

# ON OPTIMAL RECOVERY OF HOLOMORPHIC FUNCTION IN THE UNIT BALL OF $\mathbb{C}^n$

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ABSTRACT. Some problems of optimal recovery in the Hardy and Bergman spaces in the unit ball of  $\mathbb{C}^n$  are solved. In particular, some variants of the Schwartz Lemma follow.

## INTRODUCTION

Let  $\Omega$  be a subset of  $\mathbb{C}^n$  and  $\mu$  is a nonnegative measure on  $\Omega$ . For  $1 \leq p \leq \infty$  let  $L_p(\Omega, \mu)$  be the Lebesgue space of complex-valued functions on  $\Omega$  with the usual norm

$$\|f\|_p = \left( \int_{\Omega} |f(z)|^p d\mu(z) \right)^{1/p}, \quad 1 \leq p < \infty,$$
$$\|f\|_{\infty} = \operatorname{ess\,sup}_{z \in \Omega} |f(z)|.$$

Denote  $BX$  the closed unit ball of the normed space  $X$ . Consider a linear subspace  $X_p \subset L_p(\Omega, \mu)$  with the induced norm. We define an information operator as a mapping  $I: BX_p \rightarrow Y$  where  $Y$  is some set. Let  $L$  be a functional on  $X_p$  and  $S: Y \rightarrow \mathbb{C}$  is a function on  $Y$ . The function  $S$  defines a recovery algorithm  $Lf \approx SIf$ .

Consider the following problem of optimal recovery

$$(1) \quad E(L, I, X_p) = \inf_S \sup_{f \in BX_p} |Lf - SIf|.$$

$E(L, I, X_p)$  is called the intrinsic error of recovery. If

$$E(L, I, X_p) = \sup_{f \in BX_p} |Lf - S_*If| = |Lf_* - S_*If| \quad (f_* \in B_p),$$

then  $S_*$  is said to be an optimal algorithm and  $f_*$  is a worst function.

There are a lot of papers devoted to the recovery problems. The detailed bibliography and the history of the subject can be found in Micchelli and Rivlin [3], [4] and also in Traub and Wozniakowski [8].

In this paper we solve some recovery problems in the Hardy and Bergman spaces in the unit ball of  $\mathbb{C}^n$ . In particular, some variants of the Schwartz Lemma follow. In the unit disk, similar results were previously obtained for the Hardy spaces in Fisher and Micchelli [1] and for the Bergman spaces in Osipenko and Stessin [6].

In Section 1 we prove the general theorem which connects the worst functions with reproducing kernels in some weighted spaces, where weights are associated with an information operator. In Section 2 this

theorem is applied to the recovery problem in the Hardy spaces  $H_p(B_n)$  ( $B_n$  is the unit ball of  $\mathbb{C}^n$ ) when the information operator is the operator of restriction on some affine subset of  $B_n$ . Here there arises the astonishing fact that the case  $n \leq 5$  differs from  $n \geq 6$ . The last section is devoted to proving the analogous results for the Bergman spaces.

## 1. OPTIMAL RECOVERY AND REPRODUCING KERNELS

Let  $\Omega$  be a simply connected domain in  $\mathbb{C}^n$ ,  $X_p$  be the space of holomorphic functions in  $\Omega$  which belong to  $L_p(\overline{\Omega}, \mu)$  (we suppose that for the holomorphic function  $f \not\equiv 0$  it follows that  $\|f\|_p \neq 0$ ). If  $\mu(\partial\Omega) \neq 0$ , only those holomorphic functions are considered which have boundary values almost everywhere with respect to measure  $\mu$ . Let  $\varphi \in X_p$ . Suppose that there exists the reproducing kernel  $K_\varphi(z, w)$  with the weight  $|\varphi|^{p-2}$ , i.e., for every  $f \in X_p$ ,

$$f(z) = \int_{\overline{\Omega}} \overline{K_\varphi(z, w)} f(w) |\varphi(w)|^{p-2} d\mu(w)$$

and  $K_\varphi(z, \cdot) \in X_p$  for every  $z \in \Omega$ . Note that for every  $z \in \Omega$  we have  $K_\varphi(z, \cdot) \in X_2(\varphi, p)$ , where  $X_2(\varphi, p)$  is the Hilbert space of holomorphic functions in  $\Omega$  which satisfy

$$\|f\|_{X_2(\varphi, p)} = \left( \int_{\overline{\Omega}} |f(w)|^2 |\varphi(w)|^{p-2} d\mu(w) \right)^{1/2} < \infty.$$

Let  $A \subset \Omega$ . Define the information operator  $I$  as the operator of restriction on  $A$ :  $If = f|_A$ . Put  $Lf = f(a)$  where  $a \in \Omega \setminus A$ . In this case, denote by  $E(a, A, X_p)$  the intrinsic error from (1). Let  $M_A$  be the closed linear subspace of  $X_2(\varphi, p)$  spanned by  $K_\varphi(z, \cdot)$  for every  $z \in A$ :

$$M_A = \begin{cases} \text{cl}(\text{span}\{K_\varphi(z, \cdot), z \in A\}), & A \neq \emptyset, \\ \{0\}, & A = \emptyset, \end{cases}$$

and let  $\text{Pr}_A$  be the orthogonal projection  $\text{Pr}_A: X_2(\varphi, p) \rightarrow M_A$ . It follows from the Riesz Theorem that for every  $f \in X_2(\varphi, p)$  there exists a complex measure  $m_f$  such that

$$(\text{Pr}_A f)(\cdot) = \int_A K_\varphi(z, \cdot) dm_f(z).$$

Denote by  $m_a$  the appropriate measure for  $f = K_\varphi(a, \cdot)$  and put

$$(2) \quad \phi(w) = K_\varphi(a, w) - \text{Pr}_A K_\varphi(a, w) = K_\varphi(a, w) - \int_A K_\varphi(z, w) dm_a(z).$$

We call  $\varphi$  regular with respect to the pair  $(a, A)$  if:

- (i)  $I\varphi = \varphi|_A \equiv 0$ ;
- (ii)  $\chi = \phi/\varphi \in X_\infty$ ,  $\chi^{-1} = \varphi/\phi \in X_\infty$ .

**Theorem 1.** *Let  $a \in \Omega$ ,  $A \subset \Omega$ , and  $\varphi$  is regular with respect to the pair  $(a, A)$ . Put  $\alpha = [\chi(a)]^{(2-p)/p}$ ,  $g = \chi^{2/p}\varphi$ . Then:*

- (i)  $f(a) \approx S_*If = \alpha \int_A [\chi(z)]^{(p-2)/p} f(z) \overline{dm_a(z)}$  is an optimal recovery algorithm;
- (ii)  $f^*(w) = |\phi(a)|^{-1/p} g(w)$  is a worst function;
- (iii)  $E(a, A, X_p) = |\phi(a)|^{1/p} |\varphi(a)|^{(p-2)/p}$ .

*Proof.* Since  $\varphi$  is regular we have  $g \in X_p$  and  $\chi^{(p-2)/p} f \in X_p$  for every  $f \in X_p$ . Hence

$$\begin{aligned}
(3) \quad & \alpha \int_{\overline{\Omega}} \overline{g(w)} |g(w)|^{p-2} f(w) d\mu(w) \\
&= \alpha \int_{\overline{\Omega}} \overline{\phi(w)} [\chi(w)]^{(p-2)/p} f(w) |\varphi(w)|^{p-2} d\mu(w) \\
&= \alpha \int_{\overline{\Omega}} \left[ \overline{K_{\varphi}(a, w)} - \int_A \overline{K_{\varphi}(z, w)} dm_a(z) \right] [\chi(w)]^{(p-2)/p} f(w) \\
&\quad \times |\varphi(w)|^{p-2} d\mu(w) = f(a) - S_*If.
\end{aligned}$$

It immediately follows from (3) that

$$\begin{aligned}
E(a, A, X_p) &\leq \sup_{f \in BX_p} |f(a) - S_*If| \\
&= \sup_{f \in BX_p} \left| \alpha \int_{\overline{\Omega}} \overline{g(w)} |g(w)|^{p-2} f(w) d\mu(w) \right| \leq |\alpha| \|g\|_p^{p-1}
\end{aligned}$$

and  $\alpha \|g\|_p^p = g(a)$ . Thus  $\|g\|_p = |g(a)/\alpha|^{1/p} = |\phi(a)|^{1/p}$ , and  $\|f^*\|_p = 1$  (i.e.,  $f^* \in BX_p$ ). As  $If^* = f^*|_A \equiv 0$  we have for every algorithm S

$$\begin{aligned}
|f^*(a) - SIf^*| + |-f^*(a) - S(-f^*)| &= |f^*(a) - S(0)| + |-f^*(a) - S(0)| \\
&\geq 2|f^*(a)|,
\end{aligned}$$

and therefore  $\sup_{f \in BX_p} |f(a) - SIf| \geq |f^*(a)|$ ,

$$E(a, A, X_p) \geq |f^*(a)| = \frac{|g(a)|}{\|g\|_p} = |\alpha| \|g\|_p^{p-1} = |\phi(a)|^{1/p} |\varphi(a)|^{(p-2)/p}.$$

The theorem is proved.  $\square$

**Corollary 1.** Let  $K(z, w)$  is a reproducing kernel for functions from  $X_p$ ,  $a \in \Omega$ , and for every  $w \in \Omega$ ,  $K(a, w) \neq 0$ . Then

$$\sup_{f \in BX_p} |f(a)| = (K(a, a))^{1/p}.$$

*Proof.* Put  $\varphi(w) \equiv 1$ . As  $K(a, w) \neq 0$ ,  $\varphi$  is regular with respect to  $(a, \emptyset)$ . It follows from Theorem 1 that

$$\sup_{f \in BX_p} |f(a)| = \frac{[K(a, a)]^{2/p}}{\|(K(a, \cdot))^{2/p}\|_p}.$$

Now since

$$\|(K(a, \cdot))^{2/p}\|_p = \left( \int_{\overline{\Omega}} K(a, w) \overline{K(a, w)} d\mu(w) \right)^{1/p} = (K(a, a))^{1/p}$$

the result follows.  $\square$

*Remark 1.* If the conditions of Theorem 1 hold, then

$$(4) \quad E(a, A, X_p) = \sup_{\substack{f \in BX_p \\ f|_A \equiv 0}} |f(a)|.$$

Indeed,

$$\begin{aligned} \sup_{\substack{f \in BX_p \\ f|_A \equiv 0}} |f(a)| &\geq |f^*(a)| = E(a, A, X_p) = \sup_{f \in BX_p} |f(a) - S_* I f| \\ &\geq \sup_{\substack{f \in BX_p \\ f|_A \equiv 0}} |f(a)|. \end{aligned}$$

*Remark 2.* If  $p = 2$  the regularity condition is unnecessary because  $\chi^{(p-2)/p} \equiv 1$ , and  $\chi^{2/p}$  is correctly defined even in the case when  $\chi$  has zeros in  $\Omega$ . More precisely, let  $K(z, w)$  be a reproducing kernel with weight 1 for functions from  $X_2$ ,  $A$  is a subset of  $\Omega$ , and  $a \in \Omega$ . By analogy with (2) we define

$$\phi(w) = K(a, w) - \Pr_A K(a, w) = K(a, w) - \int_A K(z, w) dm_a(z).$$

The next theorem holds.

**Theorem 1'.**

- (i)  $f(a) \approx \int_A f(z) \overline{dm_a(z)}$  is an optimal algorithm in the recovery problem in  $X_2$ ;
- (ii)  $f^* = |\phi(a)|^{-1/2} \phi$  is a worst function;
- (iii) the intrinsic error is

$$E(a, A, X_2) = |\phi(a)|^{1/2}.$$

The meaning of Theorem 1 is that, in some cases, it reduces the problem of finding a worst function and an optimal algorithm to the problem of finding the divisor of a worst function, which connects with the information operator. In fact, the divisor of the worst function is the largest extension of the information set, which does not lead to decreasing the intrinsic error.

## 2. RECOVERY IN $H_p(B_n)$

Let  $B_n$  be the unit ball of  $\mathbb{C}^n$

$$B_n = \left\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z|^2 = \sum_{k=1}^n |z_k|^2 < 1 \right\}.$$

The Hardy space  $H_p(B_n)$  is the set of holomorphic functions in  $B_n$  which satisfy the condition

$$\|f\|_p = \sup_{0 < r < 1} \left( \int_{|z|=1} |f(rz)|^p d\sigma_n(z) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty = \sup_{z \in B_n} |f(z)| < \infty,$$

where  $\sigma_n$  is the probability measure on sphere  $S^{2n-1}$  which is invariant with respect to the orthogonal group  $O(2n)$ . It is known (see Rudin [7]) that functions from  $H_p(B_n)$  have finite boundary values almost everywhere, and so  $H_p(B_n)$  can be considered as a subspace of  $L_p(\bar{B}_n, \mu)$  where  $\mu$  is supported on  $\partial B_n = S^{2n-1}$  and coincides with  $\sigma_n$  in  $S^{2n-1}$ .

Let  $L$  be an affine plane in  $\mathbb{C}^n$  and set  $A = L \cap B_n$ . In this section we deal with the recovery of functions from  $H_p(B_n)$  at some point  $a \in B_n \setminus A$  in the case when  $If = f|_A$ . Initially, we consider the case

$$L = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : z_n = 0\}.$$

Let  $u \in \mathbb{C}$ ,  $|u| < 1$ ,  $\alpha > 1$ , and  $m$  is a nonnegative integer. Put

$$F_m(\alpha, u) = \sum_{k=0}^{\infty} \frac{\Gamma(m+k+\alpha)}{\Gamma(k+\alpha)} u^k.$$

We can easily prove that

$$(5) \quad F_m(\alpha, u) = \frac{(\alpha-1)\alpha \dots (\alpha+m-1)}{(1-u)^{m+1}} \sum_{k=0}^m \frac{(-1)^k}{\alpha+k-1} \binom{m}{k} u^k.$$

**Proposition 1.** *Let  $1 \leq p < \infty$ . Then*

$$(6) \quad K_{z_n}(z, w) = \frac{1}{(n-1)!(1 - \langle w'_1, z'_1 \rangle)^{n+p/2-1}} F_{n-1} \left( \frac{p}{2}, \frac{\bar{z}_n w_n}{1 - \langle w'_1, z'_1 \rangle} \right),$$

where  $x'_1 = (x_1, \dots, x_{n-1}, 0)$  for  $x = (x_1, \dots, x_n)$  is the reproducing kernel for functions from  $H_p(B_n)$  with the weight  $|z_n|^{p-2}$ .

*Proof.* As polynomials are dense in  $H_p(B_n)$  it is sufficient to prove that (6) is the reproducing kernel for the polynomials of the special kind  $f(z) = f_1(z'_1)z_n^k$ , where  $f_1(z'_1)$  is a polynomial which depends only on  $z_1, \dots, z_{n-1}$ . Let  $\nu_n$  be the normalized Lebesgue measure on  $B_n$ . We

have

$$\begin{aligned}
& \int_{|z|=1} \overline{K_{z_n}(z, w)} f_1(w'_1) w_n^k |w_n|^{p-2} d\sigma(w) \\
&= \frac{1}{(n-1)!} \int_{B_{n-1}} \frac{(1-|w'_1|^2)^{(p-2)/2} f_1(w'_1)}{(1-\langle z'_1, w'_1 \rangle)^{n+p/2-1}} d\nu_{n-1}(w'_1) \\
&\times \frac{1}{2\pi i} \int_{|w_n|=(1-|w'_1|^2)^{1/2}} w_n^k \sum_{m=0}^{\infty} \frac{\Gamma(n-1+p/2+m) z_n^m w_n^{-m} dw_n}{\Gamma(p/2+m)(1-\langle z'_1, w'_1 \rangle)^m w_n} \\
&= \frac{\Gamma(n+(p-2)/2+k)}{(n-1)! \Gamma((p-2)/2+k+1)} z_n^k \\
&\times \int_{B_{n-1}} \frac{(1-|w'_1|^2)^{k+(p-2)/2} f_1(w'_1)}{(1-\langle z'_1, w'_1 \rangle)^{n+k+(p-2)/2}} d\nu_{n-1}(w'_1) = f_1(z'_1) z_n^k.
\end{aligned}$$

In the last equality, we used that

$$\frac{\Gamma(n+s+1)}{\Gamma(n+1)\Gamma(s+1)} \frac{(1-|w|^2)^s}{(1-\langle z, w \rangle)^{n+s+1}}$$

is the reproducing kernel in  $H_p(B_n)$  (see Rudin [7, p. 121]). Thus the proposition is proved.  $\square$

In view of  $K_{z_n}(a'_1, w) \in M_A$  and

$$\langle K_{z_n}(b, \cdot), K_{z_n}(a, \cdot) - K_{z_n}(a'_1, \cdot) \rangle_{X_2(z_n, p)} = K_{z_n}(b, a) - K_{z_n}(b, a'_1) = 0$$

for every  $b = (b_1, \dots, b_{n-1}, 0)$ , we have

$$\text{Pr}_A K_{z_n}(a, w) = K_{z_n}(a'_1, w),$$

$$(7) \quad \phi(w) = K_{z_n}(a, w) - K_{z_n}(a'_1, w)$$

$$= \frac{1}{(n-1)!} \frac{\bar{a}_n w_n}{(1-\langle w'_1, a'_1 \rangle)^{n+p/2}} F_{n-1} \left( \frac{p}{2} + 1, \frac{\bar{a}_n w_n}{1-\langle w'_1, a'_1 \rangle} \right).$$

It follows from the last expression that  $\phi(w) = w_n$  is regular with respect to the pair  $(a, \{w_n = 0\})$  if and only if  $F_{n-1}(p/2 + 1, \bar{a}_n w_n / (1 - \langle w'_1, z'_1 \rangle))$  does not have a zero in  $B_n$ . For every  $n$  and  $p$  we shall find the values of  $a$  where regularity holds.

### Proposition 2.

- (i) If  $n \leq 5$ , then for every  $1 \leq p < \infty$  and  $a \in B_n$  the function  $F_{n-1}(p/2 + 1, \bar{a}_n w_n / (1 - \langle w'_1, z'_1 \rangle))$  does not have a zero in  $B_n$ .
- (ii) For every  $n > 5$  there exist  $a \in B_n$  and  $p$ ,  $1 \leq p < \infty$ , such that the zero set of  $F_{n-1}(p/2 + 1, \bar{a}_n w_n / (1 - \langle w'_1, z'_1 \rangle))$  has a nonempty intersection with  $B_n$ .

We shall prove this proposition at the end of this section, because the proof is quite technical.

Now we note that

$$\sup_{w \in B_n} \left| \frac{\bar{a}_n w_n}{1 - \langle w'_1, a'_1 \rangle} \right| = \frac{|a_n|}{\sqrt{1 - |a'_1|^2}}$$

and, therefore, in view of (5), the zero-set of  $F_{n-1}$  has an empty intersection with  $B_n$  if and only if the polynomial

$$P_{n-1} \left( \frac{p}{2} + 1, u \right) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k + p/2} \binom{n-1}{k} u^k$$

has no zeros in the disk  $\{|u| < |a_n|/\sqrt{1 - |a'_1|^2}\}$ .

Put

$$\begin{aligned} \lambda_n(p) &= \min\{|u| : P_{n-1}(p/2 + 1, u) = 0\}, \\ \Delta_n^1(p) &= \{a \in B_n : |a_n|^2 < \lambda_n^2(p)(1 - |a'_1|^2)\}. \end{aligned}$$

It follows from Proposition 2 that  $\Delta_n^1(p) = B_n$  if  $n \leq 5$ .

The next lemma immediately follows from Theorem 1, Proposition 1, and (7).

**Lemma 1.** *Let  $1 \leq p < \infty$  and  $a \in \Delta_n^1(p)$ . Then:*

(i) *the algorithm*

$$f(a) \approx (D_{n1}(p, a))^{(p-2)/p} f(a'_1),$$

where

$$D_{n1}(p, a) = \frac{(1 - |a|^2)^n}{\sum_{k=0}^{n-1} \frac{(-1)^k}{2k/p + 1} \binom{n-1}{k} |a_n|^{2k} (1 - |a'_1|^2)^{n-k}}$$

is an optimal one;

(ii)

$$\begin{aligned} f_1^*(p, a, w) &= \left( \frac{\Gamma(n + p/2) D_{n1}(p, a)}{\Gamma(n) \Gamma(p/2 + 1)} (1 - |a'_1|^2)^{n+p/2} \right)^{1/p} \\ &\quad \times \frac{w_n}{(1 - \langle w'_1, a'_1 \rangle)(1 - \langle w, a \rangle)^{2n/p}} \\ &\quad \times \left( \sum_{k=0}^{n-1} \frac{(-1)^k}{2k/p + 1} \binom{n-1}{k} \left( \frac{\bar{a}_n w_n}{1 - \langle w'_1, a'_1 \rangle} \right)^k \right)^{2/p} \end{aligned}$$

is a worst function;

(iii)

$E(a, \{z_n = 0\}, H_p(B_n))$

$$= \left( \frac{\Gamma(n + p/2)}{\Gamma(n) \Gamma(p/2 + 1) D_{n1}(p, a)} \right)^{1/p} \frac{|a_n|}{(1 - |a'_1|^2)^{n/p+1/2}}.$$

When proving Proposition 2 we shall see that  $\lambda_n(2) = 2 \sin \pi/n$  ( $n > 1$ ). Now we shall provide a lower estimate  $\lambda_n(p)$  for every  $n$  and  $p$ . Although our bound is not sharp, it will give us the opportunity to show that for every  $n$  and  $a \in B_n$  there exists  $p_0$  such that  $a \in \Delta_n^1(p)$  if  $p > p_0$ .

**Proposition 3.**

$$\lambda_n(p) \geq \frac{p+2}{p+2n}.$$

*Proof.* In view of (5) we must prove that  $F_{n-1}(p/2+1, u)$  has no zeros in the disk  $\{|u| < (p+2)/(p+2n)\}$ . Consider the function

$$\phi(v) = \sum_{k=0}^{\infty} \frac{\Gamma(n+k+p/2)}{\Gamma(k+p/2+1)} \left( \frac{p+2}{p+2n} v \right)^k = \sum_{k=0}^{\infty} d_k v^k.$$

We have

$$\frac{d_{k+1}}{d_k} = \frac{(n+k+p/2)(p+2)}{(k+p/2+1)(p+2n)} \leq 1.$$

It is well known (see Khavinson [2]) that  $d_k$  being positive and  $d_{k+1} \leq d_k$ ,  $\phi(v)$  has no zeros in the unit disk  $\{|v| < 1\}$ . Hence the statement follows and the proposition is proved.  $\square$

**Corollary 2.** *If*

$$p > p_0 = 2 \frac{n|a_n| - \sqrt{1 - |a'_1|^2}}{\sqrt{1 - |a'_1|^2} - |a_n|}$$

*then*  $a \in \Delta_n^1(p)$ .

Indeed,

$$\lambda_n(p) \geq \frac{p+2}{p+2n} > \frac{p_0+2}{p_0+2n} = \frac{|a_n|}{\sqrt{1 - |a'_1|^2}}.$$

**Lemma 2.** *Let*  $a \in B_n$  *and*  $p = \infty$ . *Then:*

(i) *the algorithm*

$$f(a) \approx \frac{1 - |a|^2}{1 - |a'_1|^2} f(a'_1)$$

*is an optimal one;*

(ii)

$$f_1^*(\infty, a, w) = \frac{\sqrt{1 - |a'_1|^2} w_n}{1 - \langle w'_1, a'_1 \rangle}$$

*is a worst function;*

(iii)

$$E(a, \{w_n = 0\}, H_\infty(B_n)) = \frac{|a_n|}{\sqrt{1 - |a'_1|^2}}.$$

*Proof.* Let  $f \in BH_\infty(B_n)$ . Consider the function  $\varphi(u) = f(a_1, \dots, a_{n-1}, u)$ .  $\varphi$  is holomorphic in the disk  $\{|u| < \sqrt{1 - |a'_1|^2}\}$  and  $\sup |\varphi(u)| \leq 1$ . It follows from Osipenko [5] that

$$\left| \varphi(a_n) - \left(1 - \frac{|a_n|^2}{1 - |a'_1|^2}\right) \varphi(0) \right| = \left| f(a) - \frac{1 - |a|^2}{1 - |a'_1|^2} f(a'_1) \right| \leq \frac{|a_n|}{\sqrt{1 - |a'_1|^2}},$$

and so the error of the algorithm

$$f(a) \approx \frac{1 - |a|^2}{1 - |a'_1|^2} f(a'_1)$$

is not greater than  $|a_n|/\sqrt{1 - |a'_1|^2}$ . On the other hand,  $f_1^*(\infty, a, w) \in BH_\infty(B_n)$ ,  $I f_1^* = f_1^*|_{w_n=0} = 0$ , and hence the intrinsic error can be evaluated as

$$E(a, \{w_n = 0\}, H_\infty(B_n)) \geq |f_1^*(\infty, a, a)| = \frac{|a_n|}{\sqrt{1 - |a'_1|^2}}.$$

The proof is complete.  $\square$

Now consider the case

$$A = \{z = (z_1, \dots, z_n) \in B_n : z_{n-k+1} = \dots = z_n = 0\}.$$

For  $w = (w_1, \dots, w_n) \in \mathbb{C}^n$  put  $w'_k = (w_1, \dots, w_{n-k}, 0, \dots, 0)$ ,  $w''_k = w - w'_k = (0, \dots, 0, w_{n-k+1}, \dots, w_n)$ . Denote  $\Delta_n^k(p) = \{z \in B_n : |z'_k| < \lambda_n(p) \sqrt{1 - |z'_k|^2}\}$ ,  $\Delta_n^k(\infty) = B_n$ . As in the case  $k = 1$ , we have  $\Delta_n^k(p) = B_n$  for every  $p$  if  $n \leq 5$ .

**Lemma 3.** *Let  $a \in \Delta_n^k(p) \setminus A$ . Then:*

(i) *the algorithm*

$$f(a) \approx \begin{cases} (D_{nk}(p, a))^{(p-2)/p} f(a'_k), & 1 \leq p < \infty, \\ \frac{1 - |a|^2}{1 - |a'_k|^2} f(a'_k), & p = \infty, \end{cases}$$

where

$$D_{nk}(p, a) = \frac{(1 - |a|^2)^n}{\sum_{j=0}^{n-1} \frac{(-1)^j}{2j/p + 1} \binom{n-1}{j} |a''_k|^{2j} (1 - |a'_k|^2)^{n-j}}$$

is an optimal one;

(ii)

$$\begin{aligned}
f_k^*(p, a, w) &= \left( \frac{\Gamma(n + p/2) D_{nk}(p, a)}{\Gamma(n) \Gamma(p/2 + 1)} (1 - |a'_k|^2)^{n+p/2} \right)^{1/p} \\
&\quad \times \frac{\langle w''_k, a''_k \rangle}{|a''_k| (1 - \langle w'_k, a'_k \rangle) (1 - \langle w, a \rangle)^{2n/p}} \\
&\quad \times \left( \sum_{j=0}^{n-1} \frac{(-1)^j}{2j/p + 1} \binom{n-1}{j} \left( \frac{\langle w''_k, a''_k \rangle}{1 - \langle w'_k, a'_k \rangle} \right)^j \right)^{2/p}, \quad 1 \leq p < \infty, \\
f_k^*(\infty, a, w) &= \frac{\sqrt{1 - |a'_k|^2}}{|a''_k|} \frac{\langle w''_k, a''_k \rangle}{1 - \langle w'_k, a'_k \rangle}
\end{aligned}$$

is a worst function;

(iii)

$$\begin{aligned}
&E(a, A, H_p(B_n)) \\
&= \begin{cases} \left( \frac{\Gamma(n + p/2)}{\Gamma(n) \Gamma(p/2 + 1) D_{nk}(p, a)} \right)^{1/p} \frac{|a''_k|}{(1 - |a'_k|^2)^{n/p+1/2}}, & 1 \leq p < \infty, \\ \frac{|a''_k|}{\sqrt{1 - |a'_k|^2}}, & p = \infty. \end{cases}
\end{aligned}$$

*Proof.* Let  $U$  be the  $n \times n$  matrix,

$$U = \begin{pmatrix} 1 & & & & \\ & \ddots & & & 0 \\ & & 1 & & \\ & & & \boxed{\phantom{C}} & \\ 0 & & & & C \end{pmatrix},$$

where  $C$  is the unitary  $k \times k$  matrix which transforms  $(a_{n-k+1}, \dots, a_n)$  into  $(0, \dots, 0, |a''_k|)$ . Since the measure  $\sigma_n$  being invariant with respect to  $U$ , the mapping  $f(z) \rightarrow f(Uz)$  is an isometry of  $H_p(B_n)$ . Using the unitary transformation with matrix  $U$  we reduce the problem of recovering  $f(a)$  to the one of recovering  $f(Ua)$ , using the information provided by operator  $I_U$ , given by  $I_U f = f|_{UA} = f|_A = If$ . ( $I_U$  coincides with  $I$  because  $A$  is a  $U$ -invariant set.) Let  $\tilde{I}$  be the information operator  $\tilde{I}f = f|_{(Uz)_n=0}$ . As  $\{(Uz)_n = 0\} \supset A$ , we evidently have

$$E(a, A, H_p(B_n)) \geq E(Ua, \{(Uz)_n = 0\}, H_p(B_n)).$$

Note that  $(Ua)'_1 = a'_k \in A$ . In accordance with the statement of Lemmas 1 and 2, the optimal algorithm in the recovery problem with the information operator  $\tilde{I}$  is  $\text{const } f((Ua)'_1) = \text{const } f(a'_k)$ . This algorithm uses only the information  $If = f|_A$ , and so it is an optimal

one for the recovery problem with information operator  $I$ . To finish the proof we note that  $\langle (Uw)'_1, (Ua)'_1 \rangle = \langle w'_k, a'_k \rangle$ ,  $(Uw)_n(\overline{Ua})_n = \langle Uw, Ua \rangle - \langle (Uw)'_1, (Ua)'_1 \rangle = \langle w, a \rangle - \langle w'_k, a'_k \rangle = \langle w''_k, a''_k \rangle$ . Now the required statements immediately follow from Lemmas 1 and 2.  $\square$

By analogy with Corollary 2,  $a \in \Delta_n^k(p)$  if

$$p > 2 \frac{n|a''_k| - \sqrt{1 - |a'_k|^2}}{\sqrt{1 - |a'_k|^2} - |a''_k|}.$$

Now let  $A$  be an arbitrary linear affine subset of  $B_n$ . In an appropriate coordinate system  $A$  has the following form

$$(8) \quad A = \{z \in B_n : z_{n-k+1} = c_{n-k+1}, \dots, z_n = c_n\}, \quad 1 \leq k \leq n,$$

$$(9) \quad \sum_{l=n-k+1}^n |c_l|^2 = |c'_k|^2 < 1 \text{ where } c = (0, \dots, 0, c_{n-k+1}, \dots, c_n) \in B_n.$$

Recall (see Rudin [7, p. 158]) that every automorphism  $\varphi$  of  $B_n$  determines the isometry  $\Phi_\varphi$  of  $H_p(B_n)$ ,

$$(10) \quad (\Phi_\varphi f)(z) = \frac{f(\varphi(z))(1 - |b|^2)^{n/p}}{(1 - \langle z, b \rangle)^{2n/p}}, \quad b = \varphi^{-1}(0),$$

and every automorphism of  $B_n$  is the superposition of some unitary transformation and special mapping  $\varphi_b$

$$\varphi_b(z) = \frac{b - P_b z - \sqrt{1 - |b|^2} Q_b z}{1 - \langle z, b \rangle}, \quad b \in B_n,$$

where

$$P_b z = \begin{cases} \frac{\langle z, b \rangle}{\langle b, b \rangle} b, & b \neq 0, \\ 0, & b = 0, \end{cases} \quad Q_b z = z - P_b z.$$

**Proposition 4.** *Let  $a \in B_n$ ,  $A \subset B_n$ ,  $\varphi \in \text{Aut}(B_n)$ ,  $D = \varphi(A)$ ,  $d = \varphi(a)$ , and  $I$  and  $I_1$  are the information operators  $I f = f|_A$ ,  $I_1 f = f|_D$ . If the algorithm  $f(a) \approx \int_A f(w) d\mu^*(w)$  is an optimal one in the recovery problem*

$$E(a, A, H_p(B_n)) = \inf_S \sup_{f \in BH_p(B_n)} |f(a) - S I f|$$

and  $f^*(w)$  is a worst function in this problem, then  $f(d) \approx \int_D f(w) d\mu_1^*(w)$  where

$$d\mu_1^*(w) = \begin{cases} \left( \frac{1 - \langle a, \varphi^{-1}(0) \rangle}{1 - \langle \varphi^{-1}(w), \varphi^{-1}(0) \rangle} \right)^{2n/p} d(\mu^* \circ \varphi^{-1})(w), & 1 \leq p < \infty, \\ d(\mu^* \circ \varphi^{-1})(w), & p = \infty, \end{cases}$$

is an optimal algorithm in the problem

$$E(d, D, H_p(B_n)) = \inf_S \sup_{f \in BH_p(B_n)} |f(d) - SI_1 f|,$$

$$f_1^*(w) = \frac{f^*(\varphi^{-1}(w))(1 - |\varphi(0)|^2)^{n/p}}{(1 - \langle w, \varphi(0) \rangle)^{2n/p}}$$

is a worst function for the last problem, and

$$E(d, D, H_p(B_n)) = \begin{cases} \frac{(1 - |\varphi(0)|^2)^{n/p}}{|1 - \langle \varphi(a), \varphi(0) \rangle|^{2n/p}}, & 1 \leq p < \infty, \\ E(a, A, H_p(B_n)), & p = \infty. \end{cases}$$

*Proof.* Let  $f \in H_p(B_n)$ ,  $1 \leq p < \infty$ . Put  $g(z) = (\Phi_\varphi f)(z)$ . In view of (10) we have  $\|g\|_p = \|f\|_p$  and

$$f(z) = \frac{g(\varphi^{-1}(z))(1 - |\varphi(0)|^2)^{n/p}}{(1 - \langle z, \varphi(0) \rangle)^{2n/p}},$$

because for every  $\varphi \in \text{Aut}(B_n)$

$$1 - \langle \varphi(z), \varphi(w) \rangle = \frac{(1 - |\varphi^{-1}(0)|^2)(1 - \langle z, w \rangle)}{(1 - \langle z, \varphi^{-1}(0) \rangle)(1 - \langle \varphi^{-1}(0), w \rangle)}$$

(see Rudin [7, p. 26]). Let  $S$  be some algorithm for the information operator  $I_1$ , then

$$f(d) - SI_1 f = \frac{(1 - |\varphi(0)|^2)^{n/p}}{(1 - \langle \varphi(a), \varphi(0) \rangle)^{2n/p}} (g(a) - \tilde{S}I_1 g),$$

where

$$\tilde{S}I_1 g = \frac{(1 - \langle \varphi(a), \varphi(0) \rangle)^{2n/p}}{(1 - |\varphi(0)|^2)^{n/p}} SI_1(\Phi_{\varphi^{-1}} g).$$

Hence

$$E(d, D, H_p(B_n)) \geq \frac{(1 - |\varphi(0)|^2)^{n/p}}{(1 - \langle \varphi(a), \varphi(0) \rangle)^{2n/p}} E(a, A, H_p(B_n)).$$

On the other hand,

$$\begin{aligned} E(d, D, H_p(B_n)) &\leq \sup_{f \in BH_p(B_n)} \left| f(d) - \int_D f(w) d\mu_1^*(w) \right| \\ &= \sup_{f \in BH_p(B_n)} \left| \frac{g(a)(1 - |\varphi(0)|^2)^{n/p}}{(1 - \langle \varphi(a), \varphi(0) \rangle)^{2n/p}} \right. \\ &\quad \left. - \int_A \frac{g(w)(1 - |\varphi(0)|^2)^{n/p}}{(1 - \langle \varphi(w), \varphi(0) \rangle)^{2n/p}} \left( \frac{1 - \langle a, \varphi^{-1}(0) \rangle}{1 - \langle w, \varphi^{-1}(0) \rangle} \right)^{2n/p} d\mu^*(w) \right| \\ &= \frac{(1 - |\varphi(0)|^2)^{n/p}}{|1 - \langle \varphi(a), \varphi(0) \rangle|^{2n/p}} \sup_{g \in BH_p(B_n)} \left| g(a) - \int_A g(w) d\mu^*(w) \right| \\ &= \frac{(1 - |\varphi(0)|^2)^{n/p}}{|1 - \langle \varphi(a), \varphi(0) \rangle|^{2n/p}} E(a, A, H_p(B_n)). \end{aligned}$$

The case  $p = \infty$  may be worked out in the same way. The proof is complete.  $\square$

Now we shall apply the last proposition to the recovery problem corresponding to the information operator, which is one of the restrictions to the set (8). Note that  $\varphi_c$  ( $c$  is defined in (9)) transforms the set  $D = \{z \in B_n : z_{n-k+1} = \dots = z_n = 0\}$  into the set (8). Put

$$\begin{aligned} \Delta_n^k(p, c) &= \varphi_c(\Delta_n^k(p)) \\ &= \left\{ z \in B_n : \left| \left( \sqrt{1 - |c|^2} z - c + \frac{\langle z, c \rangle}{1 + \sqrt{1 - |c|^2}} c \right)_k \right| \right. \\ &\quad \left. < \lambda_n(p) \sqrt{|1 - \langle z, c \rangle|^2 + (1 - |c|^2) |z'_k|^2} \right\}, \end{aligned}$$

$$a_c = \varphi_c(a), \quad \tilde{a}_c = \varphi_c((a_c)'_k) = \varphi_c \left( -\frac{\sqrt{1 - |c|^2}}{1 - \langle a, c \rangle} a'_k \right) = \frac{1 - |c|^2}{1 - \langle a, c \rangle} a'_k + c.$$

It follows from Proposition 2 that  $\Delta_n^k(p, c) = B_n$  if  $n < 5$ .

**Theorem 2.** *Let  $a \in \Delta_n^k(p, c) \setminus A$ . Then:*

(i) *the algorithm*

$$f(a) \approx \begin{cases} \left( \frac{1 - |c|^2}{1 - \langle a, c \rangle} \right)^{2n/p} (D_{nk}(p, a_c))^{(p-2)/p} f(\tilde{a}_c), & 1 \leq p < \infty, \\ \frac{1 - |a_c|^2}{1 - |(a_c)'_k|^2} f(\tilde{a}_c), & p = \infty, \end{cases}$$

*is an optimal one;*

(ii)

$$f_k^*(c; p, a, w) = \begin{cases} \frac{(1 - |c|^2)^{n/p}}{(1 - \langle w, c \rangle)^{2n/p}} f_k^*(p, a_c, \varphi_c(w)), & 1 \leq p < \infty, \\ f_k^*(\infty, a_c, \varphi_c(w)), & p = \infty, \end{cases}$$

*is a worst function;*

(iii)

$$E(a, A, H_p(B_n))$$

$$= \begin{cases} \left( \frac{(1 - |c|^2)^n \Gamma(n + p/2)}{|1 - \langle a, c \rangle|^{2n} \Gamma(n) \Gamma(p/2 + 1) D_{nk}(p, a_c)} \right)^{1/p} \\ \quad \times \frac{|(a_c)''_k|}{(1 - |(a_c)'_k|^2)^{n/p+1/2}}, & 1 \leq p < \infty, \\ \frac{|(a_c)''_k|}{\sqrt{1 - |(a_c)'_k|^2}}, & p = \infty. \end{cases}$$

*Proof.* As  $\varphi_c \circ \varphi_c = \text{id}$  (see Rudin [7, p. 26]), we have  $\varphi_c(\varphi_c(a)) = a$ ,  $\varphi_c^{-1}(0) = \varphi_c(0) = c$ . By Proposition 4 we can reduce the problem of recovery at the point  $d = \varphi_c(a)$  and information on the operator to be  $If = f|_D$  to recovery at the point  $a$  and information on the set  $A$ . We obtain from Proposition 4 that an optimal algorithm is

$$F(a) \approx \gamma^{2n/p} (D_{nk}(p, a_c))^{(p-2)/p} f(\tilde{a}_c),$$

where

$$\gamma = \frac{1 - \langle \varphi_c(a), \varphi_c^{-1}(0) \rangle}{1 - \langle \varphi_c^{-1}(\tilde{a}_c), \varphi_c^{-1}(0) \rangle}$$

and

$$E(a, A, H_p(B_n)) = \frac{(1 - |\varphi_c(0)|^2)^{n/p}}{|1 - \langle \varphi_c(a_c), \varphi_c(0) \rangle|^{2n/p}} E(a_c, D, H_p(B_n)).$$

Note that

$$\gamma = \frac{1 - \langle \varphi_c(a), \varphi_c(0) \rangle}{1 - \langle (a_c)_k, \varphi_c(0) \rangle} = 1 - \langle \varphi_c(a), \varphi_c(0) \rangle = \frac{1 - |c|^2}{1 - \langle a, c \rangle},$$

$1 - |\varphi_c(0)|^2 = 1 - |c|^2$ ,  $1 - \langle \varphi_c(a), \varphi_c(0) \rangle = 1 - \langle a, c \rangle$ , and now the required statement follows for  $1 \leq p < \infty$ . The case  $p = \infty$  is quite analogous. The theorem is proved.  $\square$

The following theorem is a generalization of the Schwartz Lemma for functions from  $H_p(B_n)$ .

**Theorem 3.** *Let  $f \in BH_p(B_n)$ ,  $f(0) = 0$ ,  $1 \leq p < \infty$ ,  $a \in \Delta_n^n(p) = \{z \in B_n : |z| < \lambda_n(p)\}$ . Then*

$$\begin{aligned} & |f(a)| \\ & \leq |a| \left( \frac{\Gamma(n + p/2)}{\Gamma(n)\Gamma(p/2 + 1)(1 - |a|^2)^n} \sum_{j=0}^{n-1} \frac{(-1)^j}{2j/p + 1} \binom{n-1}{j} |a|^{2j} \right)^{1/p}. \end{aligned}$$

*Remark.* The case  $p = \infty$  is obvious:  $|f(a)| \leq |a|$  is the classical Schwartz Lemma.

*Proof.* From (4) we have

$$|f(a)| \leq E(a, \{0\}, H_p(B_n)) = \left( \frac{\Gamma(n + p/2)}{\Gamma(n)\Gamma(p/2 + 1)D_{nn}(p, a)} \right)^{1/p} |a|.$$

Now the required statement follows from the definition of  $D_{nn}(p, a)$ .  $\square$

Using Theorem 1' instead of Theorem 1 in the case  $p = 2$  we can eliminate the condition  $a \in \Delta_n^k(2, c)$ . The following two theorems may be proved using the same arguments as in Theorems 2 and 3, respectively.

**Theorem 2'.** *Let  $a \in B_n$  and  $p = 2$ . Then:*

(i) the algorithm

$$f(a) \approx \left( \frac{1 - |c|^2}{1 - \langle a, c \rangle} \right)^n f(\tilde{a}_c)$$

is an optimal one;

(ii) a worst function is

$$f_k^*(c; 2, a, w) = \frac{(1 - |c|^2)^{n/2}}{(1 - \langle w, c \rangle)^n} f_k^*(2, a_c, \varphi_c(w)),$$

where

$$f_k^*(2, a, z) = \frac{(1 - \langle z, a \rangle)^{-n} - (1 - \langle z'_k, a'_k \rangle)^{-n}}{\sqrt{(1 - |a|^2)^{-n} - (1 - |a'_k|^2)^{-n}}};$$

(iii)

$$E(a, A, H_2(B_n)) = \sqrt{(1 - |a|^2)^{-n} - \left( \frac{|1 - \langle a, c \rangle|^2}{1 - |c|^2} - |a'_k|^2 \right)^{-n}}.$$

**Theorem 3'.** Let  $a \in B_n$ ,  $f \in BH_2(B_n)$ ,  $f(0) = 0$ . Then

$$|f(a)| \leq \sqrt{(1 - |a|^2)^{-n} - 1}.$$

Now we can easily demonstrate the difference between the cases  $n \geq 6$  and  $n < 6$ . Let us consider recovery in  $H_2(B_n)$  when the information operator is, for example,  $If = f|_{\{z_n=0\}}$ . If  $n < 6$  the set  $\{z_n = 0\}$  is the zero-set of the worst function, but if  $n > 6$  there are extra sets: the divisor of the worst function  $f_1^*(2, a, w)$  in  $B_n$  is (see Lemma 1)

$$D_n = \bigcup_{\substack{k \in \mathbb{Z} \\ |k| < n/6}} \{w \in B_n : \langle w, a \rangle - e^{2\pi ki/n} \langle w'_1, a'_1 \rangle = 1 - e^{2\pi ki/n}\}.$$

This means that in the case  $n \leq 6$  any additional information will decrease the intrinsic error. On the contrary, if  $n > 6$  there exists an extension of the information operator, which does not decrease the intrinsic error of recovery.

Now we shall prove Proposition 2.

*Proof of Proposition 2.* To prove (i) we must prove that  $P_k(\alpha, u)$ ,  $k = 1, 2, 3, 4$ , has no zeros in the unit disk if  $\alpha > \frac{3}{2}$ . We have

$$\begin{aligned} P_1(\alpha, u) &= \frac{1}{\alpha - 1} - \frac{u}{2}, \\ P_2(\alpha, u) &= \frac{1}{\alpha - 1} - \frac{2u}{\alpha} + \frac{u^2}{\alpha + 1}, \\ P_3(\alpha, u) &= \frac{1}{\alpha - 1} - \frac{3u}{\alpha} + \frac{3u^2}{\alpha + 1} - \frac{u^3}{\alpha + 2}, \\ P_4(\alpha, u) &= \frac{1}{\alpha - 1} - \frac{4u}{\alpha} + \frac{6u^2}{\alpha + 1} - \frac{4u^3}{\alpha + 2} + \frac{u^4}{\alpha + 3}. \end{aligned}$$

It is easy to see that  $P_1(\alpha, u)$  and  $P_2(\alpha, u)$  have no zeros in the unit disk. Next we observe that  $P_3(\alpha, u)$  has only one real zero which belongs to the interval  $(1, (\alpha + 2)/(\alpha + 1))$ , and so the moduli of the other zeros of  $P_3(\alpha, u)$  are greater than 1.

Now let us consider  $P_4(\alpha, u)$ . We can easily check that

$$P_4(\alpha, u) = \int_0^1 t^{\alpha-2}(1-ut)^4 dt,$$

which yields that  $P_4(\alpha, u)$  has no real zeros. Let  $x_1, \tilde{x}_1, x_2, \tilde{x}_2$  be the roots of  $P_4(\alpha, u)$ . In view of the Viète theorem,

$$\begin{cases} \operatorname{Re} x_1 + \operatorname{Re} x_2 = 2\frac{\alpha+3}{\alpha+2}, \\ |x_1|^2 + |x_2|^2 + 4\operatorname{Re} x_1 \operatorname{Re} x_2 = 6\frac{\alpha+3}{\alpha+1}, \\ |x_1|^2 \operatorname{Re} x_2 + |x_2|^2 \operatorname{Re} x_1 = 2\frac{\alpha+3}{\alpha}, \\ |x_1|^2 |x_2|^2 = \frac{\alpha+3}{\alpha+1}. \end{cases}$$

Suppose  $|x_1| = 1$ . From the first, third, and fourth equations of this system we find that  $\operatorname{Re} x_1 = (\alpha - 1)(\alpha + 3)\alpha^{-1}(\alpha + 2)^{-1}$ ,  $\operatorname{Re} x_2 = (\alpha + 1)(\alpha + 3)\alpha^{-1}(\alpha + 2)^{-1}$ , and then from the second equation we obtain  $\alpha^2 + 2\alpha + 3 = 0$ . This contradiction shows that  $P_4(\alpha, u)$  has no roots, which modulus is equal to 1. Let  $\alpha = \frac{3}{2}$ . In this case,

$$\begin{cases} \operatorname{Re} x_1 + \operatorname{Re} x_2 = \frac{18}{7}, \\ |x_1|^2 + |x_2|^2 + 4\operatorname{Re} x_1 \operatorname{Re} x_2 = \frac{54}{5}, \\ |x_1|^2 \operatorname{Re} x_2 + |x_2|^2 \operatorname{Re} x_1 = 6, \\ |x_1|^2 |x_2|^2 = 9. \end{cases}$$

It follows from the first, third, and fourth equations of this last system that  $\operatorname{Re} x_1 = 6 \left( |x_2|^2 - \frac{27}{7} \right) / (|x_2|^4 - 9)$ . If  $|x_1| < 1$ , then  $|x_2|^2 > 9$  and therefore  $\operatorname{Re} x_1 > 0$ ,  $\operatorname{Re} x_2 > \frac{18}{7} - |x_1| > 0$ . Now  $|x_1| < 1$  yields  $|x_1|^2 + |x_2|^2 = |x_1|^2 + 9/|x_1|^2 > 10$ ,  $0 < \operatorname{Re} x_1 \operatorname{Re} x_2 < \frac{1}{5}$ . As  $\operatorname{Re} x_1 \operatorname{Re} x_2 > 0$  we have  $|x_1|^2 + 9/|x_1|^2 < \frac{54}{5}$  and hence  $|x_1|^2 > (27 - 6\sqrt{14})/5 > 0.9$ ,  $|x_2|^2 < 10$ ,  $\operatorname{Re} x_2 + 10 \operatorname{Re} x_1 > |x_1|^2 \operatorname{Re} x_2 + |x_2|^2 \operatorname{Re} x_1 = 6$ . Combining this equality with the equation  $\operatorname{Re} x_2 + \operatorname{Re} x_1 = \frac{18}{7}$ , we find

$$\operatorname{Re} x_1 > \frac{8}{21} > \frac{1}{5}, \quad \operatorname{Re} x_2 = \frac{18}{7} - \operatorname{Re} x_1 > 1,$$

which contradicts  $\operatorname{Re} x_1 \operatorname{Re} x_2 < \frac{1}{5}$ . Thus we have proved that  $P_4\left(\frac{3}{2}, u\right)$  has no roots in the unit disk.

Finally, suppose the set of those  $\alpha > \frac{3}{2}$  such that  $P_4(\alpha, u)$  has zeros inside the unit disk is not empty. Let  $\alpha_0$  be the infimum of those  $\alpha$ . We can easily check that  $P_4(\alpha, u)$  has no multiple zeros and hence the zeros of  $P_4(\alpha, u)$  are differentiable functions of  $\alpha$ . Therefore  $P_4(\alpha_0, u)$  has a root, with modulus equal to 1, but we proved previously that this is impossible and so (i) is proved. To prove (ii) note that

$$P_{n-1}(2, u) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} \binom{n-1}{k} u^k = \frac{1 - (1-u)^n}{nu}.$$

Thus  $u_{n-1} = 1 - e^{2\pi i/n}$  is the root of  $P_{n-1}(2, u)$ . As  $|1 - e^{2\pi i/n}| = 2 \sin(\pi/n)$ ,  $|u_{n-1}| < 1$  if  $n \geq 7$ , we must only prove that  $P_5(\alpha, u)$  has zeros inside the unit disk for some  $\alpha \geq \frac{3}{2}$ . Let  $u(\alpha)$  be the root of  $P_5(\alpha, u)$  so that  $P_5(\alpha, u(\alpha)) \equiv 0$ . Hence it follows that

$$\begin{aligned} \frac{du(\alpha)}{d\alpha} = & \frac{\frac{u^5(\alpha)}{(\alpha+4)^2} - 5\frac{u^4(\alpha)}{(\alpha+3)^2} + 10\frac{u^3(\alpha)}{(\alpha+2)^2} - 10\frac{u^2(\alpha)}{(\alpha+1)^2}}{5\frac{u^4(\alpha)}{\alpha+4} - 4\frac{u^3(\alpha)}{\alpha+3} + 6\frac{u^2(\alpha)}{\alpha+2} - 4\frac{u(\alpha)}{\alpha+1} + \frac{1}{\alpha}} \\ & + \frac{5\frac{u(\alpha)}{\alpha^2} - \frac{1}{(\alpha-1)^2}}{5\frac{u^4(\alpha)}{\alpha+4} - 4\frac{u^3(\alpha)}{\alpha+3} + 6\frac{u^2(\alpha)}{\alpha+2} - 4\frac{u(\alpha)}{\alpha+1} + \frac{1}{\alpha}}. \end{aligned}$$

$u_5 = e^{-\pi i/3}$  is a root of  $P_5(2, u)$ . Substituting  $u(\alpha) = e^{-\pi i/3}$ ,  $\alpha = 2$ , in the last expression we obtain

$$\left. \frac{du(\alpha)}{d\alpha} \right|_{\alpha=2} = \frac{119 + 56i\sqrt{3}}{360},$$

and, therefore, if  $\alpha > 2$ ,  $\alpha$  is sufficiently close to 2 then  $|u(\alpha)| < 1$ . The proposition is proved.  $\square$

### 3. RECOVERY IN THE BERGMAN SPACE

The Bergman space  $A_p(B_n)$  is the set of holomorphic functions in  $B_n$  which satisfy

$$\|f\|_{A_p} = \left( \int_{B_n} |f(z)|^p d\nu_n(z) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

where  $\nu_n$  is the normalized Lebesgue measure in  $B_n$ .  $A_\infty(B_n)$  is the same as  $H_\infty(B_n)$ . In this section we shall extend the results of the

previous section to the Bergman space  $A_p(B_n)$ . Let the linear affine  $M$  subspace be given by the equations  $z_{n-k+1} = c_{n-k+1}, \dots, z_n = c_n$ ,  $1 \leq k \leq n$ , where  $c = (0, \dots, 0, c_{n-k+1}, \dots, c_n) \in B_n$ .

Let us consider the recovery problem (1), where  $X_p = A_p(B_n)$ ,  $Lf = f(a)$ ,  $a \in B_n$ , and  $If = f|_A$  where  $A = M \cap B_n$ . Let  $e$  be the standard embedding  $e: \mathbb{C}^n \rightarrow \mathbb{C}^{n+1}$ ,  $ez = (0, z)$ . Denote  $\tilde{\Delta}_n^k(p, c) = \{z \in B_n : ez \in \Delta_{n+1}^k(p, ec)\}$ . Note that  $\tilde{\Delta}_n^k(p, c) = B_n$ , if  $n \leq 4$ .

**Theorem 4.** *Let  $a \in \tilde{\Delta}_n^k(p, c)$  then:*

(i) *an optimal algorithm is*

$$f(a) \approx s_0 f(\tilde{a}_c),$$

where

$$s_0 = \left( \frac{1 - |c|^2}{1 - \langle a, c \rangle} \right)^{2(n+1)/p} (D_{n+1,k}(p, \varphi_{ec}(ea)))^{(p-2)/p};$$

(ii) *a worst function is*

$$g^*(w) = \frac{(1 - |c|^2)^{(n+1)/p}}{(1 - \langle w, c \rangle)^{2(n+1)/p}} f_k^*(p, \varphi_{ec}(ea), \varphi_{ec}(ew));$$

(iii) *the intrinsic error is*

$$\begin{aligned} E(a, A, A_p(B_n)) &= \left( \frac{(1 - |c|^2)^{n+1} \Gamma(n + p/2 + 1)}{|1 - \langle a, c \rangle|^{2(n+1)} \Gamma(n+1) \Gamma(p/2 + 1) D_{n+1,k}(p, \varphi_{ec}(ea))} \right)^{1/p} \\ &\quad \times \frac{|(a_c)_k''|}{(1 - |(a_c)_k'|^2)^{(n+1)/p+1/2}}. \end{aligned}$$

*Proof.* Recall (Rudin [7, p. 127]) that there exists the linear isometric embedding  $E: A_p(B_n) \rightarrow H_p(B_{n+1})$ ,

$$(Eg)(z_0, z) = g(z), \quad z_0 \in \mathbb{C}, \quad z \in B_n, \quad |z_0|^2 + |z|^2 < 1.$$

If  $g \in BA_p(B_n)$  then  $Eg \in BH_p(B_{n+1})$ . As

$$\tilde{e}a_{ec} = \varphi_{ec}((ea_{ec})'_k) = \frac{1 - |c|^2}{1 - \langle a, c \rangle} (ea'_k + ec = (0, \tilde{a}_c))$$

we have (using Theorem 2)

$$\begin{aligned} (11) \quad E(a, A, A_p(B_n)) &\leq |g(a) - s_0 g(\tilde{a}_c)| = |Eg(ea) - s_0 Eg(\tilde{e}a_{ec})| \\ &\leq E(ea, eA, H_p(B_{n+1})). \end{aligned}$$

On the other hand,  $Eg^*$  turns (11) in the equality and in view of the fact that  $Eg^*(\tilde{e}a_{ec}) = g^*(\tilde{a}_c) = 0$  we have

$$E(a, A, A_p(B_n)) \geq |g^*(a)| = |(Eg^*)(ea)| = E(ea, eA, H_p(B_{n+1})).$$

Now all required statements follow from Theorem 2. The theorem is proved.  $\square$

The following theorem is the variant of the Schwartz Lemma for the Bergman spaces. Its proof is analogous to the proof of Theorem 3.

**Theorem 5.** *Let  $g \in BA_p(B_n)$ ,  $1 \leq p < \infty$ ,  $g(0) = 0$ ,  $a \in B_n$ ,  $|a| < \lambda_{n+1}(p)$ . Then*

$$|g(a)| \leq |a| \left( \frac{\Gamma(n + p/2 + 1)}{\Gamma(n + 1)\Gamma(p/2 + 1)(1 - |a|^2)^{n+1}} \sum_{j=0}^n \frac{(-1)^j}{2j/p + 1} \binom{n}{j} |a|^{2j} \right)^{1/p}.$$

*In particular, if  $n \leq 4$  this inequality holds for every  $a \in B_n$ .*

Note that using Theorem 1' instead of Theorem 1, we can obtain the results analogous to Theorems 2' and 3' in the case of  $A_2(B_n)$ .

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