leq 1 \right\}.$$

We must recover the values of the solution at time τ , $0 < \tau < T_1$. Let x_j denote the coordinates of the solution at time T_1 . Then the condition that x_0 lies in an ellipsoid means that

$$\sum_{j=1}^n b_j e^{-2\lambda_j T_1} x_j^2 \leq 1.$$

Thus, our recovery problem reduces to the one considered above, for

$$W = \left\{ x \in l_2^n : \sum_{j=1}^n \nu_j x_j^2 \leq 1 \right\},$$

where $\nu_j = b_j e^{-2\lambda_j T_1}$, $j = 1, \dots, n$.

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ set

$$Tx = (e^{\lambda_1(T_1-\tau)} x_1, \dots, e^{\lambda_n(T_1-\tau)} x_n), \quad Ix = (x_1, \dots, x_n).$$

As in the general setting, each recovery method assigns to a random vector $y \in Y_\delta(x)$ an element of \mathbb{R}^n viewed as an approximation of Tx . To solve the problem stated we use Theorem 1.

Set

$$\gamma_j = \frac{\sqrt{\nu_j}}{e^{-\lambda_j(T_1-\tau)}} \quad \text{and} \quad \xi_j = \left(\sum_{k=1}^j \nu_k \left(\frac{\gamma_j}{\gamma_k} - 1 \right) \right)^{1/2}, \quad j = 1, \dots, n.$$

Assume that $\gamma_1 \leq \dots \leq \gamma_n$.

Theorem 4. *Let $1/\delta \in (\xi_s, \xi_{s+1}]$ for some $1 \leq s \leq n-1$, or let $1/\delta \in (\xi_n, +\infty)$ (and then set $s = n$). Then*

$$E(T, W, I, \delta) = \delta \left(\sum_{k=1}^s e^{-2\lambda_k(T_1-\tau)} \left(1 - \frac{\gamma_k}{\gamma_1} (1 - c_1) \right) \right)^{1/2},$$

where

$$c_1 = 1 - \frac{\delta^2 \gamma_1 \sum_{k=1}^s (\nu_k / \gamma_k)}{1 + \delta^2 \sum_{k=1}^s \nu_k}, \quad (4.5)$$

and the method

$$\varphi(y) = \sum_{k=1}^s \left(\left(1 - \frac{\gamma_k}{\gamma_1} (1 - c_1) \right) e^{-\lambda_k(T_1-\tau)} y_k e_k \right),$$

is optimal.

Consider the Cauchy problem for the system of linear differential equations

$$\begin{cases} \frac{dx}{dt} = Ax, \\ x(0) = x_0, \end{cases} \quad (4.6)$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, and $A = (a_{ij})$, $a_{ij} \in \mathbb{R}$.

Let the matrix A be selfadjoint,

$$\lambda_1 > \lambda_2 > \dots > \lambda_m$$

be its eigenvalues, and let r_k be the multiplicity of the eigenvalue λ_k , $k = 1, \dots, m$. Let $\{e_{kj}\}_{j=1}^{r_k}$ denote an orthonormal system of vectors corresponding to the eigenvalue λ_k . Then the vectors

$$e_{11}, \dots, e_{1r_1}, \dots, e_{m1}, \dots, e_{mr_m}$$

form an orthonormal basis of \mathbb{R}^n .

Let

$$x_0 = \sum_{k=1}^m \sum_{j=1}^{r_k} c_{kj} e_{kj}.$$

Then the solution (4.6) can be expressed in the form

$$x(t) = \sum_{k=1}^m e^{\lambda_k t} \sum_{j=1}^{r_k} c_{kj} e_{kj}.$$

Assume that we know the value, with some random error, of the solution of (4.6) at time $t = T_1$. As in the general case, a recovery method assigns to a random vector $y \in Y_\delta(x)$ an element of \mathbb{R}^n , set to be an approximation of Tx . Also assume that at the initial moment of time $\|x_0\| \leq R$ ($\|\cdot\|$ is the Euclidean norm in \mathbb{R}^n). We must recover the value of the solution at time τ , $0 < \tau < T_1$. If x_{kj} denote the coordinates of the solution at time T , then the condition $\|x_0\| \leq R$ means that

$$\sum_{k=1}^m e^{-2\lambda_k T_1} \sum_{j=1}^{r_k} x_{kj}^2 \leq R^2.$$

Thus, our recovery problem reduces to the one considered above, for

$$W = \left\{ x \in l_2^n : \sum_{k=1}^m \sum_{j=1}^{r_k} \nu_{kj} x_{kj}^2 \leq 1 \right\},$$

where $\nu_k = \nu_{kj} = R^{-2} e^{-2\lambda_k T_1}$, $k = 1, \dots, m$,

$$Tx = (e^{-\lambda_1(T_1-\tau)} x_{11}, \dots, e^{-\lambda_1(T_1-\tau)} x_{1r_1}, \dots, e^{-\lambda_m(T_1-\tau)} x_{m1}, \dots, e^{-\lambda_m(T_1-\tau)} x_{mr_m})$$

and

$$Ix = (x_{11}, \dots, x_{1r_1}, \dots, x_{m1}, \dots, x_{mr_m}).$$

Set

$$\xi_k = R^{-1} \left(\sum_{j=1}^{r_k} r_j e^{-2\lambda_j T_1} (e^{(\lambda_j - \lambda_k)\tau} - 1) \right)^{1/2}, \quad k = 1, \dots, m.$$

Theorem 5. *Let $1/\delta \in (\xi_s, \xi_{s+1}]$ for some $1 \leq s \leq m-1$, or let $1/\delta \in (\xi_m, +\infty)$ (and then set $s = m$). Then*

$$E(T, W, I, \delta) = \delta \left(\sum_{k=1}^s r_k e^{-2\lambda_k(T_1 - \tau)} (1 - e^{(\lambda_1 - \lambda_k)\tau} (1 - c_1)) \right)^{1/2},$$

where

$$c_1 = 1 - \frac{\delta^2 R^{-2} e^{-\lambda_1 \tau} \sum_{k=1}^s r_k e^{(-2T_1 + \tau)\lambda_k}}{1 + \delta^2 R^{-2} \sum_{k=1}^s r_k e^{-2\lambda_k T_1}}, \quad (4.7)$$

and the method

$$\varphi(y) = \sum_{k=1}^s (e^{-\lambda_k(T_1 - \tau)} - e^{\lambda_1 \tau - \lambda_k T_1} (1 - c_1)) \sum_{j=1}^{r_k} y_{kj} e_{kj},$$

is optimal.

§ 5. Recovering trigonometric polynomials

Let \mathcal{T}_n denote the set of trigonometric polynomials

$$p_n(t) = \frac{a_0}{2} + \sum_{j=1}^n (a_j \cos jt + b_j \sin jt). \quad (5.1)$$

Set

$$\mathcal{T}_n^r = \{p_n(\cdot) \in \mathcal{T}_n : \|p_n^{(r)}(\cdot)\|_{L_2(\mathbb{R})} \leq 1, r \geq 1\},$$

where $\mathbb{T} = [-\pi, \pi]$ (with distinguished endpoints) and

$$\|x(\cdot)\|_{L_2(\mathbb{R})} = \left(\frac{1}{\pi} \int_{\mathbb{T}} |x(t)|^2 dt \right)^{1/2}.$$

We consider the problem of the recovery of the k th derivative of a polynomial in \mathcal{T}_n^r , $0 \leq k < r$, from the coefficients of this polynomial, which are known with a random error. We reduce this to the general problem (3.1). The set \mathcal{T}_n^r is the set of polynomials (5.1) such that

$$\sum_{j=1}^n j^{2r} (a_j^2 + b_j^2) \leq 1.$$

Hence setting

$$x = (a_0, a_1, b_1, \dots, a_n, b_n), \quad W = \left\{ x \in \mathbb{R}^{2n+1} : \sum_{j=1}^n j^{2r} (a_j^2 + b_j^2) \leq 1 \right\},$$

$$Ix = x \quad \text{and} \quad Tx = \left(\frac{a_0}{\sqrt{2}} \chi_k, a_1, b_1, \dots, n^k a_n, n^k b_n \right), \quad \chi_k = \begin{cases} 1, & k = 0, \\ 0, & k \geq 1, \end{cases}$$

we arrive at (3.1). Note that the values of the operator T are coefficients of the expansion with respect to the orthonormal basis

$$\left(\frac{a_0}{\sqrt{2}}, \cos\left(t + \frac{\pi k}{2}\right), \sin\left(t + \frac{\pi k}{2}\right), \dots, \cos\left(nt + \frac{\pi k}{2}\right), \sin\left(nt + \frac{\pi k}{2}\right) \right).$$

We cannot apply Theorem 1 to this problem because here we have $\nu_1 = 0$. Thus we use a modification of this theorem. Consider a set W and an operator T in the case when $\nu_1 = 0$ and $\mu_1 \geq 0$. We introduce the notation

$$\gamma_j = \frac{\sqrt{\nu_j}}{|\mu_j|}, \quad j = 2, \dots, n, \quad \text{and} \quad \xi_j = \left(\sum_{k=2}^j \nu_k \left(\frac{\gamma_j}{\gamma_k} - 1 \right) \right)^{1/2}, \quad j = 2, \dots, n,$$

and assume that $\gamma_2 \leq \dots \leq \gamma_n$.

Theorem 6. *Let $1/\delta \in (\xi_s, \xi_{s+1}]$ for some $2 \leq s \leq n-1$, or let $1/\delta \in (\xi_n, +\infty)$ (and then set $s = n$). Then*

$$E(T, W, I, \delta) = \delta \left(|\mu_1|^2 + \sum_{k=2}^s |\mu_k|^2 \left(1 - \frac{\gamma_k(1-c_2)}{\gamma_2} \right) \right)^{1/2},$$

where

$$c_2 = 1 - \frac{\delta^2 \gamma_2 \sum_{k=2}^s (\nu_k / \gamma_k)}{1 + \delta^2 \sum_{k=2}^s \nu_k}, \quad (5.2)$$

and the method

$$\varphi(y) = \mu_1 y_1 e_1 + \sum_{k=2}^s \left(1 - \frac{\gamma_k(1-c_2)}{\gamma_2} \right) \mu_k y_k e_k$$

is optimal.

Proof. Using the same arguments as in the lower estimate in the proof of Theorem 1 we arrive at inequality (3.5).

In estimating the error of methods of the form

$$\varphi(y) = \sum_{j=1}^n \alpha_j y_j e_j$$

from above, we obtain again the equality

$$e^2(T, W, I, \delta, \varphi) = \sup_{x \in W} \|Tx - \varphi(Ix)\|_{l_2^n}^2 + \delta^2 \sum_{j=1}^n |\alpha_j|^2.$$

However, the extremal problem

$$\|Tx - \varphi(Ix)\|_{l_2^n}^2 \rightarrow \max, \quad x \in W,$$

since we now have $\nu_1 = 0$, can be written this time in the form

$$\sum_{j=1}^n |\mu_j - \alpha_j|^2 |x_j|^2 \rightarrow \max, \quad \sum_{j=2}^n \nu_j |x_j|^2 \leq 1.$$

For $\alpha_1 \neq \mu_1$ the solution of this problem is equal to ∞ , so we assume below that $\alpha_1 = \mu_1$. From the inequality

$$\begin{aligned} \sum_{j=2}^n |\mu_j - \alpha_j|^2 |x_j|^2 &= \sum_{j=2}^n \frac{|\mu_j - \alpha_j|^2}{\nu_j} \nu_j |x_j|^2 \\ &\leq \max \left\{ \frac{|\mu_2 - \alpha_2|^2}{\nu_2}, \dots, \frac{|\mu_n - \alpha_n|^2}{\nu_n} \right\} \sum_{j=2}^n \nu_j |x_j|^2 \end{aligned}$$

we obtain

$$\sup_{x \in W} \|Tx - \varphi(Ix)\|_{l_2^n}^2 \leq \max \left\{ \frac{|\mu_2 - \alpha_2|^2}{\nu_1}, \dots, \frac{|\mu_n - \alpha_n|^2}{\nu_n} \right\}.$$

Thus,

$$e^2(T, W, I, \delta, \varphi) \leq \max \left\{ \frac{|\mu_2 - \alpha_2|^2}{\nu_2}, \dots, \frac{|\mu_n - \alpha_n|^2}{\nu_n} \right\} + \delta^2 \sum_{j=1}^n |\alpha_j|^2.$$

Set

$$c_j = \frac{\alpha_j}{\mu_j}, \quad j = 2, \dots, n.$$

Then for the error of the method φ we have

$$e^2(T, W, I, \delta, \varphi) \leq \max \left\{ \frac{|1 - c_2|^2}{\gamma_1^2}, \dots, \frac{|1 - c_n|^2}{\gamma_n^2} \right\} + \delta^2 |\mu_1|^2 + \delta^2 \sum_{j=2}^n |\mu_j|^2 |c_j|^2.$$

Let $1/\delta \in (\xi_s, \xi_{s+1}]$ for some $2 \leq s \leq n-1$. Then it is easy to show that

$$\frac{1}{\gamma_{s+1}} \leq \frac{\delta^2 \sum_{k=2}^s (\nu_k / \gamma_k)}{1 + \delta^2 \sum_{k=2}^s \nu_k} < \frac{1}{\gamma_s}.$$

If we define c_2 by (5.2), then

$$\frac{1}{\gamma_{s+1}} \leq \frac{1 - c_2}{\gamma_2} < \frac{1}{\gamma_s}.$$

Let

$$c_k = 1 - \gamma_k \frac{1 - c_2}{\gamma_2}, \quad \text{for } k = 3, \dots, s, \quad \text{and} \quad c_k = 0, \quad \text{for } k = s+1, \dots, n.$$

Then we have

$$\frac{(1 - c_k)^2}{\gamma_k^2} = \frac{(1 - c_2)^2}{\gamma_2^2}, \quad k = 3, \dots, s.$$

For $k \geq s+1$

$$\frac{(1 - c_k)^2}{\gamma_k^2} = \frac{1}{\gamma_k^2} \leq \frac{1}{\gamma_{s+1}^2} \leq \frac{(1 - c_2)^2}{\gamma_2^2},$$

and therefore

$$\max \left\{ \frac{|1 - c_2|^2}{\gamma_2^2}, \dots, \frac{|1 - c_n|^2}{\gamma_n^2} \right\} = \frac{(1 - c_2)^2}{\gamma_2^2}. \quad (5.3)$$

Consequently,

$$e^2(T, W, I, \delta, \varphi) \leq \frac{(1 - c_2)^2}{\gamma_2^2} + \delta^2 |\mu_1|^2 + \delta^2 \sum_{k=2}^s |\mu_k|^2 c_k^2.$$

Using transformations similar to (3.7) we obtain

$$e^2(T, W, I, \delta, \varphi) \leq \delta^2 |\mu_1|^2 + \delta^2 \sum_{k=2}^s |\mu_k|^2 c_k.$$

Consider a vector $\hat{\tau} \in l_2^n$ of the form

$$\hat{\tau}_1 = \tau_1, \quad \hat{\tau}_k^2 = \delta^2 \left(\frac{\gamma_2}{(1 - c_2)\gamma_k} - 1 \right) \text{ for } k = 2, \dots, s, \quad \hat{\tau}_k = 0 \text{ for } k = s + 1, \dots, n.$$

Then we have

$$\begin{aligned} \sum_{k=2}^n \nu_k \hat{\tau}_k^2 &= \delta^2 \sum_{k=2}^s \nu_k \left(\frac{\gamma_2}{(1 - c_2)\gamma_k} - 1 \right) = \frac{\delta^2 \gamma_2}{1 - c_2} \sum_{k=2}^s \frac{\nu_k}{\gamma_k} - \delta^2 \sum_{k=2}^s \nu_k \\ &= \left(1 + \delta^2 \sum_{k=2}^s \nu_k \right) - \delta^2 \sum_{k=2}^s \nu_k = 1. \end{aligned}$$

Thus, $\hat{\tau} \in W$. Substituting $\hat{\tau}$ (for $\tau_1 \geq \hat{\tau}_2$) into (3.5) we obtain

$$\begin{aligned} E^2(T, W, I, \delta) &\geq \sum_{k=1}^s \frac{\delta^2 |\mu_k|^2 \hat{\tau}_k^2}{\delta^2 + \hat{\tau}_k^2} = \frac{\delta^2 |\mu_1|^2 \tau_1^2}{\delta^2 + \tau_1^2} + \sum_{k=2}^s \frac{\delta^4 |\mu_k|^2 (\gamma_2 / ((1 - c_2)\gamma_k) - 1)}{\delta^2 \gamma_2 / ((1 - c_2)\gamma_k)} \\ &= \frac{\delta^2 |\mu_1|^2 \tau_1^2}{\delta^2 + \tau_1^2} + \delta^2 \sum_{k=2}^s |\mu_k|^2 \left(1 - \frac{(1 - c_2)\gamma_k}{\gamma_2} \right) = \frac{\delta^2 |\mu_1|^2 \tau_1^2}{\delta^2 + \tau_1^2} + \delta^2 \sum_{k=1}^s |\mu_k|^2 c_k. \end{aligned}$$

Letting τ_1 tend to infinity yields

$$E^2(T, W, I, \delta) \geq \delta^2 |\mu_1|^2 + \delta^2 \sum_{k=1}^s |\mu_k|^2 c_k \geq e^2(T, W, I, \delta, \varphi).$$

Hence φ is an optimal method.

Now let $1/\delta > \xi_n$. Then

$$\frac{\delta^2 \sum_{k=2}^n (\nu_k / \gamma_k)}{1 + \delta^2 \sum_{k=2}^n \nu_k} < \frac{1}{\gamma_n}.$$

Set

$$c_2 = 1 - \gamma_2 \frac{\delta^2 \sum_{k=2}^n (\nu_k / \gamma_k)}{1 + \delta^2 \sum_{k=2}^n \nu_k}.$$

Then

$$\frac{1 - c_2}{\gamma_2} < \frac{1}{\gamma_n}.$$

Let

$$c_k = 1 - \gamma_k \frac{1 - c_2}{\gamma_2}, \quad k = 3, \dots, n.$$

Then we have

$$\frac{(1 - c_k)^2}{\gamma_k^2} = \frac{(1 - c_2)^2}{\gamma_2^2}, \quad k = 3, \dots, n.$$

Thus, as in the previous case, we obtain (5.3). Using transformations similar to (3.7) for $s = n$, we obtain

$$e^2(T, W, I, \delta, \varphi) \leq \delta^2 |\mu_1|^2 + \delta^2 \sum_{k=2}^n |\mu_k|^2 c_k.$$

Consider the vector $\hat{\tau} \in l_2^n$ of the form

$$\hat{\tau}_1 = \tau_1, \quad \hat{\tau}_k^2 = \delta^2 \left(\frac{\gamma_2}{(1 - c_2)\gamma_k} - 1 \right), \quad k = 2, \dots, n.$$

Then we have

$$\begin{aligned} \sum_{k=2}^n \nu_k \hat{\tau}_k^2 &= \delta^2 \sum_{k=2}^n \nu_k \left(\frac{\gamma_2}{(1 - c_2)\gamma_k} - 1 \right) = \frac{\delta^2 \gamma_2}{1 - c_2} \sum_{k=2}^n \frac{\nu_k}{\gamma_k} - \delta^2 \sum_{k=2}^n \nu_k \\ &= \left(1 + \delta^2 \sum_{k=2}^n \nu_k \right) - \delta^2 \sum_{k=2}^n \nu_k = 1. \end{aligned}$$

Thus, $\hat{\tau} \in W$. Plugging $\hat{\tau}$ (for $\tau_1 \geq \hat{\tau}_2$) into (3.5) we obtain

$$\begin{aligned} E^2(T, W, I, \delta) &\geq \sum_{k=1}^n \frac{\delta^2 |\mu_k|^2 \hat{\tau}_k^2}{\delta^2 + \hat{\tau}_k^2} = \frac{\delta^2 |\mu_1|^2 \tau_1^2}{\delta^2 + \tau_1^2} + \sum_{k=2}^n \frac{\delta^4 |\mu_k|^2 (\gamma_2 / ((1 - c_2)\gamma_k) - 1)}{\delta^2 \gamma_2 / ((1 - c_2)\gamma_k)} \\ &= \frac{\delta^2 |\mu_1|^2 \tau_1^2}{\delta^2 + \tau_1^2} + \delta^2 \sum_{k=2}^n |\mu_k|^2 \left(1 - \frac{(1 - c_2)\gamma_k}{\gamma_2} \right) \\ &= \frac{\delta^2 |\mu_1|^2 \tau_1^2}{\delta^2 + \tau_1^2} + \delta^2 \sum_{k=1}^n |\mu_k|^2 c_k. \end{aligned}$$

Letting τ_1 tend to infinity yields

$$E^2(T, W, I, \delta) \geq \delta^2 |\mu_1|^2 + \delta^2 \sum_{k=1}^n |\mu_k|^2 c_k \geq e^2(T, W, I, \delta, \varphi).$$

Hence φ is an optimal method.

Theorem 6 is proved.

Now we apply this theorem to the solution of our problem. Set

$$\xi_j = \left(2 \sum_{l=2}^j (l-1)^{r+k} ((j-1)^{r-k} - (l-1)^{r-k}) \right)^{1/2}, \quad j = 2, \dots, n+1.$$

Theorem 7. Let $1/\delta \in (\xi_s, \xi_{s+1}]$ for some $2 \leq s \leq n$, or let $1/\delta \in (\xi_{n+1}, +\infty)$ (and then set $s = n+1$). Then

$$E(T, W, I, \delta) = \delta \left(\frac{\chi_k^2}{2} + 2 \sum_{l=2}^s ((l-1)^{2k} - (l-1)^{r+k}(1-c_2)) \right)^{1/2},$$

where

$$c_2 = 1 - \frac{2\delta^2 \sum_{l=2}^s (l-1)^{r+k}}{1 + 2\delta^2 \sum_{l=2}^s (l-1)^{2r}}, \quad (5.4)$$

and the method

$$\begin{aligned} \varphi(y) = & \frac{\tilde{a}_0}{\sqrt{2}} \chi_k + \sum_{l=2}^s (1 - (l-1)^{r-k}(1-c_2)) \\ & \times \left((l-1)^k \tilde{a}_{l-1} \cos \left((l-1)t + \frac{\pi k}{2} \right) + (l-1)^k \tilde{b}_{l-1} \sin \left((l-1)t + \frac{\pi k}{2} \right) \right) \end{aligned}$$

where $y = (\tilde{a}_0, \tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_n, \tilde{b}_n)$, is optimal.

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Received 19/APR/24 and 6/SEP/24

Translated by N. KRUZHILIN

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