Inequalities for derivatives with the Fourier transform

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Abstract

In this paper we study sharp constants in inequalities of the following form

$$\|x^{(k)}(\cdot)\|_{L_q(\mathbb{R})} \le K\|Fx(\cdot)\|^{\alpha}_{L_p(\mathbb{R})}\|x^{(n)}(\cdot)\|^{\beta}_{L_r(\mathbb{R})},$$

where $Fx(\cdot)$ is the Fourier transform of $x(\cdot)$. The sharp value of K in the general case (that is, for all $n \in \mathbb{N}$ and $0 \le k < n$) was known only for $q = r = 2$ and $p \ge 2$. We obtain the sharp constant in the general case for $q = \infty$, $r = 2$, and $1 \le p \le \infty$. We also generalized this two cases on multidimensional situation. The sharp constants is obtained for fractional degrees of the Laplace operator $(-\Delta)^{k/2}$ and derivatives D^{α} of order $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d_+$.

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1. Introduction

Inequalities for derivatives of Kolmogorov-type on the real line are traditionally understood as multiplicative inequalities of the form

$$\|x^{(k)}(\cdot)\|_{L_q(\mathbb{R})} \le K\|x(\cdot)\|^{\alpha}_{L_p(\mathbb{R})}\|x^{(n)}(\cdot)\|^{\beta}_{L_r(\mathbb{R})}, \tag{1}$$

where $0 \le k < n$ are integers, $1 \le p, q, r \le \infty$, $\alpha, \beta \ge 0$. It is assumed that the function $x(\cdot) \in L_p(\mathbb{R})$ has the $(n-1)$-st derivative which is locally absolutely continuous on $\mathbb{R}$ and $x^{(n)}(\cdot) \in L_r(\mathbb{R})$. The problem is to find the best possible constant K (the smallest constant) in (1). Inequalities (1) with the best possible constant we call the sharp inequalities.

The first sharp inequalities of type (1) were obtained by Landau [8] (for the half-line $\mathbb{R}_+$, $n = 2$, $k = 1$, $p = q = r = \infty$) and Hadamard [5] (for the line $\mathbb{R}$, $n = 2$, $k = 1$, $p = q = r = \infty$). In the late 30s Kolmogorov [7] determined the exact constant in (1) for line, $p = q = r = \infty$ in the general case, i.e., for

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any $n \geq 2$ and $0 < k < n$. This result remains one of the most remarkable ones in this area, and the sharp inequalities of type (1) are often referred to as Kolmogorov's inequalities. Paying tribute to the first result in this area, these inequalities are also called Landau–Kolmogorov inequalities.

There are a lot of papers devoted to the search of the best constants in inequalities of such type (see survey papers [19] and [2]). However, there are only three more complete results for the line comparable to Kolmogorov's were obtained ($p = q = r = 2$ by Hardy, Littlewood, and Pólya [6], $p = q = r = 1$ by Stein [17], and $p = r = 2$, $q = \infty$ by Taikov [18]).

Sharp inequalities for derivatives are closely connected with the problems of optimal recovery of functions and their derivatives from an inaccurate information about function itself or its Fourier transform (see [13] and [10]). In the latter case sharp inequalities contain the Fourier transform of function instead of function. It turns out that inequalities of this type can be obtained even in a more general form. Namely, they are valid not only for all $n \in \mathbb{N}$ and $0 \leq k < n$ but for some family of metrics in which the Fourier transform of function is measured. One of the first result of such type was obtained in [10].

Let S be the Schwartz space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}$, S' the dual space of distributions, and $F\colon S' \to S'$ the Fourier transform. Let $1 \leq p \leq \infty$ and $n \in \mathbb{N}$. Set

$$X_p^n = \{\, x \in S' : Fx(\cdot) \in L_p(\mathbb{R}),\ x^{(n)}(\cdot) \in L_2(\mathbb{R}) \,\}.$$

It was proved in [10] that for all $x(\cdot) \in X_p^n$, $2 \leq p \leq \infty$, and all $0 \leq k < n$ the following sharp inequality holds:

$$\|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \leq K(k,n,p) \|Fx(\cdot)\|_{L_p(\mathbb{R})}^{\frac{n-k}{n+1/2-1/p}} \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{k+1/2-1/p}{n+1/2-1/p}}, \tag{2}$$

where for $2 < p \leq \infty$

$$K(k,n,p) = \sqrt{\frac{n+1/2-1/p}{k+1/2-1/p}} \left(\frac{\sqrt{k+1/2-1/p}\, B^{1/2-1/p}}{\sqrt{2\pi}(n-k)^{1-1/p}} \right)^{\frac{n-k}{n+1/2-1/p}},$$

$$B = B\left(\frac{k+1/2-1/p}{(n-k)(1-2/p)}, 2\frac{1-1/p}{1-2/p} \right),$$

and $B(\cdot,\cdot)$ is the Eiler beta-function; for $p = 2$

$$K(k,n,2) = \left(\frac{1}{2\pi} \right)^{\frac{n-k}{2n}}.$$

The case $p = 2$ immediately follows from the Hardy–Littlewood–Pólya inequality, so (2) may be considered as a generalized Hardy–Littlewood–Pólya inequality.

This paper is devoted to the study of sharp inequalities of such type, including the multidimensional situation.

2. The Carlson, Levin, and Taikov inequalities

The Carlson inequality [4]

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \le \sqrt{\pi}\|x(t)\|^{1/2}_{L_2(\mathbb{R}_+)}\|tx(t)\|^{1/2}_{L_2(\mathbb{R}_+)}, \quad \mathbb{R}_+ = [0,+\infty),$$

was generalized by many authors (see [9], [1], [3], [15]).

In particular, it was proved by Levin [9] that for $p > 1$, $q > 1$, $\lambda > 0$, $\mu > 0$ the following sharp inequality holds:

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \le K\|t^{1-\frac{1+\lambda}{p}}x(t)\|^{pa}_{L_p(\mathbb{R}_+)}\|t^{1+\frac{\mu-1}{q}}x(t)\|^{qb}_{L_q(\mathbb{R}_+)}, \tag{3}$$

where

$$a = \frac{\mu}{p\mu + q\lambda}, \quad b = \frac{\lambda}{p\mu + q\lambda},$$

and

$$K = \left(\frac{1}{pa}\right)^a \left(\frac{1}{qb}\right)^b \left(\frac{1}{\lambda+\mu}B\left(\frac{a}{1-a-b}, \frac{b}{1-a-b}\right)\right)^{1-a-b}.$$

Lemma 1 ([3]). *Let $1 < p, q < \infty$ and $\lambda, \mu > 0$. The following sharp inequality holds:*

$$\|x(t)\|_{L_1(\mathbb{R})} \le 2^{1-a-b}K\|t^{1-\frac{1+\lambda}{p}}x(t)\|^{pa}_{L_p(\mathbb{R})}\|t^{1+\frac{\mu-1}{q}}x(t)\|^{qb}_{L_q(\mathbb{R})}. \tag{4}$$

Proof. For all positive a_1, a_2, b_1, b_2 we have (see [6])

$$a_1^a b_1^b + a_2^a b_2^b \le 2^{1-a-b}(a_1 + a_2)^a (b_1 + b_2)^b. \tag{5}$$

Using this inequality we obtain

$$\begin{aligned}
\|x(\cdot)\|_{L_1(\mathbb{R})} &= \int_{-\infty}^0 |x(t)|\,dt + \int_0^\infty |x(t)|\,dt \\
&\le K\left(\int_{-\infty}^0 |t|^{p-1-\lambda}|x(t)|^p\,dt\right)^a \left(\int_{-\infty}^0 |t|^{q+\mu-1}|x(t)|^q\,dt\right)^b \\
&\quad + K\left(\int_0^\infty |t|^{p-1-\lambda}|x(t)|^p\,dt\right)^a \left(\int_0^\infty |t|^{q+\mu-1}|x(t)|^q\,dt\right)^b \\
&\qquad \le 2^{1-a-b}K\|t^{1-\frac{1+\lambda}{p}}x(t)\|^{pa}_{L_p(\mathbb{R})}\|t^{1+\frac{\mu-1}{q}}x(t)\|^{qb}_{L_q(\mathbb{R})}.
\end{aligned}$$

The sharpness of this inequality follows from the sharpness of (3) and the fact that (5) turns into equality for $a_1 = a_2$ and $b_1 = b_2$. □

Theorem 1. *Let $n \in \mathbb{N}$, $0 \le k < n$, $1 \le p \le \infty$, and $p + k > 1$. For all $x(\cdot) \in X^n_p$ the following sharp inequality holds:*

$$\|x^{(k)}(\cdot)\|_{L_\infty(\mathbb{R})} \le K_1(k,n,p)\|Fx(\cdot)\|^{\frac{n-k-1/2}{n+1/2-1/p}}_{L_p(\mathbb{R})}\|x^{(n)}(\cdot)\|^{\frac{k+1-1/p}{n+1/2-1/p}}_{L_2(\mathbb{R})}, \tag{6}$$

where

$$K_1 = \left(\frac{1}{\pi}\right)^{1-b}\left(\frac{1}{pa}\right)^{a}\left(\frac{1}{b}\right)^{b}\left(\frac{1}{n-k-1/2}B\left(\frac{1-b}{1-a-b},\frac{b}{1-a-b}\right)\right)^{1-a-b}$$

and

$$a = \frac{1}{p}\cdot\frac{n-k-1/2}{n+1/2-1/p}, \quad b = \frac{1}{2}\cdot\frac{k+1-1/p}{n+1/2-1/p}.$$

Proof. Let $1 < p < \infty$. Put $q = 2$, $\lambda = p(1+k)-1$, $\mu = 2n-2k-1$ in (4) and change $x(t)$ by $y(\xi)$. Then we obtain

$$\|y(\xi)\|_{L_1(\mathbb{R})} \le 2^{1-a-b}K\|\xi^{-k}y(\xi)\|^{pa}_{L_p(\mathbb{R})}\|\xi^{n-k}y(\xi)\|^{2b}_{L_2(\mathbb{R})}.$$

If we take $y(\xi) = \xi^k Fx(\xi)$ then we get

$$\int_{\mathbb{R}}|\xi|^k|Fx(\xi)|\,d\xi \le 2^{1-a-b}K\left(\int_{\mathbb{R}}|Fx(\xi)|^p\,d\xi\right)^a\left(\int_{\mathbb{R}}|\xi|^{2n}|Fx(\xi)|^2\,d\xi\right)^b.$$

By virtue of the Plancherel theorem we have

$$2\pi|x^{(k)}(t)| \le 2^{1-a-b}K(2\pi)^b\|Fx(\cdot)\|^{pa}_{L_p(\mathbb{R})}\|x^{(n)}(\cdot)\|^{2b}_{L_2(\mathbb{R})}.$$

It follows from properties of beta-function that

$$B\left(\frac{a}{1-a-b},\frac{b}{1-a-b}\right) = \frac{a+b}{a}B\left(\frac{1-b}{1-a-b},\frac{b}{1-a-b}\right).$$

Substituting this value in the expression for K we obtain (6).

Suppose that $x(\cdot) \in X^n_p$. If we take $\widehat{x}(\cdot)$ such that

$$F\widehat{x}(\xi) = \varepsilon(\xi)e^{-it\xi}Fx(\xi),$$

where

$$\varepsilon(\xi) = \begin{cases}(-i\,\mathrm{sign}\,\xi)^k\dfrac{\overline{Fx(\xi)}}{|Fx(\xi)|}, & Fx(\xi) \ne 0,\\ 0, & Fx(\xi) = 0,\end{cases}$$

then

$$\left|\int_{\mathbb{R}}\psi(\xi)F\widehat{x}(\xi)e^{it\xi}\,d\xi\right| = \int_{\mathbb{R}}|\psi(\xi)Fx(\xi)|\,d\xi.$$

The cases $p = 1, \infty$ were proved in [11] (see Corollaries 1, 2). □

For $p = 2$ inequality (6) has the form

$$\|x^{(k)}(\cdot)\|_{L_\infty(\mathbb{R})} \le K_2\left(\frac{1}{2\pi}\right)^{\frac{2n-2k-1}{4n}}\|Fx(\cdot)\|^{\frac{2n-2k-1}{2n}}_{L_2(\mathbb{R})}\|x^{(n)}(\cdot)\|^{\frac{2k+1}{2n}}_{L_2(\mathbb{R})},$$

where

$$K_2 = \left((2k+1)\sin\pi\frac{2k+1}{2n}\right)^{-1/2}\left(\frac{2k+1}{2n-2k-1}\right)^{\frac{2n-2k-1}{4n}}.$$

In view of Plancherel's theorem we have

$$\|x^{(k)}(\cdot)\|_{L_\infty(\mathbb{R})} \le K_2\|x(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2n-2k-1}{2n}}\|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2k+1}{2n}}.$$

This inequality was obtained by Taikov [18] (without using of the Levin inequality).

3. Multidimensional inequalities

Let $d_1(\cdot)$ and $d_2(\cdot)$ be measurable functions on $\mathbb{R}^d$. Assume that $|d_1(\cdot)|$ and $|d_2(\cdot)|$ be homogenous functions of degrees k, n, respectively (k and n are not necessarily integer), $d_j(\xi) \ne 0$, $j = 1, 2$, for almost all $\xi \in \mathbb{R}^d$. Consider the polar transformation

$$\begin{aligned}
\xi_1 &= \rho\cos\omega_1,\\
\xi_2 &= \rho\sin\omega_1\cos\omega_2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\xi_{d-1} &= \rho\sin\omega_1\sin\omega_2\ldots\sin\omega_{d-2}\cos\omega_{d-1},\\
\xi_d &= \rho\sin\omega_1\sin\omega_2\ldots\sin\omega_{d-2}\sin\omega_{d-1}.
\end{aligned}$$

Put $\omega = (\omega_1, \ldots, \omega_{d-1})$,

$$\begin{aligned}
\widetilde{d}_1(\omega) &= |d_1(\cos\omega_1, \ldots, \sin\omega_1\sin\omega_2\ldots\sin\omega_{d-2}\sin\omega_{d-1})|,\\
\widetilde{d}_2(\omega) &= |d_2(\cos\omega_1, \ldots, \sin\omega_1\sin\omega_2\ldots\sin\omega_{d-2}\sin\omega_{d-1})|,\\
J(\omega) &= \sin^{d-2}\omega_1\sin^{d-3}\omega_2\ldots\sin\omega_{d-2}.
\end{aligned}$$

Set

$$K_{pqr}(k, n, I) = \gamma^{-\frac{\gamma}{p}}(1-\gamma)^{-\frac{1-\gamma}{r}}\left(\frac{B\left(\widetilde{q}\gamma/p+1, \widetilde{q}(1-\gamma)/r\right) I}{r(n-k-d(1/q-1/r))}\right)^{1/\widetilde{q}},$$

where

$$\gamma = \frac{n-k-d(1/q-1/r)}{n+d(1/r-1/p)}, \quad \widetilde{q} = \left(\frac{1}{q} - \frac{\gamma}{p} - \frac{1-\gamma}{r}\right)^{-1}. \tag{7}$$

We will use the following result which was proved in [3, Theorem 2] (see also [15, Corollary 4]).

Theorem 2. *Let $1 \le q < p, r < \infty$, $k > d(1/p - 1/q)$, $n > k + d(1/q - 1/r)$, and*

$$\widetilde{I}_{qr} = \int_{\Pi_{d-1}} \frac{\widetilde{d}_1^{\,\widetilde{q}}(\omega)}{\widetilde{d}_2^{\,\widetilde{q}(1-\gamma)}(\omega)} J(\omega)\,d\omega < \infty, \quad \Pi_{d-1} = [0,\pi]^{d-2}\times[0,2\pi].$$

Then the following sharp inequality

$$\|d_1(\cdot)x(\cdot)\|_{L_q(\mathbb{R}^d)} \le K_{pqr}(k,n,\widetilde{I}_{qr})\|x(\cdot)\|^{\gamma}_{L_p(\mathbb{R}^d)}\|d_2(\cdot)x(\cdot)\|^{1-\gamma}_{L_r(\mathbb{R}^d)} \tag{8}$$

holds.

For the case when $p=q$ we have

Theorem 3. *Let* $1 \le p = q < r < \infty$, $k > 0$, $n > k + d(1/p - 1/r)$, *and*

$$I_{r1} = \int_{\mathbb{R}^d} |d_2(t)|^{\frac{pr}{p-r}} (|d_1(t)| - 1)_+^{\frac{p}{r-p}}\,dt < \infty,$$
$$I_{r2} = \int_{\mathbb{R}^d} |d_2(t)|^{\frac{pr}{p-r}} (|d_1(t)| - 1)_+^{\frac{r}{r-p}}\,dt < \infty$$

$(u_+ = \max\{u,0\})$. *Then the following sharp inequality*

$$\|d_1(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)} \le \widetilde{K}(p,r)\|x(\cdot)\|^{\frac{n-k-d(1/p-1/r)}{n-d(1/p-1/r)}}_{L_p(\mathbb{R}^d)}\|d_2(\cdot)x(\cdot)\|^{\frac{k}{n-d(1/p-1/r)}}_{L_r(\mathbb{R}^d)} \tag{9}$$

holds, where

$$\widetilde{K}(p,r) = I_{r1}^{-\frac{1}{p}\frac{n-k-d(1/p-1/r)}{n-d(1/p-1/r)}} I_{r2}^{-\frac{1}{r}\frac{k}{n-d(1/p-1/r)}} (I_{r1}+I_{r2})^{1/p}. \tag{10}$$

Proof. Consider the extremal problem

$$\|d_1(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)} \to \max, \quad \|x(\cdot)\|_{L_p(\mathbb{R}^d)} \le \delta, \quad \|d_2(\cdot)x(\cdot)\|_{L_r(\mathbb{R}^d)} \le 1.$$

Denote by E the value of this problem. It follows from [14, Theorem 2] that

$$E = \left(\widehat{\lambda}_1\delta^p + \frac{r}{p}\widehat{\lambda}_2\right)^{1/p},$$

where $\widehat{\lambda}_1 > 0$ satisfies the equation

$$\left(\int_{\mathbb{R}^d} |d_2(\xi)|^{\frac{pr}{p-r}} (|d_1(\xi)|^p - \widehat{\lambda}_1)_+^{\frac{p}{r-p}}\,d\xi\right)^{1/p} = \delta\left(\int_{\mathbb{R}^d} |d_2(\xi)|^{\frac{pr}{p-r}} (|d_1(\xi)|^p - \widehat{\lambda}_1)_+^{\frac{r}{r-p}}\,d\xi\right)^{1/r}$$

and

$$\widehat{\lambda}_2 = \frac{p}{r}\delta^{p-r}\left(\int_{\mathbb{R}^d} |d_2(\xi)|^{\frac{pr}{p-r}} (|d_1(\xi)|^p - \widehat{\lambda}_1)_+^{\frac{p}{r-p}}\,d\xi\right)^{\frac{r-p}{p}}.$$

Put $\widehat{\lambda}_1 = a^{pk}$ and change variables $\xi = at$. Then we obtain

$$a^{\frac{nr}{p-r}+\frac{kp}{r-p}+\frac{d}{p}} I_{r1}^{1/p} = \delta a^{\frac{np}{p-r}+\frac{kp}{r-p}+\frac{d}{r}} I_{r2}^{1/r}.$$

This implies that

$$a = \left(\frac{I_{r1}^{1/p}}{\delta I_{r2}^{1/r}} \right)^{\frac{1}{n-d(1/p-1/r)}}.$$

After the same change of variables we have

$$\widehat{\lambda}_2 = \frac{p}{r} \delta^{p-r} a^{-nr+kp+d(r/p-1)} I_{r1}^{r/p-1}.$$

Thus,

$$E = \left(a^{pk}\delta^p + \delta^{p-r} a^{-nr+kp+d(r/p-1)} I_{r1}^{r/p-1} \right)^{1/p} = \delta^{\frac{n-k-d(1/p-1/r)}{n-d(1/p-1/r)}} \widetilde{K}(p, r). \quad (11)$$

Let $\|x(\cdot)\|_{L_p(\mathbb{R}^d)} < \infty$, $\|d_2(\cdot)x(\cdot)\|_{L_r(\mathbb{R}^d)} < \infty$, and $\|d_2(\cdot)x(\cdot)\|_{L_r(\mathbb{R}^d)} \neq 0$. Put

$$\widehat{x}(t) = \frac{x(t)}{\|d_2(\cdot)x(\cdot)\|_{L_r(\mathbb{R}^d)}}, \quad \delta = \|\widehat{x}(\cdot)\|_{L_p(\mathbb{R}^d)}.$$

Then in view of (11) we have the sharp inequality

$$\|d_1(\cdot)\widehat{x}(\cdot)\|_{L_p(\mathbb{R}^d)} \le \widetilde{K}(p, r) \|\widehat{x}(\cdot)\|_{L_p(\mathbb{R}^d)}^{\frac{n-k-d(1/p-1/r)}{n-d(1/p-1/r)}}.$$

Passing to $x(\cdot)$ we obtain (9). $\square$

Set

$$Y_p = \{\, x(\cdot) \in L_2(\mathbb{R}^d) : Fx(\cdot) \in L_p(\mathbb{R}^d),\ d_2(\cdot)Fx(\cdot) \in L_2(\mathbb{R}^d) \,\}.$$

We define the operator $D_2 \colon Y_p \to L_2(\mathbb{R}^d)$ as follows

$$D_2 x(\cdot) = F^{-1}(d_2(\cdot)Fx(\cdot))(\cdot). \quad (12)$$

Assume that $d_1(\cdot)Fx(\cdot) \in L_2(\mathbb{R}^d)$ for all $x(\cdot) \in Y_p$. Put

$$D_1 x(\cdot) = F^{-1}(d_1(\cdot)Fx(\cdot))(\cdot). \quad (13)$$

We also assume that $D_1 x(\cdot) \in C(\mathbb{R}^d)$ for all $x(\cdot) \in Y_p$.

Set

$$M_{pq}(k, n, I) = \frac{K_{pq2}(k, n, I)}{(2\pi)^{d\frac{2n/q-k+(1-2/q)d/p}{2n+d(1-2/p)}}}. \quad (14)$$

Theorem 4. *Let $1 \le p < \infty$, $k > d(1/p-1)$, and $n > k + d/2$. If $p > 1$ and $\widetilde{I}_{12} < \infty$ then for all $x(\cdot) \in Y_p$ the sharp inequality*

$$\|D_1 x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \le M_{p1}(k, n, \widetilde{I}_{12}) \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\frac{n-k-d/2}{n+d(1/2-1/p)}} \|D_2 x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{k+d(1-1/p)}{n+d(1/2-1/p)}} \quad (15)$$

holds. If $p = 1$ and $I_{21}, I_{22} < \infty$ then for all $x(\cdot) \in Y_1$ the sharp inequality

$$\|D_1 x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \le \frac{\widetilde{K}(1, 2)}{(2\pi)^{d\frac{2n-k-d}{2n-d}}} \|Fx(\cdot)\|_{L_1(\mathbb{R}^d)}^{\frac{2n-2k-d}{2n-d}} \|D_2 x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2k}{2n-d}} \quad (16)$$

holds.

Proof. We have

$$D_j x(t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d_j(\xi) F x(\xi) e^{i\langle t,\xi\rangle}\, d\xi, \quad j = 1, 2,$$

where $t = (t_1, \dots, t_d)$, $\xi = (\xi_1, \dots \xi_d)$, and $\langle t, \xi\rangle = t_1\xi_1 + \dots + t_d\xi_d$. It follows from Theorem 2 that for $q = 1$ and $r = 2$ the following sharp inequality

$$\|d_1(\cdot)Fx(\cdot)\|_{L_1(\mathbb{R}^d)} \le K_{p12}(k, n, \widetilde{I}_{12})\|Fx(\cdot)\|^{\gamma}_{L_p(\mathbb{R}^d)} \|d_2(\cdot)Fx(\cdot)\|^{1-\gamma}_{L_2(\mathbb{R}^d)}$$

holds. Taking into account the Plancherel theorem we obtain (15). Using similar arguments from Theorem 3 we obtain (16).

For any $x(\cdot) \in Y_p$ we take $\widehat{x}(\cdot)$ such that

$$F\widehat{x}(\xi) = \varepsilon(\xi)e^{-i\langle t,\xi\rangle} Fx(\xi),$$

where

$$\varepsilon(\xi) = \begin{cases} \dfrac{\overline{d_1(\xi)Fx(\xi)}}{|d_1(\xi)Fx(\xi)|}, & d_1(\xi)Fx(\xi) \neq 0, \\ 0, & d_1(\xi)Fx(\xi) = 0. \end{cases}$$

Then

$$\left| \int_{\mathbb{R}^d} d_1(\xi) F\widehat{x}(\xi) e^{i\langle t,\xi\rangle}\, d\xi \right| = \int_{\mathbb{R}^d} |d_1(\xi)Fx(\xi)|\, d\xi.$$

This implies that inequalities (15) and (16) are sharp. □

4. Multidimensional generalizations of the Taikov inequality

Consider some examples. Define the operator $(-\Delta)^{n/2}$, $n \ge 0$, as follows

$$(-\Delta)^{n/2}x(\cdot) = F^{-1}(|\xi|^n Fx(\xi))(\cdot), \quad |\xi| = \sqrt{\xi_1^2 + \dots + \xi_d^2}.$$

Put $d_1(\xi) = |\xi|^k$ and $d_2(\xi) = |\xi|^n$. Then operators D_1 and D_2 defined above coincide with $(-\Delta)^{k/2}$ and $(-\Delta)^{n/2}$.

Corollary 1. *Let $1 \le p \le \infty$, $k > d(1/p - 1)$, and $n > k + d/2$. Then the sharp inequality*

$$\|(-\Delta)^{k/2}x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \le M_{p1}(k, n, I_0) \times \|Fx(\cdot)\|^{\frac{n-k-d/2}{n+d(1/2-1/p)}}_{L_p(\mathbb{R}^d)} \|(-\Delta)^{n/2}x(\cdot)\|^{\frac{k+d(1-1/p)}{n+d(1/2-1/p)}}_{L_2(\mathbb{R}^d)} \quad (17)$$

holds, where

$$I_0 = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

Proof. Let $1 < p < \infty$. Since $\widetilde{d}_1(\omega) = \widetilde{d}_2(\omega) = 1$ we have

$$\widetilde{I}_{12} = \int_{\Pi_{d-1}} J(\omega)\, d\omega = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{18}$$

Now (17) immediately follows from Theorem 4.

Let $p = 1$. We have

$$I_{21} = \int_{|t|>1} |t|^{-2n}(|t|^k - 1)\, dt, \quad I_{22} = \int_{|t|>1} |t|^{-2n}(|t|^k - 1)^2\, dt.$$

Passing to the polar transformation we get

$$I_{21} = \int_1^{+\infty} \rho^{-2n}(\rho^k - 1)\rho^{d-1}\, d\rho \widetilde{I}_{12} = \frac{k\widetilde{I}_{12}}{(2n-d)(2n-k-d)},$$

where $\widetilde{I}_{12}$ is defined by (18). The analogous calculations give

$$I_{22} = \frac{2k^2\widetilde{I}_{12}}{(2n-d)(2n-k-d)(2n-2k-d)}.$$

Substituting these values in (10) we obtain that

$$\widetilde{K}(1,2) = \left(\frac{2n-d}{2n-2k-d}\right)^{\frac{2n-k-d}{2n-d}} \left(\frac{\pi^{d/2}}{\Gamma(d/2)(2n-k-d)}\right)^{\frac{k}{2n-d}}.$$

It is easy to verify that

$$\widetilde{K}(1,2) = K_{112}(k,n,I_0).$$

The case $p = \infty$ was obtained in [16]. $\square$

Note that the value of $M_{21}(k,n,I_0)$ was calculated in [15]. Constants $M_{21}(k,n,I_0)$ and $M_{\infty 1}(k,n,I_0)$ (as $M_{11}(k,n,I_0)$) can be written in a simpler form

$$M_{21}(k,n,I_0) = \left(\frac{2k+d}{2n-2k-d}\right)^{\frac{2n-2k-d}{4n}} \left(\frac{\gamma_d}{\pi}(2k+d)\sin\frac{2k+d}{2n}\pi\right)^{-1/2},$$

$$M_{\infty 1}(k,n,I_0) = \frac{(n+d/2)^{\frac{k+d}{2n+d}}}{k+d} \left(\frac{2n-k}{\gamma_d(2n-2k-d)}\right)^{\frac{2n-k}{2n+d}},$$

where

$$\gamma_d = 2^{d-1}\pi^{d/2-1}\Gamma(d/2).$$

Consider one more example. Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}_+^d$. We define D^α (the derivative of order α) as follows

$$D^\alpha x(\cdot) = F^{-1}((i\xi)^\alpha F x(\xi))(\cdot),$$

where $(i\xi)^\alpha = (i\xi_1)^{\alpha_1}\dots(i\xi_d)^{\alpha_d}$. Let $D_1 = D^\alpha$ and $D_2 = (-\Delta)^{n/2}$ (thus, $d_1(\xi) = (i\xi)^\alpha$ and $d_2(\xi) = |\xi|^n$).

Set $|\alpha| = \alpha_1 + \ldots + \alpha_d$,

$$\widetilde{q}_1 = \frac{n + d(1/2 - 1/p)}{n(1 - 1/p) - |\alpha|(1/2 - 1/p)}.$$

Theorem 5. *Let $1 \le p \le \infty$, $|\alpha| \ge 0$ if $p > 1$ and $|\alpha| > 0$ if $p = 1$, $n > |\alpha| + d/2$. Then the sharp inequality*

$$\|D^\alpha x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \le M_{p1}(|\alpha|, n, I_\alpha) \times \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\frac{n-|\alpha|-d/2}{n+d(1/2-1/p)}} \|(-\Delta)^{n/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{|\alpha|+d(1-1/p)}{n+d(1/2-1/p)}} \quad (19)$$

holds, where

$$I_\alpha = 2\frac{\Gamma((\alpha_1\widetilde{q}_1 + 1)/2)\ldots\Gamma((\alpha_d\widetilde{q}_1 + 1)/2)}{\Gamma((|\alpha|\widetilde{q}_1 + d)/2)}.$$

Proof. Let $1 < p < \infty$. It follows from Theorem 4 that sharp inequality (19) holds with the constant $M_{p1}(|\alpha|, n, \widetilde{I})$, where

$$\widetilde{I} = \int_{\Pi_{d-1}} |\cos\omega_1|^{\alpha_1\widetilde{q}_1} \ldots |\sin\omega_1 \sin\omega_2 \ldots \sin\omega_{d-2}\sin\omega_{d-1}|^{\alpha_d\widetilde{q}_1} J(\omega)\, d\omega.$$

From the well-known Dirichlet formula we have

$$\int_{\substack{\xi_1\ge0,\ldots,\xi_d\ge0\\ \xi_1^2+\ldots+\xi_d^2\le1}} \xi_1^{p_1-1}\ldots\xi_d^{p_d-1}\, d\xi_1\ldots d\xi_d = \frac{\Gamma(p_1/2)\ldots\Gamma(p_d/2)}{2^d\Gamma(p_1/2 + \ldots + p_d/2 + 1)},$$

$p_1, \ldots, p_d > 0$. Passing to the polar transformation we obtain

$$\int_{\Pi_{d-1}} \Phi(\omega, p_1, \ldots, p_d) J(\omega)\, d\omega = 2\frac{\Gamma(p_1/2)\ldots\Gamma(p_d/2)}{\Gamma(p_1/2 + \ldots + p_d/2)},$$

where

$$\Phi(\omega, p_1, \ldots, p_d) = |\cos\omega_1|^{p_1-1}\ldots|\sin\omega_1\sin\omega_2\cdots\sin\omega_{d-2}\sin\omega_{d-1}|^{p_d-1}.$$

Consequently, $\widetilde{I} = I_\alpha$.

Now let $p = 1$. Let us use Theorem 3. We have

$$I_{21} = \int_{|t^\alpha|>1} |t|^{-2n}(|t^\alpha| - 1)\, dt = J_{\alpha1} - J_{\alpha0},$$

$$I_{22} = \int_{|t^\alpha|>1} |t|^{-2n}(|t^\alpha| - 1)^2\, dt = J_{\alpha2} - 2J_{\alpha1} + J_{\alpha0},$$

where

$$J_{\alpha c} = \int_{|t^\alpha|>1} |t|^{-2n}|t^{c\alpha}|\, dt.$$

Passing to the polar transformation we have

$$J_{\alpha c}=\int_{\Pi_{d-1}}\Phi(\omega,c\alpha_1+1,\dots,c\alpha_d+1)J(\omega)\,d\omega\int_{\Phi_0(\omega)}^{+\infty}\rho^{-2n+c|\alpha|+d-1}\,d\rho$$
$$=\frac{1}{2n-c|\alpha|-d}\int_{\Pi_{d-1}}\Phi(\omega,\beta_1,\dots,\beta_d)J(\omega)\,d\omega=\frac{I_n(\alpha,d)}{2n-c|\alpha|-d},$$

where $\Phi_0(\omega)=\Phi(\omega,-\alpha_1/|\alpha|+1,\dots,-\alpha_d/|\alpha|+1)$, $\beta_j=(2n-d)\alpha_j/|\alpha|+1$, $j=1,\dots,d$, and

$$I_n(\alpha,d)=\frac{2}{\Gamma(n)}\Gamma\left(\alpha_1\frac{n-d/2}{|\alpha|}+1/2\right)\dots\Gamma\left(\alpha_d\frac{n-d/2}{|\alpha|}+1/2\right).$$

Thus,

$$I_{21}=\frac{|\alpha|}{(2n-d)(2n-|\alpha|-d)}I_n(\alpha,d),$$
$$I_{22}=\frac{2|\alpha|^2}{(2n-d)(2n-|\alpha|-d)(2n-2|\alpha|-d)}I_n(\alpha,d).$$

Substituting these values in (10) we obtain

$$\widetilde{K}(1,2)=\left(\frac{2n-d}{2n-2|\alpha|-d}\right)^{\frac{2n-|\alpha|-d}{2n-d}}\left(\frac{2I_n(\alpha,d)}{2n-|\alpha|-d}\right)^{\frac{|\alpha|}{2n-d}}.$$

It may be easily verified that $\widetilde{K}(1,2)=K_{122}(|\alpha|,n,I_\alpha)$.

Now let $p=\infty$. Consider the extremal problem

$$|D^\alpha x(0)|\to\max,\quad \|Fx(\cdot)\|_{L_\infty(\mathbb{R}^d)}\le\delta,\quad \|(-\Delta)^{n/2}x(\cdot)\|_{L_2(\mathbb{R}^d)}\le 1. \tag{20}$$

Set

$$s_\alpha(\xi)=\begin{cases}\dfrac{(i\xi)^\alpha}{|\xi^\alpha|}, & \xi^\alpha\ne 0,\\ 1, & \xi^\alpha=0\end{cases}$$

(we recall that $\xi^\alpha=\xi_1^{\alpha_1}\dots\xi_d^{\alpha_d}$ and $|\xi^\alpha|=|\xi_1|^{\alpha_1}\dots|\xi_d|^{\alpha_d}$). Let $\widehat{x}(\cdot)$ be such that

$$F\widehat{x}(\xi)=\begin{cases}\delta\overline{s_\alpha(\xi)}, & |\xi^\alpha|\ge\lambda|\xi|^{2n},\\ \dfrac{\delta}{\lambda}\dfrac{\overline{(i\xi)^\alpha}}{|\xi|^{2n}}, & |\xi^\alpha|<\lambda|\xi|^{2n},\end{cases}$$

where $\lambda>0$. We prove that for all $x(\cdot)\in Y_\infty$ (for $d_2(\xi)=|\xi|^n$) the following equality

$$D^\alpha x(0)=\frac{1}{(2\pi)^d}\int_{|\xi^\alpha|\ge\lambda|\xi|^{2n}}((i\xi)^\alpha-\lambda s_\alpha(\xi)|\xi|^{2n})Fx(\xi)\,d\xi$$
$$+\frac{\lambda}{\delta}\int_{\mathbb{R}^d}(-\Delta)^{n/2}x(t)\overline{(-\Delta)^{n/2}\widehat{x}(t)}\,dt \tag{21}$$

holds. Indeed, in view of Plancherel's theorem we have

$$\int_{\mathbb{R}^d} (-\Delta)^{n/2} x(t) \overline{(-\Delta)^{n/2} \widehat{x}(t)}\, dt = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{2n} Fx(\xi) \overline{F\widehat{x}(\xi)}\, d\xi.$$

Consequently,

$$\begin{aligned}
&\frac{1}{(2\pi)^d} \int_{|\xi^\alpha| \ge \lambda |\xi|^{2n}} ((i\xi)^\alpha - \lambda s_\alpha(\xi) |\xi|^{2n}) Fx(\xi)\, d\xi \\
&\qquad + \frac{\lambda}{\delta} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{2n} Fx(\xi) \overline{F\widehat{x}(\xi)}\, d\xi \\
&\qquad = \frac{1}{(2\pi)^d} \int_{|\xi^\alpha| \ge \lambda |\xi|^{2n}} ((i\xi)^\alpha - \lambda s_\alpha(\xi) |\xi|^{2n}) Fx(\xi)\, d\xi \\
&+ \frac{1}{(2\pi)^d} \int_{|\xi^\alpha| \ge \lambda |\xi|^{2n}} \lambda s_\alpha(\xi) |\xi|^{2n} Fx(\xi)\, d\xi + \frac{1}{(2\pi)^d} \int_{|\xi^\alpha| < \lambda |\xi|^{2n}} (i\xi)^\alpha Fx(\xi)\, d\xi \\
&\qquad = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (i\xi)^\alpha Fx(\xi)\, d\xi = D^\alpha x(0).
\end{aligned}$$

Choose λ from the condition $\|(-\Delta)^{n/2} \widehat{x}(\cdot)\|_{L_2(\mathbb{R}^d)} = 1$. Then we get the following equation to find λ

$$\frac{\delta^2}{(2\pi)^d} \int_{|\xi^\alpha| \ge \lambda |\xi|^{2n}} |\xi|^{2n}\, d\xi + \frac{\delta^2}{(2\pi)^d \lambda^2} \int_{|\xi^\alpha| < \lambda |\xi|^{2n}} \frac{|\xi^{2\alpha}|}{|\xi|^{2n}}\, d\xi = 1.$$

After passing to the polar transformation we obtain

$$\begin{aligned}
&\frac{\delta^2}{(2\pi)^d} \int_{\Pi_{d-1}} J(\omega)\, d\omega \int_0^{\Phi_1(\omega)} \rho^{2n+d-1}\, d\rho \\
&+ \frac{\delta^2 \lambda^{-2}}{(2\pi)^d} \int_{\Pi_{d-1}} \Phi(\omega, 2\alpha_1 + 1, \ldots, 2\alpha_d + 1) J(\omega)\, d\omega \int_{\Phi_1(\omega)}^{+\infty} \rho^{-2n+2|\alpha|+d-1}\, d\rho \\
&= 1,
\end{aligned}$$

where

$$\Phi_1(\omega) = \lambda^{-\frac{1}{2n-|\alpha|}} \Phi\left(\omega, \frac{\alpha_1}{2n-|\alpha|} + 1, \ldots, \frac{\alpha_d}{2n-|\alpha|} + 1\right).$$

Further, we arrive at the equation

$$\frac{\delta^2}{(2\pi)^d} \lambda^{-\frac{2n+d}{2n-|\alpha|}} \frac{4n - 2|\alpha|}{(2n+d)(2n-2|\alpha|-d)} J_0 = 1,$$

in which

$$\begin{aligned}
J_0 &= \int_{\Pi_{d-1}} \Phi\left(\omega, \alpha_1 \frac{2n+d}{2n-|\alpha|} + 1, \ldots, \alpha_d \frac{2n+d}{2n-|\alpha|} + 1\right) J(\omega)\, d\omega \\
&= 2 \frac{\Gamma\left(\alpha_1 \frac{2n+d}{4n-2|\alpha|} + 1/2\right) \ldots \Gamma\left(\alpha_d \frac{2n+d}{4n-2|\alpha|} + 1/2\right)}{\Gamma\left(n \frac{|\alpha|+d}{2n-|\alpha|}\right)}.
\end{aligned}$$

Thus,

$$\lambda=\left(\frac{\delta^2(4n-2|\alpha|)}{(2\pi)^d(2n+d)(2n-2|\alpha|-d)}J_0\right)^{\frac{2n-|\alpha|}{2n+d}}. \tag{22}$$

In view of (21) for any admissible function for problem (20) we have

$$|D^\alpha x(0)|\le\frac{\delta}{(2\pi)^d}\int_{|\xi^\alpha|\ge\lambda|\xi|^{2n}}(|\xi^\alpha|-\lambda|\xi|^{2n})\,d\xi+\frac{\lambda}{\delta}. \tag{23}$$

Since for $\widehat{x}(\cdot)$ the inequality terns to equality inequality (23) is sharp. We have

$$\begin{aligned}\int_{|\xi^\alpha|\ge\lambda|\xi|^{2n}}&(|\xi^\alpha|-\lambda|\xi|^{2n})\,d\xi\\ &=\int_{\Pi_{d-1}}\Phi(\omega,\alpha_1+1,\dots,\alpha_d+1)J(\omega)\,d\omega\int_0^{\Phi_1(\omega)}\rho^{|\alpha|+d-1}\,d\rho\\ -\lambda&\int_{\Pi_{d-1}}J(\omega)\,d\omega\int_0^{\Phi_1(\omega)}\rho^{2n+d-1}\,d\rho=\lambda^{-\frac{|\alpha|+d}{2n-|\alpha|}}\frac{2n-|\alpha|}{(2n+d)(|\alpha|+d)}J_0.\end{aligned}$$

Substituting this value in (23), taking into account (22), we obtain the sharp inequality

$$|D^\alpha x(0)|\le K\delta^{\frac{n-|\alpha|-d/2}{n+d/2}},$$

where

$$K=\frac{n+d/2}{|\alpha|+d/2}\left(\frac{(n-|\alpha|/2)J_0}{(2\pi)^d(n+d/2)(n-|\alpha|-d/2)}\right)^{\frac{n-|\alpha|/2}{n+d/2}}.$$

This implies the sharp inequality

$$|D^\alpha x(0)|\le K\|Fx(\cdot)\|_{L_\infty(\mathbb{R}^d)}^{\frac{n-|\alpha|-d/2}{n+d/2}}\|(-\Delta)^{n/2}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{|\alpha|+d}{n+d/2}}.$$

Due to the invariance with respect to the shift we get

$$\|D^\alpha x(\cdot)\|_{L_\infty(\mathbb{R}^d)}\le K\|Fx(\cdot)\|_{L_\infty(\mathbb{R}^d)}^{\frac{n-|\alpha|-d/2}{n+d/2}}\|(-\Delta)^{n/2}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{|\alpha|+d}{n+d/2}}.$$

It is easy to check that $K=M_{\infty1}(|\alpha|,n,I_\alpha)$. □

For $p=2$ the constant $M_{21}(|\alpha|,n,I_\alpha)$ was found in [15]. It can be written in the form

$$M_{21}(|\alpha|,n,I_\alpha)=\left(\frac{2|\alpha|+d}{2n-2|\alpha|-d}\right)^{\frac{2n-2|\alpha|-d}{4n}}\left(\frac{2|\alpha|+d}{\Gamma_{\alpha d}}\sin\frac{2|\alpha|+d}{2n}\pi\right)^{-1/2},$$

where

$$\Gamma_{\alpha d}=\frac{\Gamma(\alpha_1+1/2)\dots\Gamma(\alpha_d+1/2)}{(2\pi)^{d-1}\Gamma(|\alpha|+d/2)}.$$

5. Multidimensional generalizations of the Hardy–Littlewood–Pólya inequality

Assume that functions $d_1(\cdot)$ and $d_2(\cdot)$ satisfy the conditions stated at the beginning of paragraph 3. We obtain an analog of Theorem 3 for the case when $q = r$.

Theorem 6. *Let* $1 \le q = r < p < \infty$, $k \ge 0$, $n > k$, *and*

$$S_{q1} = \int_{\mathbb{R}^d} (|d_1(t)|^q - |d_2(t)|^q)_+^{\frac{p}{p-q}}\, dt < \infty,$$
$$S_{q2} = \int_{\mathbb{R}^d} |d_2(t)|^q (|d_1(t)|^q - |d_2(t)|^q)_+^{\frac{q}{p-q}}\, dt < \infty.$$

Then the sharp inequality

$$\|d_1(\cdot)x(\cdot)\|_{L_q(\mathbb{R}^d)} \le L_{pq} \|x(\cdot)\|_{L_p(\mathbb{R}^d)}^{\frac{n-k}{n+d(1/q-1/p)}} \|d_2(\cdot)x(\cdot)\|_{L_q(\mathbb{R}^d)}^{\frac{k+d(1/q-1/p)}{n+d(1/q-1/p)}} \tag{24}$$

holds, where

$$L_{pq} = S_{q1}^{-\frac{1}{p}\frac{n-k}{n+d(1/q-1/p)}} S_{q2}^{-\frac{1}{q}\frac{k+d(1/q-1/p)}{n+d(1/q-1/p)}} (S_{q1} + S_{q2})^{1/q}.$$

Proof. Consider the extremal problem

$$\|d_1(\cdot)x(\cdot)\|_{L_q(\mathbb{R}^d)} \to \max, \quad \|x(\cdot)\|_{L_p(\mathbb{R}^d)} \le \delta, \quad \|d_2(\cdot)x(\cdot)\|_{L_q(\mathbb{R}^d)} \le 1.$$

Denote by E the value of this problem. It follows from [14, Theorem 1] that

$$E = \left(\frac{p}{q}\widehat{\lambda}_1 \delta^p + \widehat{\lambda}_2\right)^{1/q},$$

where $\widehat{\lambda}_2 > 0$ satisfies the equation

$$\left(\int_{\mathbb{R}^d} \left(|d_1(\xi)|^q - \widehat{\lambda}_2 |d_2(\xi)|^q\right)_+^{\frac{p}{p-q}} d\xi\right)^{1/p} = \delta\left(\int_{\mathbb{R}^d} |d_2(\xi)|^q \left(|d_1(\xi)|^q - \widehat{\lambda}_2 |d_2(\xi)|^q\right)_+^{\frac{q}{p-q}} d\xi\right)^{1/q}$$

and

$$\widehat{\lambda}_1 = \frac{q}{p}\delta^{q-p}\left(\int_{\mathbb{R}^d} \left(|d_1(\xi)|^q - \widehat{\lambda}_2 |d_2(\xi)|^q\right)_+^{\frac{p}{p-q}} d\xi\right)^{\frac{p-q}{p}}.$$

Put $\widehat{\lambda}_2 = a^{(k-n)q}$ and change variables $\xi = at$. Then we obtain

$$a^{\frac{kq}{p-q}+\frac{d}{p}} S_{q1}^{1/p} = \delta a^{n+\frac{kq}{p-q}+\frac{d}{q}} S_{q2}^{1/q}.$$

This implies that

$$a = \left(\frac{S_{q1}^{1/p}}{\delta S_{q2}^{1/q}} \right)^{\frac{1}{n+d(1/q-1/p)}}.$$

After the same change of variables we have

$$\widehat{\lambda}_1 = \frac{q}{p} \delta^{q-p} a^{q(k+d(1/q-1/p))} S_{q1}^{\frac{p-q}{p}}.$$

Thus,

$$E = L_{pq} \delta^{\frac{n-k}{n+d(1/q-1/p)}}.$$

By arguments analogous to those that were used in the proof of Theorem 3 we obtain sharp inequality (24). □

Consider the operators D_1 and D_2 defined by (13) and (12). It follows from (24) that for all $x(\cdot) \in Y_p$ we have

$$\|d_1(\cdot)Fx(\cdot)\|_{L_2(\mathbb{R}^d)} \leq L_{p2} \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\frac{n-k}{n+d(1/2-1/p)}} \|d_2(\cdot)Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{k+d(1/2-1/p)}{n+d(1/2-1/p)}}.$$

In view of Plancherel's theorem we obtain

$$\|D_1 x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \widetilde{L}_{dp} \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\frac{n-k}{n+d(1/2-1/p)}} \|D_2 x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{k+d(1/2-1/p)}{n+d(1/2-1/p)}}, \qquad (25)$$

where

$$\widetilde{L}_{dp} = \frac{L_{p2}}{(2\pi)^{d \frac{n-k}{2n+d(1-2/p)}}}.$$

Put $d_1(\xi) = |\xi|^k$ and $d_2(\xi) = |\xi|^n$. Then operators D_1 and D_2 defined above coincide with $(-\Delta)^{k/2}$ and $(-\Delta)^{n/2}$.

Theorem 7. *Let $2 \leq p \leq \infty$, $k \geq 0$, and $n > k$. Then the sharp inequality*

$$\|(-\Delta)^{k/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq M_{p2}(k, n, I_0) \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\frac{n-k}{n+d(1/2-1/p)}} \times \|(-\Delta)^{n/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{k+d(1/2-1/p)}{n+d(1/2-1/p)}}$$

holds, where $M_{p2}(k, n, I_0)$ is defined by (14) *for $2 < p \leq \infty$ and $M_{22}(k, n, I_0) = (2\pi)^{(1-k/n)d/2}$.*

Proof. The case $p = 2$ follows from [12]. Let $2 < p < \infty$. It follows from (25) that

$$\|(-\Delta)^{k/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \widetilde{L}_{dp} \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\frac{n-k}{n+d(1/2-1/p)}} \|(-\Delta)^{n/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{k+d(1/2-1/p)}{n+d(1/2-1/p)}}.$$

We calculate S_{21} and S_{22} which are used in $\widetilde{L}_{dp}$. We have

$$S_{21}=\int_{\mathbb{R}^d}(|t|^{2k}-|t|^{2n})_+^{\frac{p}{p-2}}\,dt=\int_{|t|<1}|t|^{\frac{2kp}{p-2}}(1-|t|^{2(n-k)})^{\frac{p}{p-2}}\,dt$$
$$=\int_0^1\rho^{\frac{2kp}{p-2}}(1-\rho^{2(n-k)})^{\frac{p}{p-2}}\rho^{d-1}\,d\rho\widetilde{I}_{12},$$

where $\widetilde{I}_{12}$ is defined by (18). Changing variables $x=\rho^{2(n-k)}$ we obtain

$$S_{21}=\frac{\widetilde{I}_{12}}{2(n-k)}\int_0^1 x^{\frac{2kp+d(p-2)}{2(p-2)(n-k)}-1}(1-x)^{\frac{p}{p-2}}\,dx$$
$$=\frac{\widetilde{I}_{12}}{2(n-k)}B\left(\frac{2kp+d(p-2)}{2(p-2)(n-k)},\frac{p}{p-2}+1\right).$$

The analogous calculations give

$$S_{22}=\int_{\mathbb{R}^d}|t|^{2n}(|t|^{2k}-|t|^{2n})_+^{\frac{2}{p-2}}\,dt$$
$$=\frac{\widetilde{I}_{12}}{2(n-k)}B\left(\frac{2kp+d(p-2)}{2(p-2)(n-k)}+1,\frac{p}{p-2}\right).$$

By properties of beta-function

$$S_{21}=\frac{\widetilde{I}_{12}}{2n+d(1-2/p)}B\left(\frac{2kp+d(p-2)}{2(p-2)(n-k)},\frac{p}{p-2}\right),$$
$$S_{22}=\frac{\widetilde{I}_{12}(2k+d(1-2/p))}{2(n-k)(2n+d(1-2/p))}B\left(\frac{2kp+d(p-2)}{2(p-2)(n-k)},\frac{p}{p-2}\right).$$

Hence,

$$L_{p2}=S_{21}^{-\frac{\gamma}{p}}S_{22}^{-\frac{1-\gamma}{2}}(S_{21}+S_{22})^{1/2}$$
$$=\gamma^{-\frac{\gamma}{p}}(1-\gamma)^{-\frac{1-\gamma}{2}}\left(\frac{B\left(\widetilde{q}\gamma/p+1,\widetilde{q}(1-\gamma)/2\right)\widetilde{I}_{12}}{2(n-k)}\right)^{1/\widetilde{q}},$$

where γ and $\widetilde{q}$ are defined by (7) for $q=r=2$. Consequently,

$$\frac{L_{p2}}{(2\pi)^{d\frac{n-k}{2n+d(1-2/p)}}}=M_{p2}(k,n,I_0).$$

Now let $p=\infty$. Consider the extremal problem

$$\|(-\Delta)^{k/2}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}\to\max,\quad \|Fx(\cdot)\|^2_{L_\infty(\mathbb{R}^d)}\le\delta^2,$$
$$\|(-\Delta)^{n/2}x(\cdot)\|^2_{L_2(\mathbb{R}^d)}\le 1.\quad (26)$$

Passing to the Fourier transform we have

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{2k}|Fx(\xi)|^2\,d\xi\to\max,\quad \|Fx(\cdot)\|^2_{L_\infty(\mathbb{R}^d)}\le\delta^2,$$
$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{2n}|Fx(\xi)|^2\,d\xi\le 1. \quad (27)$$

Let $\widehat{x}(\cdot)$ be such that

$$F\widehat{x}(\xi)=\begin{cases}\delta, & |\xi|<\sigma,\\ 0, & |\xi|\ge\sigma,\end{cases}$$

where $\sigma>0$ is chosen from the condition

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{2n}|F\widehat{x}(\xi)|^2\,d\xi=1.$$

Thus the following equality must be satisfied

$$\delta^2\int_{|\xi|<\sigma}|\xi|^{2n}\,d\xi=(2\pi)^d.$$

Passing to the polar transformation we obtain that

$$\sigma=\left(\frac{2^{d-1}\pi^{d/2}\Gamma(d/2)(2n+d)}{\delta^2}\right)^{\frac{1}{2n+d}}.$$

Consider the Lagrange function for problem (27)

$$\mathcal{L}(x(\cdot),\lambda_1(\cdot),\lambda_2)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(-|\xi|^{2k}+(2\pi)^d\lambda_1(\xi)+\lambda_2|\xi|^{2n})|Fx(\xi)|^2\,d\xi.$$

Put $\lambda_2=\sigma^{2(k-n)}$ and

$$\lambda_1(\xi)=\begin{cases}\dfrac{1}{(2\pi)^d}(|\xi|^{2k}-\lambda_2|\xi|^{2n}), & |\xi|<\sigma,\\ 0, & |\xi|\ge\sigma.\end{cases}$$

Then

$$\mathcal{L}(x(\cdot),\lambda_1(\cdot),\lambda_2)=\frac{1}{(2\pi)^d}\int_{|\xi|\ge\sigma}(-|\xi|^{2k}+\sigma^{2(k-n)}|\xi|^{2n})|Fx(\xi)|^2\,d\xi\ge 0$$

and $\mathcal{L}(\widehat{x}(\cdot),\lambda_1(\cdot),\lambda_2)=0$. Consequently, for any admissible function $x(\cdot)$ we have

$$
\begin{aligned}
-\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{2k}|Fx(\xi)|^2\,d\xi \geq -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{2k}|Fx(\xi)|^2\,d\xi \\
+\int_{\mathbb{R}^d}\lambda_1(\xi)(|Fx(\xi)|^2-\delta^2)\,d\xi+\lambda_2\left(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{2n}|Fx(\xi)|^2\,d\xi-1\right) \\
=\mathcal{L}(x(\cdot),\lambda_1(\cdot),\lambda_2)-\delta^2\int_{\mathbb{R}^d}\lambda_1(\xi)\,d\xi-\lambda_2 \\
\geq \mathcal{L}(\widehat{x}(\cdot),\lambda_1(\cdot),\lambda_2)-\delta^2\int_{\mathbb{R}^d}\lambda_1(\xi)\,d\xi-\lambda_2 \\
=-\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{2k}|F\widehat{x}(\xi)|^2\,d\xi.
\end{aligned}
$$

It means that the value of problem (27) (and (26)) is equal to

$$
\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{2k}|F\widehat{x}(\xi)|^2\,d\xi=\delta^2\frac{1}{(2\pi)^d}\int_{|\xi|<\sigma}|\xi|^{2k}\,d\xi=M^2\delta^{\frac{4(n-k)}{2n+d}},
$$

where

$$
M=\frac{(2n+d)^{\frac{k+d/2}{2n+d}}}{\sqrt{2k+d}}\left(\frac{1}{2^{d-1}\pi^{d/2}\Gamma(d/2)}\right)^{\frac{n-k}{2n+d}}.
$$

Thus,

$$
\|(-\Delta)^{k/2}x(\cdot)\|_{L_2(\mathbb{R}^d))}\leq M\delta^{\frac{2(n-k)}{2n+d}}.
$$

It follows from this inequality (as in the proof of Theorem 3) that

$$
\|(-\Delta)^{k/2}x(\cdot)\|_{L_2(\mathbb{R}^d)}\leq M\|Fx(\cdot)\|_{L_\infty(\mathbb{R}^d)}^{\frac{n-k}{n+d/2}}\|(-\Delta)^{n/2}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{k+d/2}{n+d/2}}.
$$

It remains to note that $M=M_{\infty 2}(k,n,I_0)$. □

Consider the case when $D_1=D^\alpha$, $D_2=(-\Delta)^{n/2}$ and $p=\infty$.

Theorem 8. *Let $n>|\alpha|$. Then the sharp inequality*

$$
\|D^\alpha x(\cdot)\|_{L_2(\mathbb{R}^d)}\leq \widetilde{M}_d\|Fx(\cdot)\|_{L_\infty(\mathbb{R}^d)}^{\frac{n-|\alpha|}{n+d/2}}\|(-\Delta)^{n/2}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{|\alpha|+d/2}{n+d/2}} \tag{28}
$$

holds, where

$$
\widetilde{M}_d=\frac{(n+d/2)^{\frac{|\alpha|+d/2}{2n+d}}}{\sqrt{|\alpha|+d/2}}J_1^{\frac{n-|\alpha|}{2n+d}}
$$

and

$$
J_1=\frac{\Gamma\left(\alpha_1\dfrac{n+d/2}{n-|\alpha|}+1/2\right)\dots\Gamma\left(\alpha_d\dfrac{n+d/2}{n-|\alpha|}+1/2\right)}{(2\pi)^d\Gamma\left(n\dfrac{|\alpha|+d/2}{n-|\alpha|}\right)}.
$$

Proof. Consider the extremal problem

$$\|D^\alpha x(\cdot)\|^2_{L_2(\mathbb{R}^d)} \to \max, \quad \|Fx(\cdot)\|^2_{L_\infty(\mathbb{R}^d)} \le \delta^2,$$
$$\|(-\Delta)^{n/2} x(\cdot)\|^2_{L_2(\mathbb{R}^d)} \le 1. \quad (29)$$

Passing to the Fourier transform we have

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi^{2\alpha}||Fx(\xi)|^2\,d\xi \to \max, \quad \|Fx(\cdot)\|^2_{L_\infty(\mathbb{R}^d)} \le \delta^2,$$
$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^{2n}|Fx(\xi)|^2\,d\xi \le 1. \quad (30)$$

Let $\widehat{x}(\cdot)$ be such that

$$F\widehat{x}(\xi) = \begin{cases} \delta, & |\xi^{2\alpha}| > \lambda|\xi|^{2n}, \\ 0, & |\xi^{2\alpha}| \le \lambda|\xi|^{2n}, \end{cases}$$

where $\lambda > 0$ is chosen from the condition

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^{2n}|F\widehat{x}(\xi)|^2\,d\xi = 1.$$

Thus the following equality is valid

$$\delta^2 \int_{|\xi^{2\alpha}|>\lambda|\xi|^{2n}} |\xi|^{2n}\,d\xi = (2\pi)^d.$$

Passing to the polar transformation we obtain

$$\delta^2 \int_{\Pi_{d-1}} J(\omega)\,d\omega \int_0^{\Phi_2(\omega)} \rho^{2n+d-1}\,d\rho = (2\pi)^d,$$

where

$$\Phi_2(\omega) = \lambda^{-\frac{1}{2(n-|\alpha|)}}\,\Phi\left(\omega, \frac{\alpha_1}{n-|\alpha|}+1, \dots, \frac{\alpha_d}{n-|\alpha|}+1\right).$$

Hence,

$$\lambda = \left(\frac{\delta^2 J_1}{n+d/2}\right)^{\frac{n-|\alpha|}{n+d/2}}.$$

Consider the Lagrange function for problem (30)

$$\mathcal{L}(x(\cdot), \lambda_1(\cdot), \lambda_2) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} (-|\xi^{2\alpha}| + (2\pi)^d\lambda_1(\xi) + \lambda_2|\xi|^{2n})|Fx(\xi)|^2\,d\xi.$$

Put $\lambda_2 = \lambda$ and

$$\lambda_1(\xi) = \begin{cases} \dfrac{1}{(2\pi)^d}(|\xi^{2\alpha}| - \lambda_2|\xi|^{2n}), & |\xi^{2\alpha}| > \lambda|\xi|^{2n}, \\ 0, & |\xi^{2\alpha}| \le \lambda|\xi|^{2n}. \end{cases}$$

Then

$$\mathcal{L}(x(\cdot),\lambda_1(\cdot),\lambda_2)=\frac{1}{(2\pi)^d}\int_{|\xi^{2\alpha}|\le\lambda|\xi|^{2n}}(-|\xi^{2\alpha}|+\lambda|\xi|^{2n})|Fx(\xi)|^2\,d\xi\ge 0$$

and $\mathcal{L}(\widehat{x}(\cdot),\lambda_1(\cdot),\lambda_2)=0$. Consequently, for any admissible function $x(\cdot)$ we have

$$\begin{aligned}-\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi^{2\alpha}||Fx(\xi)|^2\,d\xi &\ge -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi^{2\alpha}||Fx(\xi)|^2\,d\xi\\ &+\int_{\mathbb{R}^d}\lambda_1(\xi)(|Fx(\xi)|^2-\delta^2)\,d\xi+\lambda_2\Big(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{2n}|Fx(\xi)|^2\,d\xi-1\Big)\\ &=\mathcal{L}(x(\cdot),\lambda_1(\cdot),\lambda_2)-\delta^2\int_{\mathbb{R}^d}\lambda_1(\xi)\,d\xi-\lambda_2\\ &\ge\mathcal{L}(\widehat{x}(\cdot),\lambda_1(\cdot),\lambda_2)-\delta^2\int_{\mathbb{R}^d}\lambda_1(\xi)\,d\xi-\lambda_2\\ &=-\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi^{2\alpha}||F\widehat{x}(\xi)|^2\,d\xi.\end{aligned}$$

Thus, the value of problem (30) (and (29)) is equal to

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi^{2\alpha}||F\widehat{x}(\xi)|^2\,d\xi=\delta^2\frac{1}{(2\pi)^d}\int_{|\xi^{2\alpha}|>\lambda|\xi|^{2n}}|\xi^{2\alpha}|\,d\xi=\widetilde{M}_d^2\delta^{\frac{4(n-|\alpha|)}{2n+d}}.$$

Hence, for any admissible function $x(\cdot)$

$$\|D^\alpha x(\cdot)\|_{L_2(\mathbb{R}^d)}\le\widetilde{M}_d\delta^{\frac{n-|\alpha|}{n+d/2}}.$$

This implies (28). □

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