

**EXTREMAL PROBLEMS OF THE HADAMARD
THREE-CIRCLE THEOREM TYPES AND OPTIMAL
RECOVERY OF OPERATORS**

K. YU. OSIPENKO

The well-known Hadamard three-circle theorem states that if $f(z)$ is a holomorphic function on the annulus $r_1 \leq |z| \leq r_2$ and

$$M(r) = \max_{|z|=r} |f(z)|,$$

then

$$M(r) \leq M(r_1)^{\frac{\log r_2/r}{\log r_2/r_1}} M(r_2)^{\frac{\log r/r_1}{\log r_2/r_1}}$$

for any three concentric circles of radii $r_1 < r < r_2$.

In 1913 E. Landau considered a very similar problem. He took derivatives instead of circles. He proved that for all functions $x(\cdot) \in L_\infty(\mathbb{R}_+)$ with the first derivative locally absolutely continuous on \mathbb{R}_+ and $x''(\cdot) \in L_\infty(\mathbb{R}_+)$ the following exact inequality

$$\|x'(\cdot)\|_{L_\infty(\mathbb{R}_+)} \leq 2 \|x(\cdot)\|_{L_\infty(\mathbb{R}_+)}^{1/2} \|x''(\cdot)\|_{L_\infty(\mathbb{R}_+)}^{1/2}$$

holds. Then in 1914 Hadamard solved the analogous problem for \mathbb{R} .

In 1934 Hardy, Littlewood, and Pólya proved that for all integers $0 < k < r$ the exact inequality

$$(1) \quad \|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \leq \|x(\cdot)\|_{L_2(\mathbb{R})}^{1-\frac{k}{r}} \|x^{(r)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{k}{r}}$$

holds for all functions $x(\cdot) \in L_2(\mathbb{R})$ for which the $(r-1)$ -st derivative is locally absolute continuous on \mathbb{R} and $x^{(r)}(\cdot) \in L_2(\mathbb{R})$.

The exact inequality (1) may be easily obtained passing to the Fourier transforms from the following extremal problem

$$\int_{\mathbb{R}} \xi^{2k} |Fx(\xi)|^2 d\xi \rightarrow \max, \quad \int_{\mathbb{R}} |Fx(\xi)|^2 d\xi \leq \delta_1^2,$$

$$\int_{\mathbb{R}} \xi^{2n} |Fx(\xi)|^2 d\xi \leq \delta_2^2,$$

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where $Fx(\cdot)$ is the Fourier transform of $x(\cdot)$. The value of this extremal problem coincides with the value of the extremal problem

$$\int_{\mathbb{R}} \xi^{2k} |Fx(\xi)|^2 d\xi \rightarrow \max, \quad \int_{|\xi| \leq \sigma_1} |Fx(\xi)|^2 d\xi \leq \delta_1^2, \\ \int_{|\xi| \geq \sigma_2} \xi^{2n} |Fx(\xi)|^2 d\xi \leq \delta_2^2$$

for some $\sigma_1 \geq \sigma_2$. Using this fact we obtain a collection of optimal recovery methods of $x^{(k)}(\cdot)$ from inaccurate information about the Fourier transform of $x(\cdot)$.

REFERENCES

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