

# The Hardy-Littlewood-Pólya Inequality and Optimal Recovery of Derivatives in Hardy Spaces

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In 1934 Hardy, Littlewood, and Pólya proved that for all integer  $0 < k < r$  and all functions  $x(\cdot) \in L_2(\mathbb{R})$  such that  $x^{(r-1)}(\cdot)$  is locally absolutely continuous on  $\mathbb{R}$  and  $x^{(r)}(\cdot) \in L_2(\mathbb{R})$  the following exact inequality

$$\|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \leq \|x(\cdot)\|_{L_2(\mathbb{R})}^{1-\frac{k}{r}} \|x^{(r)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{k}{r}}$$

holds. We consider the analogue of this inequality for functions which are analytic in the strip  $S_\beta = \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$ . We also show that this problem is closely connected with the problem of optimal recovery of derivatives of analytic functions by inaccurate information about their traces on  $\mathbb{R}$ .

Denote by  $\mathcal{H}_2^\beta$  the Hardy space of functions  $f(\cdot)$  analytic in the strip  $S_\beta$  and satisfying the condition

$$\|f(\cdot)\|_{\mathcal{H}_2^\beta} = \left( \sup_{0 \leq \eta < \beta} \frac{1}{2} \int_{\mathbb{R}} (|f(t+i\eta)|^2 + |f(t-i\eta)|^2) dt \right)^{1/2} < \infty.$$

The Hardy-Sobolev space  $\mathcal{H}_2^{r,\beta}$  is the space of functions  $f(\cdot)$  analytic in the strip  $S_\beta$  for which  $f^{(r)}(\cdot) \in \mathcal{H}_2^\beta$ . Denote by  $\mu_{r\beta}(x)$  the unique solution of the equation

$$t^r \sqrt{\operatorname{ch} 2\beta t} = x$$

which belongs to the interval  $[0, +\infty)$ .

**Theorem.** *For all  $r, k \in \mathbb{N}$ ,  $k \leq r$ , and all functions  $f(\cdot) \in \mathcal{H}_2^{r,\beta} \cap L_2(\mathbb{R})$  which are not equivalent to zero the following exact inequalities*

$$\|f^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \leq \|f(\cdot)\|_{L_2(\mathbb{R})} \mu_{r\beta}^k \left( \frac{\|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta}}{\|f(\cdot)\|_{L_2(\mathbb{R})}} \right),$$

$$\|f^{(k)}(\cdot)\|_{\mathcal{H}_2^\beta} \leq \|f^{(r)}(\cdot)\|_{L_2(\mathbb{R})} \mu_{r\beta}^{k-r} \left( \frac{\|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta}}{\|f(\cdot)\|_{L_2(\mathbb{R})}} \right)$$

*hold.*

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