

OPTIMAL RECOVERY AND GENERALIZED CARLSON INEQUALITY FOR WEIGHTS WITH SYMMETRY PROPERTIES

K. YU. OSIPENKO

ABSTRACT. The paper concerns problems of the recovery of operators from noisy information in weighted L_q -spaces with homogeneous weights. A number of general theorems are proved and applied to finding exact constants in multidimensional Carlson type inequalities with several weights and problems of the recovery of differential operators from a noisy Fourier transform. In particular, optimal methods are obtained for the recovery of powers of generalized Laplace operators from a noisy Fourier transform in the L_p -metric.

1. INTRODUCTION

Let T be a nonempty set, Σ be the σ -algebra of subsets of T , and μ be a nonnegative σ -additive measure on Σ . We denote by $L_p(T, \Sigma, \mu)$ (or simply $L_p(T, \mu)$) the set of all Σ -measurable functions with values in \mathbb{R} or in \mathbb{C} for which

$$\|x(\cdot)\|_{L_p(T, \mu)} = \left(\int_T |x(t)|^p d\mu(t) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|x(\cdot)\|_{L_\infty(T, \mu)} = \operatorname{vraisup}_{t \in T} |x(t)| < \infty, \quad p = \infty.$$

If $T \subset \mathbb{R}^d$ and $d\mu = dt$, $t \in \mathbb{R}^d$, we put $L_p(T) = L_p(T, \mu)$.

The Carlson inequality [3]

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \leq \sqrt{\pi} \|x(t)\|_{L_2(\mathbb{R}_+)}^{1/2} \|tx(t)\|_{L_2(\mathbb{R}_+)}^{1/2}, \quad \mathbb{R}_+ = [0, +\infty),$$

was generalized by many authors (see [4], [1], [2], [8], [9]). In [8] we found sharp constants for inequalities of the form

$$\|w(\cdot)x(\cdot)\|_{L_q(T, \mu)} \leq K \|w_0(\cdot)x(\cdot)\|_{L_p(T, \mu)}^\gamma \|w_1(\cdot)x(\cdot)\|_{L_r(T, \mu)}^{1-\gamma},$$

where T is a cone in a linear space, $w(\cdot)$, $w_0(\cdot)$, and $w_1(\cdot)$ are homogenous functions and $1 \leq q < p, r < \infty$ (for $T = \mathbb{R}^d$ the sharp inequality was obtained in [2]). This problem is closely related with the following extremal problem

$$\|w(\cdot)x(\cdot)\|_{L_q(T, \mu)} \rightarrow \max, \quad \|w_0(\cdot)x(\cdot)\|_{L_p(T, \mu)} \leq \delta, \quad \|w_1(\cdot)x(\cdot)\|_{L_r(T, \mu)} \leq 1,$$

where $\delta > 0$. In this paper we study the extremal problem

$$(1) \quad \|w(\cdot)x(\cdot)\|_{L_q(T, \mu)} \rightarrow \max, \quad \|w_0(\cdot)x(\cdot)\|_{L_p(T, \mu)} \leq \delta,$$

$$\|w_j(\cdot)x(\cdot)\|_{L_r(T, \mu)} \leq 1, \quad j = 1, \dots, n,$$

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where $w(\cdot)$, $w_0(\cdot)$, and $w_j(\cdot)$, $j = 1, \dots, n$, are homogenous functions with some symmetry properties. Using the solution of this problem we obtain the sharp constant K for the inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \leq K \|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu)}^\gamma \left(\max_{1 \leq j \leq n} \|\omega_j(\cdot)x(\cdot)\|_{L_r(T,\mu)} \right)^{1-\gamma}.$$

In particular, we find the sharp constant for the inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(\mathbb{R}_+^d)} \leq C \|w_0(\cdot)x(\cdot)\|_{L_p(\mathbb{R}_+^d)}^{p\alpha} \left(\max_{1 \leq j \leq d} \|\omega_j(\cdot)x(\cdot)\|_{L_r(\mathbb{R}_+^d)} \right)^{r\beta},$$

where $w(t) = (t_1^2 + \dots + t_d^2)^{\theta/2}$, $w_0(t) = (t_1^2 + \dots + t_d^2)^{\theta_0/2}$, $w_j(t) = t_j^{\theta_1}$, $j = 1, \dots, d$, $\theta = d(1 - 1/q)$, $\theta_0 = d - (\lambda + d)/p$, $\theta_1 = d + (\mu - d)/r$,

$$\alpha = \frac{\mu}{p\mu + r\lambda}, \quad \beta = \frac{\lambda}{p\mu + r\lambda}, \quad \lambda, \mu > 0,$$

and $(p, q, r) \in P \cup P_1 \cup P_2$, where

$$P = \{(p, q, r) : 1 \leq q < p, r\}, \quad P_1 = \{(p, q, r) : 1 \leq q = r < p\}, \\ P_2 = \{(p, q, r) : 1 \leq q = p < r\}.$$

For $d = 1$, $q = 1$, and $(p, 1, r) \in P$ this result was proved in [4] (see also [2]).

It is appeared that the value of (1) is the error of optimal recovery of the operator $\Lambda x(\cdot) = w(\cdot)x(\cdot)$ on the class of functions $x(\cdot)$ such that $\|w_j(\cdot)x(\cdot)\|_{L_r(T,\mu)} \leq 1$, $j = 1, \dots, n$, by the information about the function $w_0(\cdot)x(\cdot)$ given with the error δ in L_p -norm. Therefore, in section 2 we begin with the setting of optimal recovery problem and then in section 3 we prove some general theorems. In section 4 we consider the case when weights are homogeneous in a cone of linear space and section 5 is devoted to the case of \mathbb{R}^d . In section 6 the results obtained are applied to optimal recovery and sharp inequalities of differential operators defined by Fourier transforms.

2. GENERAL SETTING

Let T_0 is not empty μ -measurable subset of T . Put

$$\mathcal{W} = \{x(\cdot) : x(\cdot) \in L_p(T_0, \mu), \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)} < \infty, j = 1, \dots, n\}, \\ W = \{x(\cdot) \in \mathcal{W} : \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T,\mu)} \leq 1, j = 1, \dots, n\},$$

where $1 \leq p, r \leq \infty$, and $\varphi_j(\cdot)$ is a measurable function on T . Consider the problem of recovery of operator $\Lambda : \mathcal{W} \rightarrow L_q(T, \mu)$, $1 \leq q \leq \infty$, defined by equality $\Lambda x(\cdot) = \psi(\cdot)x(\cdot)$, where $\psi(\cdot)$ is a measurable function on T , on the class W by the information about functions $x(\cdot) \in W$ given inaccurately (we assume that $\psi(\cdot)$ and $\varphi_j(\cdot)$, $j = 1, \dots, n$, such that Λ maps \mathcal{W} to $L_q(T, \mu)$). More precisely, we assume that for any function $x(\cdot) \in W$ we know $y(\cdot) \in L_p(T_0, \mu)$ such that $\|x(\cdot) - y(\cdot)\|_{L_p(T_0,\mu)} \leq \delta$, $\delta > 0$. We want to approximate the value $\Lambda x(\cdot)$ knowing $y(\cdot)$. As recovery methods we consider all possible mappings $m : L_p(T_0, \mu) \rightarrow L_q(T, \mu)$. The error of a method m is defined as

$$e(p, q, r, m) = \sup_{\substack{x(\cdot) \in W, y(\cdot) \in L_p(T_0,\mu) \\ \|x(\cdot) - y(\cdot)\|_{L_p(T_0,\mu)} \leq \delta}} \|\Lambda x(\cdot) - m(y(\cdot))\|_{L_q(T,\mu)}.$$

The quantity

$$(2) \quad E(p, q, r) = \inf_{m : L_p(T_0,\mu) \rightarrow L_q(T,\mu)} e(p, q, r, m)$$

is known as the optimal recovery error, and a method on which this infimum is attained is called optimal. Various settings of optimal recovery theory and examples of such problems may be found in [5], [13], [12], [6], [11].

For the lower bound of $E(p, q, r)$ we use the following result which was proved (in more or less general forms) in many papers (see, for example, [7]).

Lemma 1.

$$(3) \quad E(p, q, r) \geq \sup_{\substack{x(\cdot) \in W \\ \|x(\cdot)\|_{L_p(T_0, \mu)} \leq \delta}} \|\Lambda x(\cdot)\|_{L_q(T, \mu)}.$$

3. MAIN RESULTS

Set

$$\chi_0(t) = \begin{cases} 1, & t \in T_0, \\ 0, & t \notin T_0, \end{cases} \quad \sigma_r(t) = \sum_{j=1}^n \lambda_j |\varphi_j(t)|^r.$$

Theorem 1. *Let $1 \leq q < p, r$, $\lambda_j \geq 0$, $j = 0, 1, \dots, n$, $\lambda_0 + \sigma_r(t) \neq 0$ for almost all $t \in T_0$, $\sigma_r(t) \neq 0$ for almost all $t \in T \setminus T_0$, $\hat{x}(t) \geq 0$ be a solution of equation*

$$(4) \quad -q|\psi(t)|^q + p\lambda_0 x^{p-q}(t)\chi_0(t) + r\sigma_r(t)x^{r-q}(t) = 0,$$

$\bar{\lambda}$ such that

$$(5) \quad \int_{T_0} \hat{x}^p(t) d\mu(t) \leq \delta^p, \quad \int_T |\varphi_j(t)|^r \hat{x}^r(t) d\mu(t) \leq 1, \quad j = 1, \dots, n, \\ \lambda_0 \left(\int_{T_0} \hat{x}^p(t) d\mu(t) - \delta^p \right) = 0, \quad \lambda_j \left(\int_T |\varphi_j(t)|^r \hat{x}^r(t) d\mu(t) - 1 \right) = 0, \quad j = 1, \dots, n.$$

Then

$$(6) \quad E(p, q, r) = \left(q^{-1} p \lambda_0 \delta^p + q^{-1} r \sum_{j=1}^n \lambda_j \right)^{1/q},$$

and the method

$$(7) \quad \hat{m}(y)(t) = \begin{cases} q^{-1} p \lambda_0 \hat{x}^{p-q}(t) |\psi(t)|^{-q} \psi(t) y(t), & t \in T_0, \psi(t) \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

is optimal recovery method.

To prove this theorem we need some preliminary results. The first one is actually a sufficient condition in the Kuhn-Tucker theorem (the only difference is that we do not require convexity of functions).

Let $f_j : A \rightarrow \mathbb{R}$, $j = 0, 1, \dots, k$, be functions defined on some set A . Consider the extremal problem

$$(8) \quad f_0(x) \rightarrow \max, \quad f_j(x) \leq 0, \quad j = 1, \dots, k, \quad x \in A,$$

and write down its Lagrange function

$$\mathcal{L}(x, \lambda) = -f_0(x) + \sum_{j=1}^k \lambda_j f_j(x), \quad \lambda = (\lambda_1, \dots, \lambda_k).$$

Lemma 2. Assume that there exist $\widehat{\lambda}_j \geq 0$, $j = 1, \dots, k$, and an element $\widehat{x} \in A$, admissible for problem (8), such that

$$(a) \quad \min_{x \in A} \mathcal{L}(x, \widehat{\lambda}) = \mathcal{L}(\widehat{x}, \widehat{\lambda}), \quad \widehat{\lambda} = (\widehat{\lambda}_1, \dots, \widehat{\lambda}_k),$$

$$(b) \quad \widehat{\lambda}_j f_j(\widehat{x}) = 0, \quad j = 1, \dots, k.$$

Then \widehat{x} is an extremal element for problem (8).

Proof. For any x admissible for problem (8) we have

$$-f_0(x) \geq \mathcal{L}(x, \widehat{\lambda}) \geq \mathcal{L}(\widehat{x}, \widehat{\lambda}) = -f_0(\widehat{x}).$$

□

Put

$$F(u, v, \alpha) = -((1 - \alpha)u + \alpha v)^q + av^p + bu^r, \quad u, v \geq 0, \quad \alpha \in [0, 1],$$

where $a, b \geq 0$, and $1 \leq p, q, r < \infty$.

Lemma 3 ([8]). For all $a, b \geq 0$, $a + b > 0$, and all $1 \leq q < p, r < \infty$, there exists the unique solution $\widehat{u} > 0$ of the equation

$$-q + pau^{p-q} + rbu^{r-q} = 0.$$

Moreover, for all $u, v \geq 0$ and $\alpha = q^{-1}pa\widehat{u}^{p-q} = 1 - q^{-1}rb\widehat{u}^{r-q}$

$$F(\widehat{u}, \widehat{u}, \alpha) \leq F(u, v, \alpha).$$

In particular, for all $u \geq 0$

$$-\widehat{u}^q + a\widehat{u}^p + b\widehat{u}^r \leq -u^q + au^p + bu^r.$$

Proof of Theorem 1. 1. Lower estimate. The extremal problem on the right-hand side of (3) (for convenience, we raise the quantity to be maximized to the q -th power) is as follows:

$$(9) \quad \int_T |\psi(t)x(t)|^q d\mu(t) \rightarrow \max, \quad \int_{T_0} |x(t)|^p d\mu(t) \leq \delta^p,$$

$$\int_T |\varphi_j(t)x(t)|^r d\mu(t) \leq 1, \quad j = 1, \dots, n.$$

If $t \in T$ such that $\psi(t) = 0$, then evidently $\widehat{x}(t) = 0$. If $\psi(t) \neq 0$ we obtain by Lemma 3 that that there is the unique solution $\widehat{x}(t)$ of (4). It follows by (5) that $\widehat{x}(\cdot)$ is admissible function for problem (9). Therefore, by (3) we obtain

$$E(p, q, r) \geq \left(\int_T |\psi(t)|^q \widehat{x}^q(t) d\mu(t) \right)^{1/q}.$$

From (4) we have

$$|\psi(t)|^q \widehat{x}^q(t) = q^{-1}p\lambda_0 \widehat{x}^p(t)\chi_0(t) + q^{-1}r\sigma_r(t)\widehat{x}^r(t).$$

Integrating this equality over the set T , we obtain

$$\int_T |\psi(t)|^q \widehat{x}^q(t) d\mu(t) = q^{-1}p\lambda_0\delta^p + q^{-1}r \sum_{j=1}^n \lambda_j.$$

Thus,

$$E(p, q, r) \geq \left(q^{-1}p\lambda_0\delta^p + q^{-1}r \sum_{j=1}^n \lambda_j \right)^{1/q}.$$

2. Upper estimate. To estimate the error of method (7) we need to find the value of the extremal problem:

$$(10) \quad \int_{T_0} |\psi(t)x(t) - \psi(t)\alpha(t)y(t)|^q d\mu(t) + \int_{T \setminus T_0} |\psi(t)x(t)|^q d\mu(t) \rightarrow \max,$$

$$\int_{T_0} |x(t) - y(t)|^p d\mu(t) \leq \delta^p, \quad \int_T |\varphi_j(t)x(t)|^r d\mu(t) \leq 1, \quad j = 1, \dots, n,$$

where

$$\alpha(t) = \begin{cases} q^{-1} p \lambda_0 \widehat{x}^{p-q}(t) |\psi(t)|^{-q}, & t \in T_0, \psi(t) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Put

$$z(t) = \begin{cases} x(t) - y(t), & t \in T_0, \\ 0, & t \in T \setminus T_0. \end{cases}$$

Then (10) may be rewritten as follows:

$$\int_T |\psi(t)|^q |(1 - \alpha(t))x(t) + \alpha(t)z(t)|^q d\mu(t) \rightarrow \max,$$

$$\int_{T_0} |z(t)|^p d\mu(t) \leq \delta^p, \quad \int_T |\varphi_j(t)x(t)|^r d\mu(t) \leq 1, \quad j = 1, \dots, n.$$

The value of this problem does not exceed the value of the problem

$$(11) \quad \int_T |\psi(t)|^q ((1 - \alpha(t))u(t) + \alpha(t)v(t))^q d\mu(t) \rightarrow \max,$$

$$\int_{T_0} v^p(t) d\mu(t) \leq \delta^p, \quad \int_T |\varphi_j(t)|^r u^r(t) d\mu(t) \leq 1, \quad j = 1, \dots, n,$$

$$u(t) \geq 0, \quad v(t) \geq 0 \quad \text{for almost all } t \in T.$$

The Lagrange function for this problem is

$$\mathcal{L}(u(\cdot), v(\cdot), \bar{\lambda}) = \int_T L(t, u(t), v(t), \bar{\lambda}) d\mu(t),$$

where

$$L(t, u, v, \bar{\lambda}) = -|\psi(t)|^q ((1 - \alpha(t))u + \alpha(t)v)^q + \lambda_0 v^p \chi_0(t) + \sigma_r(t) u^r.$$

By Lemma 3 we have

$$L(t, \widehat{x}(t), \widehat{x}(t), \bar{\lambda}) \leq L(t, u(t), v(t), \bar{\lambda}).$$

Thus,

$$\mathcal{L}(\widehat{x}(\cdot), \widehat{x}(\cdot), \bar{\lambda}) \leq \mathcal{L}(u(\cdot), v(\cdot), \bar{\lambda}).$$

It follows by Lemma 2 that functions $u(\cdot) = v(\cdot) = \widehat{x}(\cdot)$ are extremal in (11). Consequently,

$$e(p, q, r, \widehat{m}) \leq \left(\int_T |\psi(t)|^q \widehat{x}^q(t) d\mu(t) \right)^{1/q} = \left(q^{-1} p \lambda_0 \delta^p + q^{-1} r \sum_{j=1}^n \lambda_j \right)^{1/q} \leq E(p, q, r).$$

It means that method (7) is optimal and equality (6) holds. \square

Denote $a_+ = \max\{a, 0\}$.

Theorem 2. Let $1 \leq q = r < p$, $\lambda_0 > 0$, $\lambda_j \geq 0$, $j = 1, \dots, n$,

$$(12) \quad \widehat{x}(t) = \begin{cases} \left(\frac{q}{p\lambda_0} (|\psi(t)|^q - \sigma_q(t))_+ \right)^{\frac{1}{p-q}}, & t \in T_0, \\ 0, & t \notin T_0, \end{cases}$$

$\bar{\lambda}$ satisfies conditions (5), and $|\psi(t)|^q - \sigma_q(t) \leq 0$ for almost all $t \notin T_0$. Then

$$(13) \quad E(p, q, q) = \left(q^{-1}p\lambda_0\delta^p + \sum_{j=1}^n \lambda_j \right)^{1/q},$$

and the method

$$(14) \quad \widehat{m}(y)(t) = \begin{cases} (1 - |\psi(t)|^{-q}\sigma_q(t))_+ \psi(t)y(t), & t \in T_0, \psi(t) \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

is optimal.

Proof. 1. Lower estimate. It follows by (5) that $\widehat{x}(\cdot)$ is admissible function for extremal problem in the right-hand side of (3). Therefore,

$$E(p, q, q) \geq \left(\int_T |\psi(t)|^q \widehat{x}^q(t) d\mu(t) \right)^{1/q}.$$

From the definition of $\widehat{x}(\cdot)$ we have

$$|\psi(t)|^q \widehat{x}^q(t) = q^{-1}p\lambda_0 \widehat{x}^p(t) \chi_0(t) + \sigma_q(t) \widehat{x}^q(t).$$

Integrating this equality, we obtain

$$\int_T |\psi(t)|^q \widehat{x}^q(t) d\mu(t) = q^{-1}p\lambda_0\delta^p + \sum_{j=1}^n \lambda_j.$$

Thus,

$$E(p, q, q) \geq \left(q^{-1}p\lambda_0\delta^p + \sum_{j=1}^n \lambda_j \right)^{1/q}.$$

2. Upper estimate. Put

$$\alpha(t) = \begin{cases} (1 - |\psi(t)|^{-q}\sigma_q(t))_+, & t \in T_0, \psi(t) \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

To estimate the error of method (14) we need to find the value of the extremal problem:

$$\begin{aligned} \int_{T_0} |\psi(t)|^q |x(t) - \alpha(t)y(t)|^q d\mu(t) + \int_{T \setminus T_0} |\psi(t)x(t)|^q d\mu(t) &\rightarrow \max, \\ \int_{T_0} |x(t) - y(t)|^p d\mu(t) \leq \delta^p, \quad \int_T |\varphi_j(t)x(t)|^q d\mu(t) \leq 1, & j = 1, \dots, n. \end{aligned}$$

Putting $z(\cdot) = x(\cdot) - y(\cdot)$ this problem may be rewritten in the following form

$$\begin{aligned} \int_{T_0} |\psi(t)|^q |(1 - \alpha(t))x(t) + \alpha(t)z(t)|^q d\mu(t) + \int_{T \setminus T_0} |\psi(t)x(t)|^q d\mu(t) &\rightarrow \max, \\ \int_{T_0} |z(t)|^p d\mu(t) \leq \delta^p, \quad \int_T |\varphi_j(t)x(t)|^q d\mu(t) \leq 1, & j = 1, \dots, n. \end{aligned}$$

The value of this problem evidently coincides with the value of the problem

$$(15) \quad \int_T |\psi(t)|^q ((1 - \alpha(t))v(t) + \alpha(t)u(t))^q d\mu(t) \rightarrow \max,$$

$$\int_{T_0} u^p(t) d\mu(t) \leq \delta^p, \quad \int_T |\varphi_j(t)|^q v^q(t) d\mu(t) \leq 1, \quad j = 1, \dots, n,$$

$$u(t), v(t) \geq 0, \quad \text{for almost all } t \in T.$$

The Lagrange function of (15) has the form

$$\mathcal{L}(u(\cdot), v(\cdot), \bar{\lambda}) = \int_T L(u(t), v(t), \bar{\lambda}) d\mu(t),$$

where

$$L(u, v, \bar{\lambda}) = \begin{cases} -|\psi(t)|^q ((1 - \alpha(t))v + \alpha(t)u)^q + \lambda_0 u^p + \sigma_q(t) v^q, & t \in T_0, \\ -|\psi(t)|^q v^q + \sigma_q(t) v^q, & t \notin T_0. \end{cases}$$

If $\alpha(t) > 0$, then

$$\frac{\partial L}{\partial v} = q(v^{q-1} - ((1 - \alpha(t))v + \alpha(t)u)^{q-1})\sigma_q(t).$$

Therefore, for $\alpha(t) > 0$ and any $u > 0$, the function $L(u, v, \bar{\lambda})$, $v \in (0, +\infty)$, reaches a minimum at $v = u$. Set $T'_0 = \{t \in T_0 : \alpha(t) > 0\}$. We have

$$\mathcal{L}(u(\cdot), v(\cdot), \bar{\lambda}) \geq \int_{T'_0} L(u(\cdot), u(\cdot), \bar{\lambda}) d\mu(t).$$

It is easily checked that for $t \in T'_0$ for all $u(t) \geq 0$

$$L(u(\cdot), u(\cdot), \bar{\lambda}) \geq L(\hat{x}(\cdot), \hat{x}(\cdot), \bar{\lambda}).$$

Consequently,

$$\mathcal{L}(u(\cdot), v(\cdot), \bar{\lambda}) \geq \int_{T'_0} L(\hat{x}(\cdot), \hat{x}(\cdot), \bar{\lambda}) d\mu(t) = \mathcal{L}(\hat{x}(\cdot), \hat{x}(\cdot), \bar{\lambda}).$$

Taking into account (5) we obtain by Lemma 2 that $u(\cdot) = v(\cdot) = \hat{x}(\cdot)$ are extremal functions in (15). Thus,

$$e^q(p, q, q, \hat{m}) = \int_T |\psi(t)\hat{x}(t)|^q d\mu(t) = q^{-1} p \lambda_0 \delta^p + \sum_{j=1}^n \lambda_j \leq E^q(p, q, q).$$

It means that the method \hat{m} is optimal and the optimal recovery error is as stated. \square

Theorem 3. Let $1 \leq q = p < r$, $\lambda_0 > 0$, $\lambda_j \geq 0$, $j = 1, \dots, n$, $\sigma_r(t) \neq 0$ for almost all $t \in T$,

$$(16) \quad \hat{x}(t) = \begin{cases} (pr^{-1}\sigma_r^{-1}(t)(|\psi(t)|^p - \lambda_0)_+)^{\frac{1}{r-p}}, & t \in T_0, \\ (pr^{-1}\sigma_r^{-1}(t)|\psi(t)|^p)^{\frac{1}{r-p}}, & t \in T \setminus T_0, \end{cases}$$

and $\bar{\lambda}$ satisfies conditions (5). Then

$$(17) \quad E(p, p, r) = \left(\lambda_0 \delta^p + \frac{r}{p} \sum_{j=1}^n \lambda_j \right)^{1/p},$$

and the method

$$(18) \quad \widehat{m}(y)(t) = \begin{cases} \alpha(t)\psi(t)y(t), & t \in T_0, \\ 0, & t \in T \setminus T_0, \end{cases}$$

where

$$\alpha(t) = \begin{cases} \min \{1, \lambda_0 |\psi(t)|^{-p}\}, & t \in T_0, \psi(t) \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

is optimal.

Proof. 1. Lower estimate. By the definition of $\widehat{x}(\cdot)$ we have

$$|\psi(t)|^p \widehat{x}^p(t) = \lambda_0 \widehat{x}^p(t) \chi_0(t) + \frac{r}{p} \sigma_r(t) \widehat{x}^r(t).$$

Using the similar arguments as in the proof of Theorem 1 we obtain

$$E(p, p, r) \geq \left(\int_T |\psi(t)|^p \widehat{x}^p(t) d\mu(t) \right)^{1/p} = \left(\lambda_0 \delta^p + \frac{r}{p} \sum_{j=1}^n \lambda_j \right)^{1/p}.$$

2. Upper estimate. To estimate the error of method (18) we need to find the value of the following extremal problem:

$$\begin{aligned} \int_{T_0} |\psi(t)|^p |x(t) - \alpha(t)y(t)|^p d\mu(t) + \int_{T \setminus T_0} |\psi(t)x(t)|^p d\mu(t) \rightarrow \max, \\ \int_{T_0} |x(t) - y(t)|^p d\mu(t) \leq \delta^p, \quad \int_T |\varphi_j(t)x(t)|^r d\mu(t) \leq 1, \quad j = 1, \dots, n. \end{aligned}$$

Putting $z(\cdot) = x(\cdot) - y(\cdot)$ this problem may be rewritten in the form

$$\begin{aligned} \int_{T_0} |\psi(t)|^p |(1 - \alpha(t))x(t) + \alpha(t)z(t)|^p d\mu(t) + \int_{T \setminus T_0} |\psi(t)x(t)|^p d\mu(t) \rightarrow \max, \\ \int_{T_0} |z(t)|^p d\mu(t) \leq \delta^p, \quad \int_T |\varphi_j(t)x(t)|^r d\mu(t) \leq 1, \quad j = 1, \dots, n. \end{aligned}$$

The value of this problem evidently coincides with the value of the problem

$$(19) \quad \begin{aligned} \int_T |\psi(t)|^p ((1 - \alpha(t))v(t) + \alpha(t)u(t))^p d\mu(t) \rightarrow \max, \\ \int_{T_0} u^p(t) d\mu(t) \leq \delta^p, \quad \int_T |\varphi_j(t)|^r v^r(t) d\mu(t) \leq 1, \quad j = 1, \dots, n, \\ u(t), v(t) \geq 0, \quad \text{for almost all } t \in T. \end{aligned}$$

The Lagrange function of (19) has the form

$$\mathcal{L}(u(\cdot), v(\cdot), \bar{\lambda}) = \int_T L(u(t), v(t), \bar{\lambda}) d\mu(t),$$

where

$$L(u, v, \bar{\lambda}) = \begin{cases} -|\psi(t)|^p ((1 - \alpha(t))v + \alpha(t)u)^p + \lambda_0 u^p + \sigma_r(t)v^r, & t \in T_0, \\ -|\psi(t)|^p v^p + \sigma_r(t)v^r, & t \in T \setminus T_0. \end{cases}$$

For $t \in T_0$ and $|\psi(t)|^p > \lambda_0$ we have

$$\frac{\partial L}{\partial u} = p\lambda_0(u^{p-1} - ((1 - \alpha(t))v + \alpha(t)u)^{p-1}).$$

Consequently, in this case for any $v > 0$ the function $L(u, v, \bar{\lambda})$, $v \in (0, +\infty)$, reaches a minimum at $v = u$. If $t \in T_0$, $0 < |\psi(t)|^p \leq \lambda_0$, then $\alpha(t) = 1$ and $L(u, v, \bar{\lambda}) \geq 0$. If $t \in T_0$ and $\psi(t) = 0$, then again $L(u, v, \bar{\lambda}) \geq 0$. Set $T_1 = \{t \in T_0 : |\psi(t)|^p > \lambda_0\}$. Then for all $u(t), v(t) \geq 0$ we have

$$\mathcal{L}(u(\cdot), v(\cdot), \bar{\lambda}) \geq \int_{T_1} L(v(\cdot), v(\cdot), \bar{\lambda}) d\mu(t) + \int_{T \setminus T_0} L(v(\cdot), v(\cdot), \bar{\lambda}) d\mu(t).$$

It is easy to check that for all $v(t) \geq 0$

$$L(v(\cdot), v(\cdot), \bar{\lambda}) \geq L(\hat{x}(\cdot), \hat{x}(\cdot), \bar{\lambda}).$$

Therefore,

$$\mathcal{L}(u(\cdot), v(\cdot), \bar{\lambda}) \geq \int_{T_1 \cup (T \setminus T_0)} L(\hat{x}(\cdot), \hat{x}(\cdot), \bar{\lambda}) d\mu(t) = \mathcal{L}(\hat{x}(\cdot), \hat{x}(\cdot), \bar{\lambda}).$$

Taking into account (5) we obtain by Lemma 2 that $u(\cdot) = v(\cdot) = \hat{x}(\cdot)$ are extremal functions in (19). Consequently,

$$e^p(p, p, r, \hat{m}) = \int_T |\psi(t)\hat{x}(t)|^q d\mu(t) = \lambda_0 \delta^p + \frac{r}{p} \sum_{j=1}^n \lambda_j \leq E^p(p, p, r).$$

It means that the method \hat{m} is optimal and the optimal recovery error is as stated. \square

Note that if conditions of Theorems 1, 2, and 3 are fulfilled, then we have

$$(20) \quad E(p, q, r) = \sup_{\substack{\|x(\cdot)\|_{L_p(T_0, \mu)} \leq \delta \\ \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T, \mu)} \leq 1, j=1, \dots, n}} \|\psi(\cdot)x(\cdot)\|_{L_q(T, \mu)}.$$

4. THE CASE OF HOMOGENOUS WEIGHT FUNCTIONS

Let T be a cone in a linear space, $T_0 = T$, $\mu(\cdot)$ be a homogenous measure of degree d , $|\psi(\cdot)|$ be homogenous function of degree η , $|\varphi_j(\cdot)|$, $j = 1, \dots, n$, be homogenous functions of degrees ν , $\psi(t) \neq 0$ and $\sum_{j=1}^n |\varphi_j(t)| \neq 0$ for almost all $t \in T$. Let assume, again, that $1 \leq p < q, r < \infty$. For $k \in [0, 1)$ the function $k^{\frac{1}{p-q}}(1-k)^{-\frac{1}{r-q}}$ increases monotonically from 0 to $+\infty$. Consequently, there exists $k(\cdot)$ such that for almost all $t \in T$

$$(21) \quad \frac{k^{\frac{1}{p-q}}(t)}{(1-k(t))^{\frac{1}{r-q}}} = s_r^{-\frac{1}{r-q}}(t) |\psi(t)|^{\frac{q(p-r)}{(p-q)(r-q)}}, \quad s_r(t) = \sum_{j=1}^n |\varphi_j(t)|^r.$$

Set

$$k(t) = \begin{cases} (1 - |\psi(t)|^{-q} s_q(t))_+, & (p, q, r) \in P_1, \\ \min\{1, |\psi(t)|^{-p}\}, & (p, q, r) \in P_2 \end{cases}$$

Theorem 4. Let $(p, q, r) \in P \cup P_1 \cup P_2$ and $\nu + d(1/r - 1/p) \neq 0$. Assume that for $(p, q, r) \in P \cup P_1$

$$I_1 = \int_T |\psi(t)|^{\frac{qp}{p-q}} k^{\frac{p}{p-q}}(t) d\mu(t) < \infty,$$

$$I_{j+1} = \int_T |\psi(t)|^{\frac{qr}{p-q}} |\varphi_j(t)|^r k^{\frac{r}{p-q}}(t) d\mu(t) < \infty, \quad j = 1, \dots, n,$$

and for $(p, q, r) \in P_2$

$$I_1 = \int_T (s_r^{-1}(t)(|\psi(t)|^p - 1)_+)^{\frac{p}{r-p}} d\mu(t) < \infty,$$

$$I_{j+1} = \int_T |\varphi_j(t)|^r (s_r^{-1}(t)(|\psi(t)|^p - 1)_+)^{\frac{r}{r-p}} d\mu(t) < \infty, \quad j = 1, \dots, n.$$

Moreover, assume that $I_2 = \dots = I_{n+1}$. Then

$$(22) \quad E(p, q, r) = \delta^\gamma I_1^{-\gamma/p} I_2^{-(1-\gamma)/r} (I_1 + nI_2)^{1/q},$$

where

$$(23) \quad \gamma = \frac{\nu - \eta - d(1/q - 1/r)}{\nu + d(1/r - 1/p)}.$$

The method

$$\widehat{m}(y)(t) = k(\xi t)\psi(t)y(t),$$

where

$$(24) \quad \xi = \left(\delta I_1^{-1/p} I_2^{1/r} \right)^{\frac{1}{\nu + d(1/r - 1/p)}},$$

is optimal.

Proof. 1. Let $(p, q, r) \in P$. Put

$$\widehat{x}(t) = \left(\frac{q|\psi(t)|^q}{p\lambda_0} \right)^{\frac{1}{p-q}} k^{\frac{1}{p-q}}(\xi t),$$

where λ_0 will be specified later. We have

$$(25) \quad p\lambda_0 \widehat{x}^{p-q}(t) = q|\psi(t)|^q k(\xi t)$$

and

$$rc_r(t) \widehat{x}^{r-q}(t) = rc_r(t) \left(\frac{q|\psi(t)|^q}{p\lambda_0} \right)^{\frac{r-q}{p-q}} k^{\frac{r-q}{p-q}}(\xi t).$$

Since $|\psi(\cdot)|$ and $|\varphi_j(\cdot)|$, $j = 1, \dots, n$, are homogenous it follows by (21) that

$$k^{\frac{r-q}{p-q}}(\xi t) = \frac{|\psi(\xi t)|^{\frac{q(p-r)}{p-q}}}{c_r(\xi t)} (1 - k(\xi t)) = \xi^{\eta \frac{q(p-r)}{p-q} - \nu r} \frac{|\psi(t)|^{\frac{q(p-r)}{p-q}}}{c_r(t)} (1 - k(\xi t)).$$

Thus,

$$rc_r(t) \widehat{x}^{r-q}(t) = r \left(\frac{q}{p\lambda_0} \right)^{\frac{r-q}{p-q}} \xi^{\eta \frac{q(p-r)}{p-q} - \nu r} |\psi(t)|^q (1 - k(\xi t)).$$

Put

$$(26) \quad \lambda = \frac{q}{r} \left(\frac{q}{p\lambda_0} \right)^{-\frac{r-q}{p-q}} \xi^{-\eta \frac{q(p-r)}{p-q} + \nu r}.$$

Then

$$(27) \quad r\lambda c_r(t) \widehat{x}^{r-q}(t) = q|\psi(t)|^q (1 - k(\xi t)).$$

Taking the sum of (25) and (27), we obtain

$$p\lambda_0 \widehat{x}^{p-q}(t) + r\lambda c_r(t) \widehat{x}^{r-q}(t) = q|\psi(t)|^q.$$

It means that $\widehat{x}(\cdot)$ satisfies (4) for $\lambda_1 = \dots = \lambda_n = \lambda$.

Now we show that for

$$(28) \quad \lambda_0 = \frac{q}{p} I_1^{\frac{p-q}{p}} \xi^{-\eta q - d \frac{p-q}{p}} \delta^{q-p}$$

the equalities

$$\int_T \widehat{x}^p(t) d\mu(t) = \delta^p, \quad \int_T |\varphi_j(t)|^r \widehat{x}^r(t) d\mu(t) = 1, \quad j = 1, \dots, n,$$

hold. In view of the definition of $\widehat{x}(\cdot)$ we need to check that

$$\begin{aligned} & \int_T \left(\frac{q|\psi(t)|^q}{p\lambda_0} \right)^{\frac{p}{p-q}} k^{\frac{p}{p-q}}(\xi t) d\mu(t) = \delta^p, \\ & \int_T |\varphi_j(t)|^r \left(\frac{q|\psi(t)|^q}{p\lambda_0} \right)^{\frac{r}{p-q}} k^{\frac{r}{p-q}}(\xi t) d\mu(t) = 1, \quad j = 1, \dots, n. \end{aligned}$$

Changing $z = \xi t$ and taking into account that functions $|\psi(\cdot)|$, $|\varphi_j(\cdot)|$, $j = 1, \dots, n$, with the measure $\mu(\cdot)$ are homogenous, we obtain

$$\left(\frac{q}{p\lambda_0} \right)^{\frac{p}{p-q}} I_1 = \delta^p \xi^{\frac{\eta q p}{p-q} + d}, \quad \left(\frac{q}{p\lambda_0} \right)^{\frac{r}{p-q}} I_{j+1} = \xi^{\frac{\eta q r}{p-q} + \nu r + d}, \quad j = 1, \dots, n.$$

The validity of these equalities immediately follows from the definitions of λ_0 and ξ .

It follows by Theorem 1, (28), (26), and (24) that

$$\begin{aligned} E^q(p, q, r) &= \frac{p\lambda_0 \delta^p + nr\lambda}{q} = I_1^{\frac{p-q}{p}} \xi^{-\eta q - d \frac{p-q}{p}} \delta^q \\ &+ n \left(\frac{p\lambda_1}{q} \right)^{\frac{r-q}{p-q}} \xi^{\nu r - \eta \frac{q(p-r)}{p-q}} = \delta^{q\gamma} I_1^{-q\gamma/p} I_2^{-q(1-\gamma)/r} (I_1 + nI_2). \end{aligned}$$

Moreover, the same theorem states that the method

$$\widehat{m}(y)(t) = q^{-1} p \lambda_0 \widehat{x}^{p-q}(t) |\psi(t)|^{-q} \psi(t) y(t) = k(\xi t) \psi(t) y(t)$$

is optimal.

2. Let $(p, q, r) \in P_1$. We use Theorem 2. Consider the function $\widehat{x}(\cdot)$ defined by (12) with $\lambda_1 = \dots = \lambda_n = \lambda$. Let us find λ_0 and λ from the conditions

$$\int_T \widehat{x}^p(t) d\mu(t) = \delta^p, \quad \int_T |\varphi_j(t)|^q \widehat{x}^q(t) d\mu(t) = 1, \quad j = 1, \dots, n.$$

Then we obtain

$$\begin{aligned} & \left(\frac{q}{p\lambda_0} \right)^{\frac{p}{p-q}} \int_T (|\psi(t)|^q - \lambda s_q(t))_+^{\frac{p}{p-q}} d\mu(t) = \delta^p, \\ & \left(\frac{q}{p\lambda_0} \right)^{\frac{q}{p-q}} \int_T |\varphi_j(t)|^q (|\psi(t)|^q - \lambda s_q(t))_+^{\frac{q}{p-q}} d\mu(t) = 1, \quad j = 1, \dots, n. \end{aligned}$$

Put $\lambda = a^{(\eta-\nu)q}$, $a > 0$. Changing $t = az$, we obtain

$$\left(\frac{q}{p\lambda_0} \right)^{\frac{p}{p-q}} a^{d + \frac{pqn}{p-q}} I_1 = \delta^p, \quad \left(\frac{q}{p\lambda_0} \right)^{\frac{q}{p-q}} a^{d+q\nu + \frac{q^2\eta}{p-q}} I_{j+1} = 1, \quad j = 1, \dots, n.$$

It is easy to check that these equalities are fulfilled for

$$a = (I_1^{1/p} I_2^{-1/q} \delta^{-1})^{\frac{1}{\nu + d(1/q - 1/p)}}, \quad \lambda_0 = \frac{q}{p} I_1 I_2^{-1} \delta^{-p} (I_1^{-q/p} I_2 \delta^q)^{\frac{\eta-\nu}{\nu + d(1/q - 1/p)}}.$$

Substituting these values in (13) and (14) we obtain the statement of the theorem in the case under consideration.

3. Let $(p, q, r) \in P_2$. Here we use Theorem 3. Put $\lambda_1 = \dots = \lambda_n = \lambda$ in the definition of $\widehat{x}(\cdot)$ (see (16)). We find λ_0 and λ from the conditions

$$\int_T \widehat{x}^p(t) d\mu(t) = \delta^p, \quad \int_T |\varphi_j(t)|^r \widehat{x}^r(t) d\mu(t) = 1, \quad j = 1, \dots, n.$$

We have

$$\begin{aligned} \left(\frac{p}{r\lambda}\right)^{\frac{p}{r-p}} \int_T (s_r^{-1}(t)(|\psi(t)|^p - \lambda_0)_+)^{\frac{p}{r-p}} d\mu(t) &= \delta^p, \\ \left(\frac{p}{r\lambda}\right)^{\frac{r}{r-p}} \int_T |\varphi_j(t)|^r (s_r^{-1}(t)(|\psi(t)|^p - \lambda_0)_+)^{\frac{r}{r-p}} d\mu(t) &= 1, \quad j = 1, \dots, n. \end{aligned}$$

Put $\lambda_0 = a^{np}$, $a > 0$. Changing $t = az$, we obtain

$$\begin{aligned} \left(\frac{p}{r\lambda}\right)^{\frac{p}{r-p}} a^{d+\frac{p^2\eta}{r-p}-\frac{pr\nu}{r-p}} I_1 &= \delta^p, \\ \left(\frac{p}{r\lambda}\right)^{\frac{r}{r-p}} a^{d+r\nu+\frac{pr\eta}{r-p}-\frac{r^2\nu}{r-p}} I_{j+1} &= 1, \quad j = 1, \dots, n. \end{aligned}$$

These equalities are valid for

$$\begin{aligned} a &= (I_1^{1/p} I_2^{-1/r} \delta^{-1})^{\frac{1}{\nu+d(1/r-1/p)}}, \\ \lambda &= \frac{p}{r} I_1^{r/p-1} \delta^{p-r} (I_1^{r/p} I_2^{-1} \delta^{-r})^{\frac{p\eta/r-\nu-d(1/r-1/p)}{\nu+d(1/r-1/p)}}. \end{aligned}$$

It remains to substitute these values into (17) and (18). \square

Corollary 1. *Assume that conditions of Theorem 4 hold. Then for all $x(\cdot) \neq 0$ such that $x(\cdot) \in L_p(T, \mu)$ and $\varphi_j(\cdot)x(\cdot) \in L_r(T, \mu)$, $j = 1, \dots, n$, the sharp inequality*

$$(29) \quad \|\psi(\cdot)x(\cdot)\|_{L_q(T, \mu)} \leq C \|x(\cdot)\|_{L_p(T, \mu)}^\gamma \left(\max_{1 \leq j \leq n} \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T, \mu)} \right)^{1-\gamma}$$

holds, where

$$C = I_1^{-\gamma/p} I_2^{-(1-\gamma)/r} (I_1 + nI_2)^{1/q}.$$

Proof. Let $x(\cdot) \in L_p(T, \mu)$, $\|\varphi_j(\cdot)x(\cdot)\|_{L_r(T, \mu)} < \infty$, $j = 1, \dots, n$ and $x(\cdot) \neq 0$. Put

$$A = \max_{1 \leq j \leq n} \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T, \mu)}.$$

Consider $\widehat{x}(\cdot) = x(\cdot)/A$. Put $\delta = \|\widehat{x}(\cdot)\|_{L_p(T, \mu)}$. Then $\|\varphi_j(\cdot)\widehat{x}(\cdot)\|_{L_r(T, \mu)} \leq 1$, $j = 1, \dots, n$. In view of (20) and Theorem 4 we have

$$\|\psi(\cdot)\widehat{x}(\cdot)\|_{L_q(T, \mu)} \leq C \|\widehat{x}(\cdot)\|_{L_p(T, \mu)}^\gamma.$$

This implies (29).

If there exists a $\widetilde{C} < C$ for which (29) holds, then

$$E(p, q, r) = \sup_{\substack{\|x(\cdot)\|_{L_p(T, \mu)} \leq \delta \\ \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T, \mu)} \leq 1, \quad j=1, \dots, n}} \|\psi(\cdot)x(\cdot)\|_{L_q(T, \mu)} \leq \widetilde{C} \delta^\gamma < C \delta^\gamma.$$

This contradicts with (22). \square

Let $|w(\cdot)|$, $|w_0(\cdot)|$ be homogenous functions of degrees θ , θ_0 , respectively and $|w_j(\cdot)|$, $j = 1, \dots, n$, be homogenous functions of degree θ_1 . We assume that $w(t), w_0(t) \neq 0$ and $\sum_{j=1}^n |w_j(t)| \neq 0$ for almost all $t \in T$.

For $(p, q, r) \in P$ we define $\tilde{k}(\cdot)$ by the equality

$$\frac{\tilde{k}^{\frac{1}{p-q}}(t)}{(1 - \tilde{k}(t))^{\frac{1}{r-q}}} = \left| \frac{w_0(t)}{w(t)} \right|^{\frac{p}{p-q}} \left(\sum_{j=1}^n \left| \frac{w_j(t)}{w(t)} \right|^r \right)^{-\frac{1}{r-q}}.$$

For $(p, q, r) \in P_1$ set

$$\tilde{k}(t) = \left(1 - |w(t)|^{-q} \sum_{j=1}^n |w_j(t)|^q \right)_+.$$

Put

$$(30) \quad \tilde{\theta} = \theta + d/q, \quad \tilde{\theta}_0 = \theta_0 + d/p, \quad \tilde{\theta}_1 = \theta_1 + d/r, \quad \tilde{\gamma} = \frac{\tilde{\theta}_1 - \tilde{\theta}}{\tilde{\theta}_1 - \tilde{\theta}_0}.$$

Corollary 2. Let $(p, q, r) \in P \cup P_1 \cup P_2$ and $\tilde{\theta}_0 \neq \tilde{\theta}_1$. Assume that for $(p, q, r) \in P \cup P_1$

$$\begin{aligned} \tilde{I}_1 &= \int_T \left| \frac{w(t)}{w_0(t)} \right|^{\frac{qp}{p-q}} \tilde{k}^{\frac{p}{p-q}}(t) d\mu(t) < \infty, \\ \tilde{I}_{j+1} &= \int_T \frac{|w(t)|^{\frac{qr}{p-q}}}{|w_0(t)|^{\frac{pr}{p-q}}} |w_j(t)|^r \tilde{k}^{\frac{r}{p-q}}(t) d\mu(t) < \infty, \quad j = 1, \dots, n, \end{aligned}$$

and for $(p, q, r) \in P_2$

$$\begin{aligned} \tilde{I}_1 &= \int_T |w_0(t)|^p \left(\frac{(|w(t)|^p - |w_0(t)|^p)_+}{\sum_{k=1}^n |w_k(t)|^r} \right)^{\frac{p}{r-p}} d\mu(t) < \infty, \\ \tilde{I}_{j+1} &= \int_T |w_j(t)|^r \left(\frac{(|w(t)|^p - |w_0(t)|^p)_+}{\sum_{k=1}^n |w_k(t)|^r} \right)^{\frac{r}{r-p}} d\mu(t) < \infty, \quad j = 1, \dots, n. \end{aligned}$$

Moreover, assume that $\tilde{I}_2 = \dots = \tilde{I}_{n+1}$. Then for all $x(\cdot) \neq 0$ such that $w_0(\cdot)x(\cdot) \in L_p(T, \mu)$ and $w_j(\cdot)x(\cdot) \in L_r(T, \mu)$, $j = 1, \dots, n$, the sharp inequality

$$(31) \quad \|w(\cdot)x(\cdot)\|_{L_q(T, \mu)} \leq \tilde{C} \|w_0(\cdot)x(\cdot)\|_{L_p(T, \mu)}^{\tilde{\gamma}} \left(\max_{1 \leq j \leq n} \|\omega_j(\cdot)x(\cdot)\|_{L_r(T, \mu)} \right)^{1-\tilde{\gamma}}$$

holds, where

$$\tilde{C} = \tilde{I}_1^{-\tilde{\gamma}/p} \tilde{I}_2^{-(1-\tilde{\gamma})/r} (\tilde{I}_1 + n\tilde{I}_2)^{1/q}.$$

Proof. Set

$$\psi(t) = \frac{w(t)}{w_0(t)}, \quad \varphi_j(t) = \frac{w_j(t)}{w_0(t)}, \quad j = 1, \dots, n.$$

Then $|\psi(\cdot)|$ is a homogenous function of degree $\eta = \theta - \theta_0$ and $|\varphi_j(\cdot)|$, $j = 1, \dots, n$, are homogenous functions of degrees $\nu = \theta_1 - \theta_0$. The quantity γ which was defined by (23) has the following form:

$$\tilde{\gamma} = \frac{\tilde{\theta}_1 - \tilde{\theta}}{\tilde{\theta}_1 - \tilde{\theta}_0}.$$

It follows by Corollary 1 that for all $y(\cdot) \neq 0$ such that $y(\cdot) \in L_p(T, \mu)$ and $\varphi_j(\cdot)y(\cdot) \in L_r(T, \mu)$, $j = 1, \dots, n$, the sharp inequality

$$\|\psi(\cdot)y(\cdot)\|_{L_q(T, \mu)} \leq \tilde{C} \|y(\cdot)\|_{L_p(T, \mu)}^{\tilde{\gamma}} \left(\max_{1 \leq j \leq n} \|\varphi_j(\cdot)y(\cdot)\|_{L_r(T, \mu)} \right)^{1-\tilde{\gamma}}.$$

holds. Substituting $y(\cdot) = w_0(\cdot)x(\cdot)$, we obtain (31). \square

5. HOMOGENOUS WEIGHTS IN \mathbb{R}^d

Let T be a cone in \mathbb{R}^d , $d\mu(t) = dt$, $|\psi(\cdot)|$ be homogenous function of degree η , $|\varphi_j(\cdot)|$, $j = 1, \dots, n$, be homogenous functions of degrees ν , $\psi(t) \neq 0$ and $\sum_{j=1}^n |\varphi_j(t)| \neq 0$ for almost all $t \in T$. Consider the polar transformation

$$\begin{aligned} t_1 &= \rho \cos \omega_1, \\ t_2 &= \rho \sin \omega_1 \cos \omega_2, \\ &\dots\dots\dots \\ t_{d-1} &= \rho \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \cos \omega_{d-1}, \\ t_d &= \rho \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \sin \omega_{d-1}. \end{aligned}$$

Set $\omega = (\omega_1, \dots, \omega_{d-1})$. For any function $f(\cdot)$ we put

$$(32) \quad \tilde{f}(\omega) = |f(\cos \omega_1, \dots, \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \sin \omega_{d-1})|.$$

Note that if $|f(\cdot)|$ is a homogenous function of degree κ , then $f(\omega) = \rho^{-\kappa} |f(t)|$. Denote by Ω the range of ω . Since T is a cone, Ω does not depend on ρ . Put

$$J(\omega) = \sin^{d-2} \omega_1 \sin^{d-3} \omega_2 \dots \sin \omega_{d-2}.$$

Assume that $\gamma \in (0, 1)$, where γ is defined by (23). Put

$$(33) \quad \frac{1}{q^*} = \frac{1}{q} - \frac{\gamma}{p} - \frac{1-\gamma}{r}.$$

It is easy to verify that $q^* > q \geq 1$. Moreover,

$$q^* = \frac{pqr(\nu + d(1/r - 1/p))}{\nu r(p - q) - \eta q(p - r)}.$$

Theorem 5. *Let $(p, q, r) \in P \cup P_1 \cup P_2$ and $\gamma \in (0, 1)$. Assume that*

$$I = \int_{\Omega} \frac{\tilde{\psi}^{q^*}(\omega)}{\tilde{s}_r^{q^*(1-\gamma)/r}(\omega)} J(\omega) d\omega < \infty,$$

and $I'_1 = \dots = I'_n$, where

$$I'_j = \int_{\Omega} \frac{\tilde{\psi}^{q^*}(\omega) \tilde{\varphi}_j^r(\omega)}{\tilde{s}_r^{q^*(1-\gamma)/r+1}(\omega)} J(\omega) d\omega, \quad j = 1, \dots, n.$$

Then

$$(34) \quad E(p, q, r) = K \delta^\gamma,$$

where

$$K = \gamma^{-\frac{\gamma}{p}} \left(\frac{1-\gamma}{n} \right)^{-\frac{1-\gamma}{r}} \left(\frac{B(q^*\gamma/p, q^*(1-\gamma)/r) I}{|\nu + d(1/r - 1/p)|(\gamma r + (1-\gamma)p)} \right)^{1/q^*},$$

where $B(\cdot, \cdot)$ is the Euler beta-function. Moreover, the method

$$\hat{m}(y)(t) = \kappa \left(\hat{\xi}^{\nu+d(1/r-1/p)} t \right) \psi(t)y(t),$$

where

$$\widehat{\xi} = \delta \gamma^{-1/p} \left(\frac{1-\gamma}{n} \right)^{1/r} \left(\frac{B(q^* \gamma/p, q^*(1-\gamma)/r) I}{|\nu + d(1/r - 1/p)|(\gamma r + (1-\gamma)p)} \right)^{1/r-1/p},$$

is optimal recovery method.

Proof. First of all, we note that $I'_1 + \dots + I'_n = I$. Consequently, $I'_j = I/n$, $j = 1, \dots, n$. We will apply Theorem 4.

1. Let $(p, q, r) \in P$. Passing to the polar transformation we obtain

$$\frac{k^{\frac{1}{p-q}}(\rho, \omega)}{(1 - k(\rho, \omega))^{\frac{1}{r-q}}} = \rho^{\frac{\eta q(p-r) - \nu r(p-q)}{(p-q)(r-q)}} \frac{\widetilde{\psi}^{\frac{q(p-r)}{(p-q)(r-q)}}(\omega)}{\widetilde{s}_r^{\frac{1}{r-q}}(\omega)}.$$

Using the same scheme of calculation of I_1 as it was given in [8, Theorem 3], we obtain

$$I_1 = \frac{\gamma}{pr|\nu + d(1/r - 1/p)|} \left(\frac{\gamma}{p} + \frac{1-\gamma}{r} \right)^{-1} B(\widehat{p}, \widehat{q})I,$$

where

$$\widehat{p} = q^* \frac{\gamma}{p}, \quad \widehat{q} = q^* \frac{1-\gamma}{r}.$$

In a similar way we calculate

$$I_{j+1} = \frac{1-\gamma}{pr|\nu + d(1/r - 1/p)|} \left(\frac{\gamma}{p} + \frac{1-\gamma}{r} \right)^{-1} B(\widehat{p}, \widehat{q})I'_j, \quad j = 1, \dots, n.$$

Thus,

$$I_2 = \frac{1-\gamma}{npr|\nu + d(1/r - 1/p)|} \left(\frac{\gamma}{p} + \frac{1-\gamma}{r} \right)^{-1} B(\widehat{p}, \widehat{q})I.$$

It remains to substitute these values into (22) and (24).

2. Let $(p, q, r) \in P_1$. Now we use the scheme of calculation of I_1 which was given in [10, Theorem 3]. We obtain

$$\begin{aligned} I_1 &= \frac{I}{|\nu - \eta|q} B(q^* \gamma/p + 2, q^*(1-\gamma)/q) \\ &= \frac{I}{|\nu - \eta|q} \frac{q^* \gamma/p + 1}{q^* \gamma/p + 1 + q^*(1-\gamma)/q} B(q^* \gamma/p + 1, q^*(1-\gamma)/q). \end{aligned}$$

Since $r = q$ we have

$$\frac{1}{q^*} = \gamma \left(\frac{1}{q} - \frac{1}{p} \right), \quad \gamma = \frac{\nu - \eta}{\nu + d(1/q - 1/p)}.$$

Therefore, $q^* \gamma/p + 1 = q^* \gamma/q$. Hence

$$\begin{aligned} I_1 &= \frac{I\gamma}{|\nu - \eta|q} B(q^* \gamma/p + 1, q^*(1-\gamma)/q) \\ &= \frac{I\gamma}{|\nu - \eta|q} \frac{q^* \gamma/p}{q^* \gamma/p + q^*(1-\gamma)/q} B(q^* \gamma/p, q^*(1-\gamma)/q) \\ &= \frac{\gamma}{pr|\nu + d(1/r - 1/p)|} \left(\frac{\gamma}{p} + \frac{1-\gamma}{r} \right)^{-1} B(\widehat{p}, \widehat{q})I. \end{aligned}$$

By the similar way we get

$$\begin{aligned}
I_{j+1} &= \frac{I'_j}{|\nu - \eta|q} B(q^*\gamma/p + 1, q^*(1 - \gamma)/q + 1) \\
&= \frac{I'_j}{|\nu - \eta|q} \frac{q^*\gamma/p}{q^*\gamma/p + q^*(1 - \gamma)/q + 1} B(q^*\gamma/p, q^*(1 - \gamma)/q + 1) \\
&= \frac{I'_j\gamma}{|\nu - \eta|p} B(q^*\gamma/p, q^*(1 - \gamma)/q + 1) = \frac{I'_j\gamma}{|\nu - \eta|p} \frac{q^*(1 - \gamma)/q}{q^*\gamma/p + q^*(1 - \gamma)/q} B(q^*\gamma/p, q^*(1 - \gamma)/q) \\
&= \frac{1 - \gamma}{npr|\nu + d(1/r - 1/p)|} \left(\frac{\gamma}{p} + \frac{1 - \gamma}{r} \right)^{-1} B(\widehat{p}, \widehat{q})I.
\end{aligned}$$

Thus, we obtain the same formulas for I_1 and I_2 as in the first case.

3. Let $(p, q, r) \in P_2$. Here we use the scheme of calculation of J_1 and J_2 which was given in [10, Theorem 3]. We obtain

$$\begin{aligned}
I_1 &= \frac{I}{|\eta|p} B(q^*\gamma/p + 1, q^*(1 - \gamma)/r + 1), \\
I_{j+1} &= \frac{I'_j}{|\eta|p} B(q^*\gamma/p, q^*(1 - \gamma)/r + 2), \quad j = 1, \dots, n.
\end{aligned}$$

Since $q = p$ we have

$$\frac{1}{q^*} = (1 - \gamma) \left(\frac{1}{p} - \frac{1}{r} \right), \quad 1 - \gamma = \frac{\eta}{\nu + d(1/r - 1/p)}.$$

Therefore, $q^*(1 - \gamma)/r + 1 = q^*(1 - \gamma)/p$. Hence

$$\begin{aligned}
I_1 &= \frac{I}{|\eta|p} \frac{q^*\gamma/p}{q^*\gamma/p + q^*(1 - \gamma)/r + 1} B(q^*\gamma/p, q^*(1 - \gamma)/r + 1) \\
&= \frac{I\gamma}{|\eta|p} B(q^*\gamma/p, q^*(1 - \gamma)/r + 1) = \frac{I\gamma}{|\eta|p} \frac{q^*(1 - \gamma)/r}{q^*\gamma/p + q^*(1 - \gamma)/r} B(q^*\gamma/p, q^*(1 - \gamma)/r) \\
&= \frac{\gamma}{pr|\nu + d(1/r - 1/p)|} \left(\frac{\gamma}{p} + \frac{1 - \gamma}{r} \right)^{-1} B(\widehat{p}, \widehat{q})I.
\end{aligned}$$

For I_{j+1} , $j = 1, \dots, n$, we have

$$\begin{aligned}
I_{j+1} &= \frac{I'_j}{|\eta|p} \frac{q^*(1 - \gamma)/r + 1}{q^*\gamma/p + q^*(1 - \gamma)/r + 1} B(q^*\gamma/p, q^*(1 - \gamma)/r + 1) \\
&= \frac{I'_j(1 - \gamma)}{|\eta|p} B(q^*\gamma/p, q^*(1 - \gamma)/r + 1) = \frac{I'_j(1 - \gamma)}{pr|\nu + d(1/r - 1/p)|} \left(\frac{\gamma}{p} + \frac{1 - \gamma}{r} \right)^{-1} B(\widehat{p}, \widehat{q}) \\
&= \frac{1 - \gamma}{npr|\nu + d(1/r - 1/p)|} \left(\frac{\gamma}{p} + \frac{1 - \gamma}{r} \right)^{-1} B(\widehat{p}, \widehat{q})I.
\end{aligned}$$

Again we obtain the same formulas for I_1 and I_2 as in the previous cases. \square

For $n = 1$ Theorem 5 was proved in [10]. Analogously to Corollary 1 we obtain

Corollary 3. *Assume that conditions of Theorem 5 hold. Then for all $x(\cdot)$ such that $x(\cdot) \in L_p(T, \mu)$ and $\varphi_j(\cdot)x(\cdot) \in L_r(T, \mu)$, $j = 1, \dots, n$, the sharp inequality*

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T, \mu)} \leq K \|x(\cdot)\|_{L_p(T, \mu)}^\gamma \left(\max_{1 \leq j \leq n} \|\varphi_j(\cdot)x(\cdot)\|_{L_r(T, \mu)} \right)^{1-\gamma}$$

For \tilde{I} we have

$$(37) \quad \tilde{I} = \int_{\Pi_+^{d-1}} \frac{J(\omega) d\omega}{\left(\sum_{k=1}^d \tilde{t}_k^{r\theta_1}(\omega)\right)^{\tilde{q}(1-\tilde{\gamma})/r}}, \quad \Pi_+^{d-1} = [0, \pi/2]^{d-1}.$$

If $r\theta_1 \leq 2$, then

$$(38) \quad \sum_{k=1}^d \tilde{t}_k^{r\theta_1}(\omega) \geq \sum_{k=1}^d \tilde{t}_k^2(\omega) = 1.$$

For $r\theta_1 > 2$ by Hölder's inequality

$$1 = \sum_{k=1}^d \tilde{t}_k^2(\omega) \leq \left(\sum_{k=1}^d \tilde{t}_k^{r\theta_1}(\omega)\right)^{\frac{2}{r\theta_1}} d^{1-\frac{2}{r\theta_1}}.$$

Thus,

$$(39) \quad \sum_{k=1}^d \tilde{t}_k^{r\theta_1}(\omega) \geq d^{1-\frac{r\theta_1}{2}}.$$

It follows by (38) and (39) that $\tilde{I} < \infty$.

For \tilde{I}'_j we have

$$\tilde{I}'_j = \int_{\Pi_+^{d-1}} \frac{\tilde{t}_j^{r\theta_1} J(\omega) d\omega}{\left(\sum_{k=1}^d \tilde{t}_k^{r\theta_1}(\omega)\right)^{\tilde{q}(1-\tilde{\gamma})/r+1}}, \quad j = 1, \dots, d.$$

Consider the integrals

$$L_j = \int_{\mathbb{R}_+^d \cap \mathbb{B}^d} \frac{\left(\sum_{k=1}^d t_k^2\right)^{\theta_1 \tilde{q}(1-\tilde{\gamma})/2} t_j^{r\theta_1}}{\left(\sum_{k=1}^d t_k^{r\theta_1}\right)^{\tilde{q}(1-\tilde{\gamma})/r+1}} dt, \quad j = 1, \dots, d,$$

where \mathbb{B}^d is the unit ball in \mathbb{R}^d . If we change variables in L_j changing places variables t_j and t_k , then L_j passes to L_k . Therefore, $L_1 = \dots = L_d$. Passing to the polar transformation we obtain that $L_j = \tilde{I}'_j/d$, $j = 1, \dots, d$. Consequently, $\tilde{I}'_1 = \dots = \tilde{I}'_d$.

Thus, we obtain

Corollary 5. *Let $(p, q, r) \in P \cup P_1 \cup P_2$, $\theta_1 > 0$, θ and θ_0 be such that $\theta_1 + d(1/r - 1/q) > \theta > \theta_0 + d(1/p - 1/q)$ or $\theta_1 + d(1/r - 1/q) < \theta < \theta_0 + d(1/p - 1/q)$. Then for weights (36) and all $x(\cdot)$ for which $w_0(\cdot)x(\cdot) \in L_p(\mathbb{R}_+^d)$ and $w_j(\cdot)x(\cdot) \in L_r(\mathbb{R}_+^d)$, $j = 1, \dots, d$, the sharp inequality*

$$\|w(\cdot)x(\cdot)\|_{L_q(\mathbb{R}_+^d)} \leq \tilde{K} \|w_0(\cdot)x(\cdot)\|_{L_p(\mathbb{R}_+^d)}^{\tilde{\gamma}} \left(\max_{1 \leq j \leq d} \|\omega_j(\cdot)x(\cdot)\|_{L_r(\mathbb{R}_+^d)} \right)^{1-\tilde{\gamma}}$$

holds, where \tilde{K} is defined by (35) in which the value \tilde{I} is defined by (37).

We give one more example.

Corollary 6. *Let $(p, q, r) \in P \cup P_1 \cup P_2$, weights $w(\cdot)$, $w_0(\cdot)$, $w_1(\cdot)$ be defined by (36) for $\theta = d(1 - 1/q)$, $\theta_0 = d - (\lambda + d)/p$, $\theta_1 = d + (\mu - d)/r$, where $\lambda, \mu > 0$. Put*

$$\alpha = \frac{\mu}{p\mu + r\lambda}, \quad \beta = \frac{\lambda}{p\mu + r\lambda}.$$

Then for all $x(\cdot)$ such that $w_0(\cdot)x(\cdot) \in L_p(\mathbb{R}_+^d)$ and $w_j(\cdot)x(\cdot) \in L_r(\mathbb{R}_+^d)$, $j = 1, \dots, d$, the sharp inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(\mathbb{R}_+^d)} \leq C \|w_0(\cdot)x(\cdot)\|_{L_p(\mathbb{R}_+^d)}^{p\alpha} \left(\max_{1 \leq j \leq d} \|\omega_j(\cdot)x(\cdot)\|_{L_r(\mathbb{R}_+^d)} \right)^{r\beta}$$

holds, where

$$C = \frac{d^\beta}{(p\alpha)^\alpha (r\beta)^\beta} \left(\frac{I}{\lambda + \mu} B \left(\frac{\alpha}{1/q - \alpha - \beta}, \frac{\beta}{1/q - \alpha - \beta} \right) \right)^{1/q - \alpha - \beta},$$

and

$$I = \int_{\Pi_+^{d-1}} \frac{J(\omega) d\omega}{\left(\sum_{k=1}^d t_k^{\tilde{r}(d-1)+\mu}(\omega) \right)^{\frac{\beta}{1/q - \alpha - \beta}}}.$$

For $d = 1$, $q = 1$, and $(p, 1, r) \in P$ the statement of Corollary 6 was proved in [4].

6. RECOVERY OF DIFFERENTIAL OPERATORS FROM A NOISY FOURIER TRANSFORM

Let T be a cone in \mathbb{R}^d , $d\mu(t) = dt$, $|\psi(\cdot)|$ be homogenous function of degree η , $|\varphi_j(\cdot)|$, $j = 1, \dots, n$, be homogenous functions of degrees ν , $\psi(t) \neq 0$ and $\sum_{j=1}^n |\varphi_j(t)| \neq 0$ for almost all $t \in T$.

Let S be the Schwartz space of rapidly decreasing C^∞ -functions on \mathbb{R}^d , S' be the corresponding space of distributions, and let $F: S' \rightarrow S'$ be the Fourier transform. Set

$$X_p = \{ x(\cdot) \in S' : \varphi_j(\cdot)Fx(\cdot) \in L_2(\mathbb{R}^d), j = 1, \dots, n, Fx(\cdot) \in L_p(\mathbb{R}^d) \}.$$

We define operators D_j , $j = 1, \dots, n$, as follows

$$D_j x(\cdot) = F^{-1}(\varphi_j(\cdot)Fx(\cdot))(\cdot), j = 1, \dots, n.$$

Put

$$(40) \quad \Lambda x(\cdot) = F^{-1}(\psi(\cdot)Fx(\cdot))(\cdot).$$

Consider the problem of the optimal recovery of values of the operator Λ on the class

$$W_p^{\mathcal{D}} = \{ x(\cdot) \in X_p : \|D_j x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq 1, j = 1, \dots, n \}, \quad \mathcal{D} = (D_1, \dots, D_n),$$

from the noisy Fourier transform of the function $x(\cdot)$. We assume that for each $x(\cdot) \in W_p^{\mathcal{D}}$ one knows a function $y(\cdot) \in L_p(\mathbb{R}^d)$ such that $\|Fx(\cdot) - y(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta$, $\delta > 0$. It is required to recover the function $\Lambda x(\cdot)$ from $y(\cdot)$. Assume that $\Lambda x(\cdot) \in L_q(\mathbb{R}^d)$ for all $x(\cdot) \in X_p$. As recovery methods we consider all possible mappings $m: L_p(\mathbb{R}^d) \rightarrow L_q(\mathbb{R}^d)$. The error of a method m is defined by

$$e_{pq}(\Lambda, \mathcal{D}, m) = \sup_{\substack{x(\cdot) \in W_p^{\mathcal{D}}, y(\cdot) \in L_p(\mathbb{R}^d) \\ \|Fx(\cdot) - y(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta}} \|\Lambda x(\cdot) - m(y)(\cdot)\|_{L_q(\mathbb{R}^d)}.$$

The quantity

$$(41) \quad E_{pq}(\Lambda, \mathcal{D}) = \inf_{m: L_p(\mathbb{R}^d) \rightarrow L_q(\mathbb{R}^d)} e_{pq}(\Lambda, \mathcal{D}, m)$$

is called the error of optimal recovery, and the method on which the infimum is attained, an optimal method.

1. **Recovery in the metric $L_2(\mathbb{R}^d)$.** By Plancherel's theorem,

$$\|\Lambda x(\cdot) - m(y)(\cdot)\|_{L_2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{d/2}} \|\tilde{L}x(\cdot) - F(m(y))(\cdot)\|_{L_2(\mathbb{R}^d)},$$

where $\tilde{L}x(\cdot) = \psi(\cdot)Fx(\cdot)$. Moreover,

$$\|D_j x(\cdot)\|_{L_2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{d/2}} \|\varphi_j(\cdot)Fx(\cdot)\|_{L_2(\mathbb{R}^d)}, \quad j = 1, \dots, n.$$

So, the problem under consideration coincides, up to a factor of $(2\pi)^{-d/2}$, with problem (2) for $q = r = 2$ with $\varphi_j(\cdot)$ replaced by $(2\pi)^{-d/2}\varphi_j(\cdot)$, $j = 1, \dots, n$.

For $q = r = 2$ we denote by $\hat{\gamma}$ and \hat{q}^* the values γ and q^* , which were defined by (23) and (33):

$$\hat{\gamma} = \frac{\nu - \eta}{\nu + d(1/2 - 1/p)}, \quad \hat{q}^* = \frac{1}{\hat{\gamma}(1/2 - 1/p)}.$$

Set

$$C_p(\nu, \eta) = \hat{\gamma}^{-\frac{\hat{\gamma}}{p}} \left(\frac{1 - \hat{\gamma}}{n} \right)^{-\frac{1-\hat{\gamma}}{2}} \left(\frac{B(\hat{q}^*\hat{\gamma}/p + 1, \hat{q}^*(1 - \hat{\gamma})/2)}{2|\nu - \eta|} \right)^{1/\hat{q}^*}.$$

Theorem 6. Let $2 < p \leq \infty$, $\hat{\gamma} \in (0, 1)$. Assume that

$$(42) \quad I = \int_{\Pi^{d-1}} \frac{\tilde{\psi}^{\hat{q}^*}(\omega)}{\tilde{s}_2^{\hat{q}^*(1-\hat{\gamma})/2}(\omega)} J(\omega) d\omega < \infty, \quad \Pi^{d-1} = [0, \pi]^{d-2} \times [0, 2\pi]$$

and $I'_1 = \dots = I'_n$, where

$$(43) \quad I'_j = \int_{\Pi^{d-1}} \frac{\tilde{\psi}^{\hat{q}^*}(\omega) \tilde{\varphi}_j^2(\omega)}{\tilde{s}_2^{\hat{q}^*(1-\hat{\gamma})/2+1}(\omega)} J(\omega) d\omega, \quad j = 1, \dots, n.$$

Then

$$(44) \quad E_{p2}(\Lambda, \mathcal{D}) = \frac{1}{(2\pi)^{d\hat{\gamma}/2}} C_p(\nu, \eta) I^{1/\hat{q}^*} \delta^{\hat{\gamma}}.$$

The method

$$(45) \quad \hat{m}(y)(t) = F^{-1} \left(\left(1 - \beta \frac{s_2(t)}{|\psi(t)|^2} \right)_+ \psi(t)y(t) \right),$$

where

$$\beta = \frac{1 - \hat{\gamma}}{n(2\pi)^{d\hat{\gamma}}} C_p^2(\nu, \eta) \left(\delta I^{1/2-1/p} \right)^{2\hat{\gamma}},$$

is optimal.

Moreover, the sharp inequality

$$(46) \quad \|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{d\hat{\gamma}/2}} C_p(\nu, \eta) I^{1/\hat{q}^*} \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\hat{\gamma}} \left(\max_{1 \leq j \leq n} \|D_j x(\cdot)\|_{L_2(\mathbb{R}^d)} \right)^{1-\hat{\gamma}}$$

holds.

Proof. Let $2 < p < \infty$. By Theorem 5 we have

$$E_{p2}(\Lambda, \mathcal{D}) = \frac{1}{(2\pi)^{d\hat{\gamma}/2}} K \delta^{\hat{\gamma}},$$

where

$$K = \hat{\gamma}^{-\frac{\hat{\gamma}}{p}} \left(\frac{1 - \hat{\gamma}}{n} \right)^{-\frac{1-\hat{\gamma}}{2}} \left(\frac{B(\hat{q}^* \hat{\gamma}/p, \hat{q}^*(1-\hat{\gamma})/2) I}{|\nu + d(1/2 - 1/p)|(2\hat{\gamma} + (1-\hat{\gamma})p)} \right)^{1/\hat{q}^*}.$$

From the properties of the beta-function we find that

$$(47) \quad \begin{aligned} & \frac{B(\hat{q}^* \hat{\gamma}/p, \hat{q}^*(1-\hat{\gamma})/2)}{|\nu + d(1/2 - 1/p)|(2\hat{\gamma} + (1-\hat{\gamma})p)} \\ &= \frac{B(\hat{q}^* \hat{\gamma}/p + 1, \hat{q}^*(1-\hat{\gamma})/2) (\hat{q}^* \hat{\gamma}/p + \hat{q}^*(1-\hat{\gamma})/2)}{|\nu + d(1/2 - 1/p)|(2\hat{\gamma} + (1-\hat{\gamma})p) \hat{q}^* \hat{\gamma}/p} \\ &= \frac{B(\hat{q}^* \hat{\gamma}/p + 1, \hat{q}^*(1-\hat{\gamma})/2)}{2|\nu - \eta|}. \end{aligned}$$

Thus, equality (44) holds.

It follows by Theorem 5 that the method

$$\hat{m}(y)(t) = \left(1 - \frac{\hat{\xi}^{2\hat{\gamma}} c_2(t)}{(2\pi)^d |\psi(t)|^2} \right)_+ \psi(t) y(t),$$

where

$$\hat{\xi} = \delta(2\pi)^{d \frac{1-\hat{\gamma}}{2\hat{\gamma}}} \hat{\gamma}^{-1/p} \left(\frac{1-\hat{\gamma}}{n} \right)^{1/2} \left(\frac{B(\hat{q}^* \hat{\gamma}/p, \hat{q}^*(1-\hat{\gamma})/2) I}{|\nu + d(1/2 - 1/p)|(2\hat{\gamma} + (1-\hat{\gamma})p)} \right)^{1/2-1/p},$$

is optimal. In view of (47) we obtain

$$\begin{aligned} \frac{\hat{\xi}^{2\hat{\gamma}}}{(2\pi)^d} &= \frac{\delta^{2\hat{\gamma}} \hat{\gamma}^{-2\hat{\gamma}/p}}{(2\pi)^{d\hat{\gamma}}} \left(\frac{1-\hat{\gamma}}{n} \right)^{\hat{\gamma}} \left(\frac{B(\hat{q}^* \hat{\gamma}/p + 1, \hat{q}^*(1-\hat{\gamma})/2) I}{2|\nu - \eta|} \right)^{2\hat{\gamma}(1/2-1/p)} \\ &= \frac{1-\hat{\gamma}}{n(2\pi)^{d\hat{\gamma}}} C_p^2(\nu, \eta) \left(\delta I^{1/2-1/p} \right)^{2\hat{\gamma}}. \end{aligned}$$

Inequality (46) follows from Corollary 3. Consider the case $p = \infty$. It follows by Lemma 1 that

$$(48) \quad E_{\infty 2}(\Lambda, \mathcal{D}) \geq \sup_{\substack{x(\cdot) \in W_{\infty}^{\mathcal{D}} \\ \|Fx(\cdot)\|_{L_{\infty}(\mathbb{R}^d)} \leq \delta}} \|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)}.$$

Let $\hat{x}(\cdot)$ be such that

$$F\hat{x}(\xi) = \begin{cases} \delta, & |\psi(\xi)| > \lambda \sqrt{s_2(\xi)}, \\ 0, & |\psi(\xi)| \leq \lambda \sqrt{s_2(\xi)}. \end{cases}$$

We show that $\lambda > 0$ may be selected from the condition

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\varphi_j(\xi)|^2 |F\hat{x}(\xi)|^2 d\xi = 1, \quad j = 1, \dots, n.$$

Thus, $\lambda > 0$ should be chosen from the condition

$$\delta^2 \int_{|\psi(\xi)| > \lambda \sqrt{s_2(\xi)}} |\varphi_j(\xi)|^2 d\xi = (2\pi)^d.$$

Passing to the polar transformation for $\nu > \eta$ we obtain

$$\delta^2 \int_{\Pi_{d-1}} \tilde{\varphi}_j^2(\omega) J(\omega) d\omega \int_0^{\Phi_1(\omega)} \rho^{2\nu+d-1} d\rho = (2\pi)^d, \quad \Phi_1(\omega) = \left(\frac{\tilde{\psi}(\omega)}{\lambda \sqrt{\tilde{s}_2(\xi)}} \right)^{\frac{1}{\nu-\eta}}.$$

If $\nu < \eta$, then $2\nu + d < 0$ (since $\hat{\gamma} \in (0, 1)$) and we have

$$\delta^2 \int_{\Pi_{d-1}} \tilde{\varphi}_j^2(\omega) J(\omega) d\omega \int_{\Phi_1(\omega)}^{+\infty} \rho^{2\nu+d-1} d\rho = (2\pi)^d.$$

Hence

$$\frac{\delta^2}{|2\nu + d|} \lambda^{-\frac{2\nu+d}{\nu-\eta}} I_j' = (2\pi)^d.$$

As already noted, it follows from the equality $I_1' + \dots + I_n' = I$ that $I_j' = I/n$, $j = 1, \dots, n$. Consequently,

$$\lambda = \left(\frac{\delta^2 I}{(2\pi)^d n |2\nu + d|} \right)^{\frac{\nu-\eta}{2\nu+d}}.$$

It is easily checked that

$$C_\infty^2(\nu, \eta) = \frac{1}{|2\eta + d|} (n|2\nu + d|)^{\frac{\eta+d/2}{\nu+d/2}}.$$

As a result, $\lambda^2 = \beta$. In view of (48), using calculations similar to those that were above, we obtain

$$(49) \quad E_{\infty 2}^2(\Lambda, \mathcal{D}) \geq \|\Lambda \hat{x}(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{\delta^2}{(2\pi)^d} \int_{|\psi(\xi)| > \lambda \sqrt{s_2(\xi)}} |\psi(\xi)|^2 d\xi \\ = \frac{\delta^2}{|2\eta + d| (2\pi)^d} \lambda^{-\frac{2\eta+d}{\nu-\eta}} I = \frac{1}{(2\pi)^{d\hat{\gamma}}} C_\infty^2(\nu, \eta) I^{2/\hat{\gamma}} \delta^{2\hat{\gamma}}.$$

We estimate the error of the method (45). Put

$$a(\xi) = \left(1 - \beta \frac{s_2(\xi)}{|\psi(\xi)|^2} \right)_+.$$

Taking the Fourier transform we obtain

$$\|\Lambda x(\cdot) - \hat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\psi(\xi)|^2 |Fx(\xi) - a(\xi)y(\xi)|^2 d\xi.$$

We set $z(\cdot) = Fx(\cdot) - y(\cdot)$ and note that

$$\|z(\cdot)\|_{L_\infty(\mathbb{R}^d)} \leq \delta, \quad \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\varphi_j(\xi)|^2 |Fx(\xi)|^2 d\xi \leq 1, \quad j = 1, \dots, n.$$

Hence

$$\|\Lambda x(\cdot) - \hat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\psi(\xi)|^2 |(1 - a(\xi)) Fx(\xi) + a(\xi)z(\xi)|^2 d\xi.$$

The integrand can be written as

$$\left| \frac{|\psi(\xi)|(1 - a(\xi))\sqrt{\beta s_2(\xi)} Fx(\xi)}{\sqrt{\beta s_2(\xi)}} + \sqrt{a(\xi)}\sqrt{a(\xi)}|\psi(\xi)|z(\xi) \right|^2.$$

Using the Cauchy-Bunyakovskii-Schwarz inequality

$$|ab + cd|^2 \leq (|a|^2 + |c|^2)(|b|^2 + |d|^2)$$

we obtain the estimate

$$\|\Lambda x(\cdot) - \widehat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \leq \operatorname{vraisup}_{\xi \in \mathbb{R}^d} S(\xi) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (\beta s_2(\xi) |Fx(\xi)|^2 + a(\xi) |\psi(\xi)|^2 |z(\xi)|^2) d\xi,$$

where

$$S(\xi) = \frac{|\psi(\xi)|^2 (1 - a(\xi))^2}{\beta s_2(\xi)} + a(\xi).$$

If $|\psi(\xi)|^2 \leq \beta s_2(\xi)$, then $a(\xi) = 0$ and $S(\xi) \leq 1$. If $|\psi(\xi)|^2 > \beta s_2(\xi)$, then $S(\xi) = 1$. So we have

$$\begin{aligned} e_{\infty 2}^2(\Lambda, \mathcal{D}, \widehat{m}) &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (\beta s_2(\xi) |Fx(\xi)|^2 + a(\xi) |\psi(\xi)|^2 |z(\xi)|^2) d\xi \leq n\beta \\ &+ \frac{\delta^2}{(2\pi)^d} \int_{|\psi(\xi)| > \lambda \sqrt{s_2(\xi)}} (|\psi(\xi)|^2 - \beta s_2(\xi)) d\xi = n\beta + \frac{\delta^2}{(2\pi)^d} \int_{|\psi(\xi)| > \lambda \sqrt{s_2(\xi)}} |\psi(\xi)|^2 d\xi \\ &- \beta \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} s_2(\xi) |F\widehat{x}(\xi)|^2 d\xi = \frac{\delta^2}{(2\pi)^d} \int_{|\psi(\xi)| > \lambda \sqrt{s_2(\xi)}} |\psi(\xi)|^2 d\xi \leq E_{\infty 2}^2(\Lambda, \mathcal{D}). \end{aligned}$$

It follows that the method $\widehat{m}(y)(\cdot)$ is optimal. Moreover, by (49) we have

$$E_{\infty 2}^2(\Lambda, \mathcal{D}) = \frac{\delta^2}{(2\pi)^d} \int_{|\psi(\xi)| > \lambda \sqrt{s_2(\xi)}} |\psi(\xi)|^2 d\xi = \frac{1}{(2\pi)^{d\widehat{\gamma}}} C_{\infty}^2(n, k) I^{2/\widehat{q}^*} \delta^{2\widehat{\gamma}}.$$

Similar to the proof of Corollary 1 we prove that for $p = \infty$ inequality (46) is sharp. \square

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}_+^d$. We define the operator D^α (the derivative of order α) by

$$D^\alpha x(\cdot) = F^{-1}((i\xi)^\alpha Fx(\xi))(\cdot),$$

where $(i\xi)^\alpha = (i\xi_1)^{\alpha_1} \dots (i\xi_d)^{\alpha_d}$.

Consider problem (41) for $D_j = D^{\nu e_j}$, $j = 1, \dots, d$, where e_j , $j = 1, \dots, d$, is a standard basis in \mathbb{R}^d , and Λ defined by (40). Assume that $\psi(\cdot)$ has the following symmetry property

$$\psi(\dots, \xi_j, \dots, \xi_m, \dots) = \psi(\dots, \xi_m, \dots, \xi_j, \dots), \quad 1 \leq j, m \leq d.$$

Moreover, we assume that $\widetilde{\psi}(\cdot)$ is continuous function on Π^{d-1} .

In this case for (42) and (43) we have

$$(50) \quad \begin{aligned} I &= \int_{\Pi^{d-1}} \frac{\widetilde{\psi}^{\widehat{q}^*}(\omega) J(\omega) d\omega}{\left(\sum_{k=1}^d \widetilde{t}_k^{2\nu}(\omega)\right)^{\widehat{q}^*(1-\widehat{\gamma})/2}}, \\ I'_j &= \int_{\Pi^{d-1}} \frac{\widetilde{\psi}^{\widehat{q}^*}(\omega) \widetilde{t}_j^{2\nu}(\omega) J(\omega) d\omega}{\left(\sum_{k=1}^d \widetilde{t}_k^{2\nu}(\omega)\right)^{\widehat{q}^*(1-\widehat{\gamma})/2+1}}, \quad j = 1, \dots, d. \end{aligned}$$

Similar to how it was done for weights (36) we prove that $I < \infty$ and $I'_1 = \dots = I'_d$. Thus, from Theorem 6 we obtain

Corollary 7. *Let $2 < p \leq \infty$ and $\nu > \eta \geq 0$. Then*

$$E_{p2}(\Lambda, (D^{\nu e_1}, \dots, D^{\nu e_d})) = \frac{1}{(2\pi)^{d\widehat{\gamma}/2}} C_p(\nu, \eta) I^{1/\widehat{q}^*} \delta^{\widehat{\gamma}},$$

where I is defined by (50). The method

$$\widehat{m}(y)(t) = F^{-1} \left(\left(1 - \beta \frac{\sum_{j=1}^d |t_j|^{2\nu}}{|\psi(t)|^2} \right)_+ \psi(t) y(t) \right),$$

where

$$\beta = \frac{1 - \widehat{\gamma}}{d(2\pi)^{d\widehat{\gamma}}} C_p^2(\nu, \eta) \left(\delta I^{1/2-1/p} \right)^{2\widehat{\gamma}},$$

is optimal.

The sharp inequality

$$\|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{d\widehat{\gamma}/2}} C_p(\nu, \eta) I^{1/\widehat{q}^*} \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\widehat{\gamma}} \left(\max_{1 \leq j \leq d} \|D^{\nu e_j} x(\cdot)\|_{L_2(\mathbb{R}^d)} \right)^{1-\widehat{\gamma}}$$

holds.

As functions $\psi(\cdot)$ defining the operator Λ we can consider the functions

$$\psi_\theta(\xi) = (|\xi_1|^\theta + \dots + |\xi_d|^\theta)^{2/\theta}, \quad \theta > 0.$$

The corresponding operator is denoted by Λ_θ . In particular, $\Lambda_2 = -\Delta$, where Δ is the Laplace operator. We denote by $\Lambda_\theta^{\eta/2}$ the operator Λ which is defined by $\psi(\cdot) = \psi_\theta^{\eta/2}(\cdot)$.

Now we consider the case when $p = 2$.

Theorem 7. *Let $\nu > \eta > 0$, $\nu \geq 1$, and $0 < \theta \leq 2\nu$. Then*

$$(51) \quad E_{22}(\Lambda_\theta^{\eta/2}, (D^{\nu e_1}, \dots, D^{\nu e_d})) = d^{\eta/\theta} \left(\frac{\delta}{(2\pi)^{d/2}} \right)^{1-\eta/\nu},$$

and all methods

$$(52) \quad \widehat{m}(y)(t) = F^{-1} \left(a(t) \psi_\theta^{\eta/2}(t) y(t) \right),$$

where $a(\cdot)$ are measurable functions satisfying the condition

$$(53) \quad \psi_\theta^\eta(\xi) \left(\frac{|1 - a(\xi)|^2}{\lambda_2 \sum_{j=1}^d |\xi_j|^{2\nu}} + \frac{|a(\xi)|^2}{(2\pi)^d \lambda_1} \right) \leq 1,$$

in which

$$\lambda_1 = \frac{d^{2\eta/\theta}}{(2\pi)^d} \left(1 - \frac{\eta}{\nu}\right) \left(\frac{(2\pi)^d}{\delta^2}\right)^{\eta/\nu}, \quad \lambda_2 = \frac{\eta}{\nu} d^{2\eta/\theta-1} \left(\frac{(2\pi)^d}{\delta^2}\right)^{\eta/\nu-1},$$

are optimal.

The sharp inequality

$$(54) \quad \|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \frac{d^{\eta/\theta}}{(2\pi)^{d(1-\eta/\nu)/2}} \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{\eta/\nu} \left(\max_{1 \leq j \leq d} \|D^{\nu e_j} x(\cdot)\|_{L_2(\mathbb{R}^d)} \right)^{1-\eta/\nu}$$

holds.

Proof. It follows by Lemma 1 that

$$(55) \quad E_{22}(\Lambda_\theta^{\eta/2}, (D^{\nu e_1}, \dots, D^{\nu e_d})) \geq \sup_{\substack{x(\cdot) \in W_2^{(D^{\nu e_1}, \dots, D^{\nu e_d})} \\ \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \delta}} \|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}.$$

Given $0 < \varepsilon < (2\pi)^{d/(2\nu)} \delta^{-1/\nu}$, we set

$$\widehat{\xi}_\varepsilon = \left(\frac{(2\pi)^d}{\delta^2} \right)^{\frac{1}{2\nu}} (1, \dots, 1) - (\varepsilon, \dots, \varepsilon), \quad B_\varepsilon = \{\xi \in \mathbb{R}^d : |\xi - \widehat{\xi}_\varepsilon| < \varepsilon\}.$$

Consider a function $x_\varepsilon(\cdot)$ such that

$$(56) \quad Fx_\varepsilon(\xi) = \begin{cases} \frac{\delta}{\sqrt{\text{mes } B_\varepsilon}}, & \xi \in B_\varepsilon, \\ 0, & \xi \notin B_\varepsilon. \end{cases}$$

Then $\|Fx_\varepsilon(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \delta^2$ and

$$\|D^{\nu e_j} x_\varepsilon(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{\delta^2}{(2\pi)^d \text{mes } B_\varepsilon} \int_{B_\varepsilon} |\xi_j|^{2\nu} d\xi \leq 1, \quad j = 1, \dots, d.$$

By virtue of (55) we have

$$\begin{aligned} E_{22}^2(\Lambda_\theta^{\eta/2}, (D^{\nu e_1}, \dots, D^{\nu e_d})) &\geq \|\Lambda_\theta^{\eta/2} x_\varepsilon(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \\ &= \frac{\delta^2}{(2\pi)^d \text{mes } B_\varepsilon} \int_{B_\varepsilon} \psi_\theta^\eta(\xi) d\xi = \frac{\delta^2}{(2\pi)^d} \psi_\theta^\eta(\tilde{\xi}_\varepsilon), \quad \tilde{\xi}_\varepsilon \in B_\varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we obtain the estimate

$$(57) \quad E_{22}^2(\Lambda_\theta^{\eta/2}, (D^{\nu e_1}, \dots, D^{\nu e_d})) \geq d^{2\eta/\theta} \left(\frac{\delta^2}{(2\pi)^d} \right)^{1-\eta/\nu}.$$

We will find optimal methods among methods (52). Passing to the Fourier transform we have

$$\|\Lambda_\theta^{\eta/2} x(\cdot) - \widehat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi_\theta^\eta(\xi) |Fx(\xi) - a(\xi)y(\xi)|^2 d\xi.$$

We set $z(\cdot) = Fx(\cdot) - y(\cdot)$ and note that

$$\int_{\mathbb{R}^d} |z(\xi)|^2 d\xi \leq \delta^2, \quad \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi_j|^{2\nu} |Fx(\xi)|^2 d\xi \leq 1, \quad j = 1, \dots, d.$$

Then

$$\|\Lambda_\theta^{\eta/2} x(\cdot) - \widehat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi_\theta^\eta(\xi) |(1-a(\xi))Fx(\xi) + a(\xi)z(\xi)|^2 d\xi.$$

We write the integrand as

$$\psi_\theta^\eta(\xi) \left| \frac{(1-a(\xi))\sqrt{\lambda_2} \left(\sum_{j=1}^d |\xi_j|^{2\nu} \right)^{1/2} Fx(\xi)}{\sqrt{\lambda_2} \left(\sum_{j=1}^d |\xi_j|^{2\nu} \right)^{1/2}} + \frac{a(\xi)}{(2\pi)^{d/2} \sqrt{\lambda_1}} (2\pi)^{d/2} \sqrt{\lambda_1} z(\xi) \right|^2.$$

Applying the Cauchy-Bunyakovskii-Schwarz inequality we obtain the estimate

$$\begin{aligned} &\|\Lambda_\theta^{\eta/2} x(\cdot) - \widehat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \\ &\leq \text{vraisup}_{\xi \in \mathbb{R}^d} S(\xi) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\lambda_2 \sum_{j=1}^d |\xi_j|^{2\nu} |Fx(\xi)|^2 + (2\pi)^d \lambda_1 |z(\xi)|^2 \right) d\xi, \end{aligned}$$

where

$$S(\xi) = \psi_\theta^\eta(\xi) \left(\frac{|1-a(\xi)|^2}{\lambda_2 \sum_{j=1}^d |\xi_j|^{2\nu}} + \frac{|a(\xi)|^2}{(2\pi)^d \lambda_1} \right).$$

If we assume that $S(\xi) \leq 1$ for almost all ξ , then taking into account (57), we get

$$\begin{aligned} e_{22}^2(\Lambda_\theta^{\eta/2}, (D^{\nu e_1}, \dots, D^{\nu e_d}), \widehat{m}) &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\lambda_2 \sum_{j=1}^d |\xi_j|^{2\nu} |Fx(\xi)|^2 + (2\pi)^d \lambda_1 |z(\xi)|^2 \right) d\xi \\ &\leq \lambda_2 d + \lambda_1 \delta^2 = d^{2\eta/\theta} \left(\frac{\delta^2}{(2\pi)^d} \right)^{1-\eta/\nu} \leq E_{22}^2(\Lambda_\theta^{\eta/2}, (D^{\nu e_1}, \dots, D^{\nu e_d})). \end{aligned}$$

This proves (51) and shows that the methods under consideration are optimal.

It remains to verify that the set of functions $a(\cdot)$ satisfying (53) is nonempty. Put

$$a(\xi) = \frac{(2\pi)^d \lambda_1}{(2\pi)^d \lambda_1 + \lambda_2 \sum_{j=1}^d |\xi_j|^{2\nu}}.$$

Then

$$S(\xi) = \frac{\psi_\theta^\eta(\xi)}{(2\pi)^d \lambda_1 + \lambda_2 \sum_{j=1}^d |\xi_j|^{2\nu}}.$$

Since $\theta \leq 2\nu$ by Hölder's inequality

$$\sum_{j=1}^d |\xi_j|^\theta \leq \left(\sum_{j=1}^d |\xi_j|^{2\nu} \right)^{\theta/(2\nu)} d^{1-\theta/(2\nu)}.$$

Putting $\rho = (|\xi_1|^\theta + \dots + |\xi_d|^\theta)^{1/\theta}$, we obtain

$$\sum_{j=1}^d |\xi_j|^{2\nu} \geq \rho^{2\nu} d^{1-2\nu/\theta}.$$

Thus,

$$S(\xi) \leq \frac{\rho^{2\eta}}{(2\pi)^d \lambda_1 + \lambda_2 \rho^{2\nu} d^{1-2\nu/\theta}}.$$

It is easily checked that the function $f(\rho) = (2\pi)^d \lambda_1 + \lambda_2 \rho^{2\nu} d^{1-2\nu/\theta} - \rho^{2\eta}$ reaches a minimum on $[0, +\infty)$ at

$$\rho_0 = d^{1/\theta} \left(\frac{(2\pi)^d}{\delta^2} \right)^{1/(2\nu)}.$$

Moreover, $f(\rho_0) = 0$. Consequently, $f(\rho) \geq 0$ for all $\rho \geq 0$. Hence $S(\xi) \leq 1$ for all ξ .

Inequality (54) is proved by the analogy with the proof of Corollary 1. \square

2. Recovery in the metric $L_\infty(\mathbb{R}^d)$. Put

$$\begin{aligned} \gamma_1 &= \frac{\nu - \eta - d/2}{\nu + d(1/2 - 1/p)}, \quad q_1 = \frac{1}{1/2 + \gamma_1(1/2 - 1/p)}, \\ \widetilde{C}_p(\nu, \eta) &= \gamma_1^{-\frac{21}{p}} \left(\frac{1 - \gamma_1}{n} \right)^{-\frac{1-\gamma_1}{2}} \left(\frac{B(q_1 \gamma_1/p + 1, q_1(1 - \gamma_1)/2)}{2|\nu - \eta - d/2|} \right)^{1/q_1}. \end{aligned}$$

For $1 < p < \infty$ we define $k(\cdot)$ by the equality

$$\frac{k(t)}{(1 - k(t))^{p-1}} = (2\pi)^d \frac{|\psi(t)|^{p-2}}{s_2^{p-1}(t)}.$$

We set

$$k(t) = \begin{cases} \min \{1, (2\pi)^d |\psi(t)|^{-1}\}, & p = 1, \\ (1 - s_2(t) |\psi(t)|^{-1})_+, & p = \infty. \end{cases}$$

Theorem 8. Let $1 \leq p \leq \infty$, $\gamma_1 \in (0, 1)$. Assume that

$$I = \int_{\Pi^{d-1}} \frac{\tilde{\psi}^{q_1}(\omega)}{\tilde{s}_2^{q_1(1-\gamma_1)/2}(\omega)} J(\omega) d\omega < \infty$$

and $I'_1 = \dots = I'_n$, where

$$I'_j = \int_{\Pi^{d-1}} \frac{\tilde{\psi}^{q_1}(\omega) \tilde{\varphi}_j^2(\omega)}{\tilde{s}_2^{q_1(1-\gamma_1)/2+1}(\omega)} J(\omega) d\omega, \quad j = 1, \dots, n.$$

Then

$$E_{p\infty}(\Lambda, \mathcal{D}) = \frac{1}{(2\pi)^{d(1+\gamma_1)/2}} \tilde{C}_p(\nu, \eta) I^{1/q_1} \delta^{\gamma_1}.$$

The method

$$\hat{m}(y)(t) = F^{-1} \left(k \left(\xi_1^{\frac{n+d(1/2-1/p)}{2}} t \right) \psi(t) y(t) \right),$$

where

$$\xi_1 = \delta \gamma_1^{-\frac{q_1}{2p}} \left(\frac{(1-\gamma_1) \tilde{C}_p(\nu, \eta) I^{1/q_1}}{n(2\pi)^{d(1+\gamma_1)/2}} \right)^{q_1(1/2-1/p)},$$

is optimal.

The sharp inequality

(58)

$$\|\Lambda x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{d(1+\gamma_1)/2}} \tilde{C}_p(\nu, \eta) I^{1/q_1} \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\gamma_1} \left(\max_{1 \leq j \leq n} \|D_j x(\cdot)\|_{L_2(\mathbb{R}^d)} \right)^{1-\gamma_1}$$

holds.

Proof. Using an estimate similar to (48) we have

$$E_{p\infty}(\Lambda, \mathcal{D}) \geq \sup_{\substack{x(\cdot) \in W_p^{\mathcal{D}} \\ \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta}} \|\Lambda x(\cdot)\|_{L_\infty(\mathbb{R}^d)}.$$

Assume that $x(\cdot) \in W_p^{\mathcal{D}}$ and $\|Fx(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta$. If $\hat{x}(\cdot)$ is such that $F\hat{x}(\xi) = \varepsilon(\xi) e^{-i\langle t, \xi \rangle} Fx(\xi)$, where

$$\varepsilon(\xi) = \begin{cases} \frac{\overline{\psi(\xi) Fx(\xi)}}{|\psi(\xi) Fx(\xi)|}, & \psi(\xi) Fx(\xi) \neq 0, \\ 0, & \psi(\xi) Fx(\xi) = 0, \end{cases}$$

then we obtain $\hat{x}(\cdot) \in W_p^{\mathcal{D}}$, $\|F\hat{x}(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta$ and

$$\left| \int_{\mathbb{R}^d} \psi(\xi) F\hat{x}(\xi) e^{i\langle t, \xi \rangle} d\xi \right| = \int_{\mathbb{R}^d} |\psi(\xi) Fx(\xi)| d\xi.$$

Hence

$$(59) \quad E_{p\infty}(\Lambda, \mathcal{D}) \geq \frac{1}{(2\pi)^d} \sup_{\substack{x(\cdot) \in W_p^{\mathcal{D}} \\ \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta}} \int_{\mathbb{R}^d} |\psi(\xi) Fx(\xi)| d\xi.$$

Let $1 \leq p < \infty$. It follows from (20) that

$$E_{p\infty}(\Lambda, \mathcal{D}) \geq E(p, 1, 2),$$

where, in the problem of the evaluation of $E(p, 1, 2)$, the functions $\varphi_j(\cdot)$ should be replaced by the function $(2\pi)^{-d/2}\varphi_j(\cdot)$, and the function $\psi(\cdot)$ by $(2\pi)^{-d}\psi(\cdot)$. From Theorem 5 we obtain

$$E_{p\infty}(\Lambda, \mathcal{D}) \geq \frac{1}{(2\pi)^{d(1+\gamma_1)/2}} K \delta^{\gamma_1},$$

where

$$K = \gamma_1^{-\frac{\gamma_1}{p}} \left(\frac{1-\gamma_1}{n} \right)^{-\frac{1-\gamma_1}{2}} \left(\frac{B(q_1\gamma_1/p, q_1(1-\gamma_1)/2) I}{|\nu + d(1/2 - 1/p)|(2\gamma_1 + (1-\gamma_1)p)} \right)^{1/q_1}.$$

From the properties of the beta-function

$$\begin{aligned} & \frac{B(q_1\gamma_1/p, q_1(1-\gamma_1)/2)}{|\nu + d(1/2 - 1/p)|(2\gamma_1 + (1-\gamma_1)p)} \\ &= \frac{B(q_1\gamma_1/p + 1, q_1(1-\gamma_1)/2) (q_1\gamma_1/p + q_1(1-\gamma_1)/2)}{|\nu + d(1/2 - 1/p)|(2\gamma_1 + (1-\gamma_1)p) q_1\gamma_1/p} \\ &= \frac{B(q_1\gamma_1/p + 1, q_1(1-\gamma_1)/2)}{2|\nu - \eta - d/2|}. \end{aligned}$$

Thus,

$$E_{p\infty}(\Lambda, \mathcal{D}) \geq \frac{1}{(2\pi)^{d(1+\gamma_1)/2}} \tilde{C}_p(\nu, \eta) I^{1/q_1} \delta^{\gamma_1}.$$

Moreover, it follows from the same Theorem 5 that

$$\int_{\mathbb{R}^d} \left| \frac{1}{(2\pi)^d} \psi(\xi) F(\xi) - m(y)(\xi) \right| d\xi \leq E(p, 1, 2),$$

where

$$m(y)(t) = \frac{1}{(2\pi)^d} k \left(\xi_1^{\nu+d(1/2-1/p)} t \right) \psi(t) y(t),$$

and

$$\begin{aligned} \xi_1 &= \frac{\delta}{\gamma_1^{1/p}} \left(\frac{1-\gamma_1}{n} \right)^{1/2} \left(\frac{B(q_1\gamma_1/p, q_1(1-\gamma_1)/2) I (2\pi)^{-dq_1(1+\gamma_1)/2}}{|\nu + d(1/2 - 1/p)|(2\gamma_1 + (1-\gamma_1)p)} \right)^{1/2-1/p} \\ &= \delta \gamma_1^{-\frac{q_1}{2p}} \left(\frac{(1-\gamma_1) \tilde{C}_p(\nu, \eta) I^{1/q_1}}{n(2\pi)^{d(1+\gamma_1)/2}} \right)^{q_1(1/2-1/p)}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi(\xi) F(\xi) e^{i\langle t, \xi \rangle} d\xi - \int_{\mathbb{R}^d} m(y)(\xi) e^{i\langle t, \xi \rangle} d\xi \right| \\ & \leq \int_{\mathbb{R}^d} \left| \frac{1}{(2\pi)^d} \psi(\xi) F(\xi) - m(y)(\xi) \right| d\xi \leq E(p, 1, 2) \leq E_{p\infty}(\Lambda, \mathcal{D}). \end{aligned}$$

It follows that the method $\hat{m}(y)(\cdot)$ is optimal, and the error of optimal recovery coincides with $E(p, 1, 2)$.

Now we consider the case when $p = \infty$. Put

$$s(\xi) = \begin{cases} \frac{\psi(\xi)}{|\psi(\xi)|}, & \psi(\xi) \neq 0, \\ 1, & \psi(\xi) = 0. \end{cases}$$

Let $\widehat{x}(\cdot)$ be such that

$$F\widehat{x}(\xi) = \begin{cases} \overline{\delta s(\xi)}, & |\psi(\xi)| \geq \lambda s_2(\xi), \\ \frac{\delta \overline{\psi(\xi)}}{\lambda s_2(\xi)}, & |\psi(\xi)| < \lambda s_2(\xi). \end{cases}$$

We choose $\lambda > 0$ such that

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\varphi_j(\xi)|^2 |F\widehat{x}(\xi)|^2 d\xi = 1, \quad j = 1, \dots, n.$$

Now, to find λ we have the equation

$$\frac{\delta^2}{(2\pi)^d} \int_{|\psi(\xi)| \geq \lambda s_2(\xi)} |\varphi_j(\xi)|^2 d\xi + \frac{\delta^2 \lambda^{-2}}{(2\pi)^d} \int_{|\psi(\xi)| < \lambda s_2(\xi)} \frac{|\varphi_j(\xi)|^2 |\psi(\xi)|^2}{s_2^2(\xi)} d\xi = 1.$$

If $\nu > \eta + d/2$, then from the fact that $\gamma_1 \in (0, 1)$ it follows that $\eta > -d$. In this case it is easy to check that $2\nu > \eta$ and $2\nu + d > 0$. Passing to the polar transformation we obtain

$$\begin{aligned} \frac{\delta^2}{(2\pi)^d} \int_{\Pi_{d-1}} \widetilde{\varphi}_j^2(\omega) J(\omega) d\omega \int_0^{\Phi_2(\omega)} \rho^{2\nu+d-1} d\rho \\ + \frac{\delta^2 \lambda^{-2}}{(2\pi)^d} \int_{\Pi_{d-1}} \frac{\widetilde{\varphi}_j^2(\omega) \widetilde{\psi}^2(\omega)}{\widetilde{s}_2^2(\omega)} J(\omega) d\omega \int_{\Phi_2(\omega)}^{+\infty} \rho^{-2\nu+2\eta+d-1} d\rho = 1, \end{aligned}$$

where

$$\Phi_2(\omega) = \left(\frac{\widetilde{\psi}(\omega)}{\lambda \widetilde{s}_2(\omega)} \right)^{\frac{1}{2\nu-\eta}}.$$

Thus,

$$\frac{\delta^2}{(2\pi)^d} \lambda^{-\frac{2\nu+d}{2\nu-\eta}} \frac{4\nu - 2\eta}{(2\nu + d)(2\nu - 2\eta - d)} I_j = 1.$$

If $\nu < \eta + d/2$, then it follows from $\gamma_1 \in (0, 1)$ that $\eta < -d$, $2\nu < \eta$, and $2\nu + d < 0$. Passing to the polar transformation we obtain

$$\begin{aligned} \frac{\delta^2}{(2\pi)^d} \int_{\Pi_{d-1}} \widetilde{\varphi}_j^2(\omega) J(\omega) d\omega \int_{\Phi_2(\omega)}^{+\infty} \rho^{2\nu+d-1} d\rho \\ + \frac{\delta^2 \lambda^{-2}}{(2\pi)^d} \int_{\Pi_{d-1}} \frac{\widetilde{\varphi}_j^2(\omega) \widetilde{\psi}^2(\omega)}{\widetilde{s}_2^2(\omega)} J(\omega) d\omega \int_0^{\Phi_2(\omega)} \rho^{-2\nu+2\eta+d-1} d\rho = 1, \end{aligned}$$

For this case we have

$$\frac{\delta^2}{(2\pi)^d} \lambda^{-\frac{2\nu+d}{2\nu-\eta}} \frac{2\eta - 4\nu}{(2\nu + d)(2\nu - 2\eta - d)} I_j = 1.$$

Combining both of these cases and taking into account that $I_j = I/n$, $j = 1, \dots, n$, we get

$$\lambda = \left(\frac{2\delta^2 |2\nu - \eta| I}{(2\pi)^d n (2\nu + d)(2\nu - 2\eta - d)} \right)^{\frac{2\nu-\eta}{2\nu+d}}.$$

It follows by (59) that

$$\begin{aligned} E_{\infty\infty}(\Lambda, \mathcal{D}) \geq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\psi(\xi) F\widehat{x}(\xi)| d\xi = \frac{\delta}{(2\pi)^d} \int_{|\psi(\xi)| \geq \lambda s_2(\xi)} |\psi(\xi)| d\xi \\ + \frac{\delta}{\lambda (2\pi)^d} \int_{|\psi(\xi)| < \lambda s_2(\xi)} \frac{|\psi(\xi)|^2}{s_2(\xi)} d\xi. \end{aligned}$$

Using calculations similar to those that were above, we obtain

$$E_{\infty\infty}(\Lambda, \mathcal{D}) \geq \frac{\delta|2\nu - \eta|\lambda^{-\frac{\eta+d}{2\nu-\eta}}I}{(2\pi)^d(\eta+d)(2\nu-2\eta-d)} = E_0,$$

where

$$E_0 = \frac{(n|\nu + d/2|)^{\frac{\eta+d}{2\nu+d}}}{\eta+d} \left(\frac{(2\nu-\eta)I}{(2\pi)^d(2\nu-2\eta-d)} \right)^{\frac{2\nu-\eta}{2\nu+d}} \delta^{\frac{2\nu-2\eta-d}{2\nu+d}}.$$

We prove that for all $x(\cdot) \in X_\infty$ the equality

$$(60) \quad \Lambda x(t) = \frac{1}{(2\pi)^d} \int_{|\psi(\xi)| \geq \lambda s_2(\xi)} (\psi(\xi) - \lambda s(\xi) s_2(\xi)) Fx(\xi) e^{i\langle t, \xi \rangle} d\xi \\ + \frac{\lambda}{\delta(2\pi)^d} \int_{\mathbb{R}^d} s_2(\xi) Fx(\xi) \overline{F\widehat{x}(\xi)} e^{i\langle t, \xi \rangle} d\xi.$$

holds. Indeed,

$$\frac{1}{(2\pi)^d} \int_{|\psi(\xi)| \geq \lambda s_2(\xi)} (\psi(\xi) - \lambda s(\xi) s_2(\xi)) Fx(\xi) e^{i\langle t, \xi \rangle} d\xi \\ + \frac{\lambda}{\delta(2\pi)^d} \int_{\mathbb{R}^d} s_2(\xi) Fx(\xi) \overline{F\widehat{x}(\xi)} e^{i\langle t, \xi \rangle} d\xi \\ = \frac{1}{(2\pi)^d} \int_{|\psi(\xi)| \geq \lambda s_2(\xi)} ((\psi(\xi) - \lambda s(\xi) s_2(\xi)) Fx(\xi) e^{i\langle t, \xi \rangle} d\xi \\ + \frac{1}{(2\pi)^d} \int_{|\psi(\xi)| \geq \lambda s_2(\xi)} \lambda s(\xi) s_2(\xi) Fx(\xi) e^{i\langle t, \xi \rangle} d\xi + \frac{1}{(2\pi)^d} \int_{|\psi(\xi)| < \lambda s_2(\xi)} \psi(\xi) Fx(\xi) e^{i\langle t, \xi \rangle} d\xi \\ = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi(\xi) Fx(\xi) e^{i\langle t, \xi \rangle} d\xi = \Lambda x(t).$$

We estimate the error of the method

$$m(y)(t) = \frac{1}{(2\pi)^d} \int_{|\psi(\xi)| \geq \lambda s_2(\xi)} (\psi(\xi) - \lambda s(\xi) s_2(\xi)) y(\xi) e^{i\langle t, \xi \rangle} d\xi.$$

We have

$$|\Lambda x(t) - m(y)(t)| \leq \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi(\xi) Fx(\xi) e^{i\langle t, \xi \rangle} d\xi \right. \\ \left. - \frac{1}{(2\pi)^d} \int_{|\psi(\xi)| \geq \lambda s_2(\xi)} (\psi(\xi) - \lambda s(\xi) s_2(\xi)) Fx(\xi) e^{i\langle t, \xi \rangle} d\xi \right| \\ + \frac{1}{(2\pi)^d} \int_{|\psi(\xi)| \geq \lambda s_2(\xi)} |\psi(\xi) - \lambda s(\xi) s_2(\xi)| |Fx(\xi) - y(\xi)| d\xi.$$

If $x(\cdot)$ such that

$$\|Fx(\cdot) - y(\cdot)\|_{L_\infty(\mathbb{R}^d)} \leq \delta, \quad \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\varphi_j(\xi)|^2 |Fx(\xi)|^2 d\xi \leq 1, \quad j = 1, \dots, n,$$

then, taking into account (60), we obtain

$$|\Lambda x(t) - m(y)(t)| \leq \frac{\lambda}{\delta(2\pi)^d} \int_{\mathbb{R}^d} s_2(\xi) |Fx(\xi)| |F\widehat{x}(\xi)| d\xi + \mu \leq \frac{n\lambda}{\delta} + \mu,$$

where

$$\mu = \frac{\delta}{(2\pi)^d} \int_{|\psi(\xi)| \geq \lambda s_2(\xi)} (|\psi(\xi)| - \lambda s_2(\xi)) d\xi.$$

Passing to the polar transformation we find

$$\begin{aligned} \frac{\delta}{(2\pi)^d} \int_{|\psi(\xi)| \geq \lambda s_2(\xi)} |\psi(\xi)| d\xi &= \frac{\delta \lambda^{-\frac{\eta+d}{2\nu-\eta}}}{(2\pi)^d |\eta+d|} I, \\ \frac{\delta \lambda}{(2\pi)^d} \int_{|\psi(\xi)| \geq \lambda s_2(\xi)} s_2(\xi) d\xi &= \frac{\delta \lambda^{-\frac{\eta+d}{2\nu-\eta}}}{(2\pi)^d |2\nu+d|} I. \end{aligned}$$

Hence

$$\mu = \frac{\delta \lambda^{-\frac{\eta+d}{2\nu-\eta}} |2\nu-\eta|}{(2\pi)^d (\eta+d)(2\nu+d)} I.$$

It is easily checked that $n\lambda/\delta + \mu = E_0$, and therefore

$$e_{\infty\infty}(\Lambda, \mathcal{D}, m) \leq E_0 \leq E_{\infty\infty}(\Lambda, \mathcal{D}).$$

It follows that $m(y)(\cdot)$ is an optimal method, and the error of optimal recovery is E_0 . It is easily checked that for $p = \infty$

$$\frac{1}{(2\pi)^{d(1+\gamma_1)/2}} \tilde{C}_\infty(\nu, \eta) I^{1/q_1} \delta^{\gamma_1} = E_0.$$

We evaluate ξ_1 for $p = \infty$. We have

$$(61) \quad \xi_1 = \delta \left(\frac{(1-\gamma_1) \tilde{C}_\infty(\nu, \eta) I^{1/q_1}}{n(2\pi)^{d(1+\gamma_1)/2}} \right)^{q_1/2} = \lambda^{\frac{\nu+d/2}{2\nu-\eta}}.$$

The method $m(y)(\cdot)$ can be written as

$$m(y)(t) = F^{-1} \left(\left(1 - \lambda \frac{s_2(\xi)}{|\psi(t)|} \right)_+ \psi(t)y(t) \right).$$

In view of (61) we have

$$m(y)(t) = F^{-1} \left(k \left(\xi_1^{\frac{1}{n+d/2}} t \right) \psi(t)y(t) \right) = \hat{m}(y)(t).$$

Inequality (58) is proved by the analogy with the proof of Corollary 1. \square

It is not difficult to formulate a corollary from Theorem 8 analogous to Corollary 7 for the same Λ and $\mathcal{D} = (D^{\nu e_1}, \dots, D^{\nu e_d})$.

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MOSCOW STATE UNIVERSITY, MOSCOW AND INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS, RUSSIAN ACADEMY OF SCIENCES, MOSCOW

E-mail address: `kosipenko@yahoo.com`