# ON SOME PROBLEMS OF OPTIMAL RECOVERY OF ANALYTIC AND HARMONIC FUNCTIONS FROM INACCURATE DATA

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Abstract. We consider some problems of optimal recovery of holomorphic and harmonic functions in the unit disc. We obtain extensions of Schwartz's Lemma and optimal formulas for numerical differentiation.

## Introduction

Let $X$ be a linear space, let $Y$ and $Z$ be normed spaces, and let $W \subset X$. Consider the problem of optimal recovery of the operator $L\colon W \to Z$ using the values of the information operator $I\colon W \to Y$ in the case where those values are inaccurate ones. More precisely, let us consider the extremal problem

$$E(L, I, \delta) = \inf_{S} \sup_{\substack{x \in W \\ \|Ix - y\| \le \delta}} \|Lx - Sy\|, \tag{1}$$

where $S\colon Y \to Z$ is some mapping (algorithm). $E(L, I, \delta)$ is called the intrinsic error of recovery. An algorithm $S_0$ is called an optimal one if

$$\sup_{\substack{x \in W \\ \|Ix - y\| \le \delta}} \|Lx - S_0 y\| = E(L, I, \delta).$$

If $S_0$ is an optimal algorithm, $x_0 \in W$, and

$$\sup_{\|Ix_0 - y\| \le \delta} \|Lx_0 - S_0 y\| = E(L, I, \delta),$$

then $x_0$ is called a worst element.

The investigations of the problem (1) were initiated in [1] for the case $\dim Y < \infty$. The case $\dim Y = \infty$ was worked out in [2] (see also [3]). In this paper we consider some problems of optimal recovery of analytic functions from the Hardy space $H_p$ and the Bergman space $A_p$, $1 \le p \le \infty$. We also consider the same problems for harmonic functions from similar classes $h_p$ and $a_p$. Some results related to $H_\infty$, can be found in [4–7].

In Section 1 we prove some general theorems on optimal recovery from inaccurate data, which closely relate to results obtained in [2, 3, 8, 9]. In Section 2 we apply these theorems to finding optimal recovery algorithms for functions from $H_p$, $A_p$, $h_p$, and $a_2$ in some point of the unit disc of $\mathbb{C}$, when the disposed data is the inaccurate value of these functions in some other point. In particular we obtain some generalizations of Schwartz's Lemma. In the last section we find optimal algorithms of recovery of $f'(0)$ from inaccurate data $f(-h)$ and $f(h)$, $h \in (0,1)$ in $H_p$ spaces. In $H_\infty$ we also find the optimal value of $h$ for which the intrinsic error is minimal.

## 1. Some General Theorems on Recovery from Inaccurate Data

Now our aim is proving the sufficiency of some conditions for the $S_0$ to be an optimal algorithm and $x_0$ to be a worst element. These conditions were originally found by Micchelli and Rivlin [2]. Though it is closely connected with Micchelli and Rivlin's result the theorem we need is slightly different.

**Theorem 1.** *Let $x_0 \in W$, $-x_0 \in W$, $L(-x_0) = -Lx_0$, $\|Ix_0\| \le \delta$, $\|I(-x_0)\| \le \delta$, and*

$$\sup_{\substack{x \in W \\ \|Ix-y\| \le \delta}} \|Lx - S_0 y\| = \|Lx_0\|.$$

*Then*

(i) *$S_0$ is an optimal algorithm,*
(ii) *$x_0$ is a worst element,*
(iii) *the intrinsic error is $E(L,I,\delta) = \|Lx_0\|$.*

*Proof.* It follows from (1) that

$$E(L,I,\delta) \le \sup_{\substack{x \in W \\ \|Ix-y\| \le \delta}} \|Lx - S_0 y\| = \|Lx_0\|.$$

On the other hand, for any algorithm $S$ we have

$$\|Lx_0 - S(0)\| + \|L(-x_0) - S(0)\| \ge 2\|Lx_0\| \tag{2}$$

and therefore

$$\sup_{\substack{x \in W \\ \|Ix-y\| \le \delta}} \|Lx - Sy\| \ge \max\{\|Lx_0 - S(0)\|, \|L(-x_0) - S(0)\|\} \ge \|Lx_0\|.$$

Thus $E(L,I,\delta) = \|Lx_0\|$ and $S_0$ is an optimal algorithm. Now suppose that $x_0$ is not a worst element, i.e.,

$$\sup_{\|Ix_0-y\| \le \delta} \|Lx_0 - S_0 y\| < \sup_{\substack{x \in W \\ \|Ix-y\| \le \delta}} \|Lx - S_0 y\| = \|Lx_0\|.$$

Then

$$\|Lx_0 - S_0(0)\| < \|Lx_0\|, \quad \|L(-x_0) - S_0(0)\| \le \|Lx_0\|,$$

which contradicts (2). □

**Corollary 1.** *Let* $x_0 \in W$, $-x_0 \in W$, $L(-x_0) = -Lx_0$, $\|Ix_0\| \le \delta$, $\|I(-x_0)\| \le \delta$, *let* $S_0$ *be a linear operator, and let*

$$\sup_{x\in W} \|Lx - S_0Ix\| = \|Lx_0\| - \delta\|S_0\|.$$

*Then*

(i) $S_0$ *is an optimal algorithm,*
(ii) $x_0$ *is a worst element,*
(iii) *the intrinsic error is* $E(L, I, \delta) = \sup\limits_{\substack{x\in W\\ \|Ix\|\le\delta}} \|Lx\| = \|Lx_0\|$.

*Proof.* Note that

$$\sup_{\substack{x\in W\\ \|Ix-y\|\le\delta}} \|Lx - S_0y\| = \sup_{\substack{x\in W\\ \|Ix-y\|\le\delta}} \|Lx - S_0Ix + S_0(Ix - y)\|$$
$$\le \sup_{x\in W} \|Lx - S_0Ix\| + \delta\|S_0\| = \|Lx_0\|.$$

Since $\|Ix_0\| \le \delta$ we have

$$\sup_{\substack{x\in W\\ \|Ix\|\le\delta}} \|Lx\| \ge \|Lx_0\| \ge \sup_{\substack{x\in W\\ \|Ix-y\|\le\delta}} \|Lx - S_0y\| \ge \sup_{\substack{x\in W\\ \|Ix\|\le\delta}} \|Lx\|.$$

Thus

$$\sup_{\substack{x\in W\\ \|Ix-y\|\le\delta}} \|Lx - S_0y\| = \|Lx_0\| = \sup_{\substack{x\in W\\ \|Ix\|\le\delta}} \|Lx\|.$$

Now the corollary follows from Theorem 1. □

Let $\Omega$ be a subset of $\mathbb{C}^n$ and $\mu$ be a nonnegative measure on $\Omega$. Denote by $L_p(\Omega, \mu)$ the Lebesgue space of complex- (or real-) valued functions with the usual norm

$$\|f\|_p = \left(\int_\Omega |f(z)|^p \, d\mu(z)\right)^{1/p}, \quad 1 \le p < \infty,$$
$$\|f\|_\infty = \operatorname*{ess\,sup}_{z\in\Omega} |f(z)|.$$

Let $X_p$ be some linear subspace of $L_p(\Omega, \mu)$ and $BX_p = \{f \in X_p : \|f\|_p \le 1\}$. Consider the problem (1) for $X = X_p$, $W = BX_p$ and $Z = \mathbb{C}(\mathbb{R})$.

The following theorem is a generalization of the appropriate results from [8,9] obtained for the case $\delta = 0$.

**Theorem 2.** *Let* $g \in X_p$, $\|g\|_p \ne 0$, $g_0 = g/\|g\|_p$. *Also let* $L$ *be a functional on* $X_p$, $L(-g_0) = -Lg_0$, $\|Ig_0\| \le \delta$, $\|I(-g_0)\| \le \delta$, *and* $S_0$ *a linear functional. Let* $S_0Ig_0 = \delta\|S_0\|$ *and for every* $f \in BX_p$ *let*

$$(3) \qquad Lf - S_0If = \begin{cases} \alpha \displaystyle\int_\Omega \overline{g(z)}|g(z)|^{p-2} f(z)\, d\mu(z), & 1 \le p < \infty, \\ \displaystyle\int_\Omega \overline{g(z)}|\varphi(z)| f(z)\, d\mu(z), & p = \infty, \end{cases}$$

*where $\alpha > 0$, $\varphi \in L_1(\Omega, \mu)$ and if $p = \infty$ then $|g(z)| = 1$ almost everywhere on $\Omega$ with respect to measure $\mu$. Then*

(i) *$S_0$ is an optimal algorithm,*
(ii) *$g_0$ is a worst element,*
(iii) *the intrinsic error is*

$$E(L, I, \delta) = \sup_{\substack{f \in BX_p \\ \|If\| \le \delta}} |Lf| = Lg_0 = \begin{cases} \alpha\|g\|_p^{p-1} + \delta\|S_0\|, & 1 \le p < \infty, \\ \|\varphi\|_1 + \delta\|S_0\|, & p = \infty. \end{cases}$$

*Proof.* For every $f \in BX_p$ from (3) and the Hölder inequality we have

$$|Lf - S_0 If| \le \begin{cases} \alpha\|g\|_p^{p-1}, & 1 \le p < \infty, \\ \|\varphi\|_1, & p = \infty. \end{cases}$$

On the other hand we obtain

$$Lg_0 - \delta\|S_0\| = Lg_0 - S_0 Ig_0 = \begin{cases} \alpha\|g\|_p^{p-1}, & 1 \le p < \infty, \\ \|\varphi\|_1, & p = \infty. \end{cases}$$

Hence

$$\sup_{f \in BX_p} |Lf - S_0 If| = Lg_0 - \delta\|S_0\|.$$

Now the theorem follows from Corollary 1. □

Let $l_q^m$ be the space $\mathbb{C}^m$ supplied with the norm

$$\|a\|_q = \|(a_1, \dots, a_m)\|_q = \begin{cases} \left(\sum_{j=1}^m |a_j|^q\right)^{1/q}, & 1 \le q < \infty, \\ \max_{1 \le j \le m} |a_j|, & q = \infty, \end{cases}$$

and by $(a, b)$ denote the Hermitian inner product

$$(a, b) = \sum_{j=1}^m a_j \bar{b}_j.$$

Let $a \neq 0$, $a^* \in Bl_q^m$ and

$$(a, a^*) = \|a\|_{q'}, \quad 1/q + 1/q' = 1.$$

It is easy to see that

$$a_j^* = \frac{a_j |a_j|^{q'-2}}{\|a\|_{q'}^{q'-1}}, \quad 1 \le q' < \infty,$$

and for $q' = \infty$

$$a_j^* = \begin{cases} 0, & j \neq j_0, \\ \dfrac{a_{j_0}}{|a_{j_0}|}, & j = j_0, \end{cases}$$

where $j_0$, such that $|a_{j_0}| = \max_{1 \le j \le m} |a_j|$.

**Corollary 2.** *Let* $I\colon BX_p \to l_q^m$, $S_0 y = (y, a)$, $a \in \mathbb{C}^m$, $g \in X_p$, $\|g\|_p \neq 0$, $g_0 = g/\|g\|_p$, $L$ *be a functional,* $L(-g_0) = -Lg_0$, *and* $\|I(-g_0)\|_q \le \delta$. *Suppose that for every* $f \in BX_p$ *the equality* (3) *holds and* $Ig_0 = \delta a^*$ *if* $a \neq 0$, *or* $\|Ig_0\|_q \le \delta$ *if* $a = 0$. *Then*

(i) $S_0$ *is an optimal algorithm,*
(ii) $g_0$ *is a worst element,*
(iii) *the intriilsic error is*

$$E(L, I, \delta) = \sup_{\substack{f \in BX_p \\ \|If\|_q \le \delta}} |Lf| = Lg_0 = \begin{cases} \alpha \|g\|_p^{p-1} + \delta \|a\|_{q'}, & 1 \le p < \infty, \\ \|\varphi\|_1 + \delta \|a\|_{q'}, & p = \infty. \end{cases}$$

## 2. Optimal Recovery of Analytic and Harmonic Functions

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $H_p$ be the Hardy space, i.e., the space of functions which are analytic in $D$ and for which

$$\begin{aligned} \|f\|_{H_p} &= \sup_{0<r<1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty, \quad 1 \le p < \infty, \\ \|f\|_{H_\infty} &= \sup_{z \in D} |f(z)| < \infty. \end{aligned} \tag{4}$$

It is well known that the functions from $H_p$ have boundary values almost everywhere and therefore $H_p$ can be considered as a subspace of $L_p(\Omega, \mu)$ for $\Omega = \{z \in \mathbb{C} : |z| = 1\}$ and $d\mu(e^{i\theta}) = (1/2\pi) d\theta$.

Recall that the Bergman space $A_p$ is the space of analytic functions which satisfy the inequality

$$\|f\|_{A_p} = \left( \frac{1}{\pi} \int_D |f(z)|^p \, d\sigma(z) \right)^{1/p} < \infty, \quad 1 \le p < \infty, \tag{5}$$

where $\sigma(z)$ is the Lebesgue measure on $D$ (for $p = \infty$ $A_\infty = H_\infty$). Thus the space $A_p$ is the subspace of $L_p(D, \mu)$ for $d\mu(z) = (1/\pi) d\sigma(z)$.

Denote by $h_p$ and $a_p$ the spaces of harmonic functions in $D$ which satisfy (4) and (5), respectively.

Consider the problem (1) when $X$ is one of the spaces $H_p$, $A_p$, $h_p$, or $a_p$, $W = BX$, $Lf = f(\xi)$, $If = f(z_1)$, $\xi$ and $z_1$ are distinct points in $D$. The relative intrinsic error will be denoted by $E(\xi, z_1, \delta, X)$.

Put

$$\rho = \left| \frac{\xi - z_1}{1 - \overline{z}_1 \xi} \right|, \quad W(z) = e^{i\varphi} \frac{z - z_1}{1 - \overline{z}_1 z}, \tag{6}$$

where $\varphi$ is determined from the condition $W(\xi) = \rho$,

$$\delta_1 = \left(\frac{1-\rho}{2(1-|z_1|^2)}\right)^{1/p}, \quad \delta_2 = \left(\frac{1-\rho^2}{1-|z_1|^2}\right)^{1/p},$$

$$h(z) = \begin{cases} 1, & 0 \le \delta \le \delta_1, \\ (W(z)+a)/(1+aW(z)), & \delta_1 < \delta < \delta_2, \\ W(z), & \delta \ge \delta_2, \end{cases}$$

where $a \in [0,1]$ and satisfies the equation

$$\frac{a\delta_2}{h(z_1)(1+a\rho+a^2)^{1/p}} = \delta, \quad 0 \le \delta < \delta_2. \tag{7}$$

(The existence of a solution follows from the continuity of the function from the left hand side of (7).) Put $a = 0$ for $\delta \ge \delta_2$.

**Proposition 1.** *Let $X = H_p$. Then for every $1 \le p < \infty$ and $\delta \ge 0$*

(i)

$$S_0 y = \frac{h(z_1)(1-\rho^2)}{h(\xi)(1+a\rho)^{2(p-1)/p}} \left(\frac{1-\overline{\xi} z_1}{1-|\xi|^2}\right)^{2/p} y$$

*is an optimal algorithm,*

(ii)

$$g_0(z) = \left(\frac{1-|\xi|^2}{1+2a\rho+a^2}\right)^{1/p} \frac{(W(z)+a)(1+aW(z))^{(2-p)/p}}{h(z)(1-\overline{\xi} z)^{2/p}} \tag{8}$$

*is a worst function,*

(iii) *the intrinsic error is*

$$E(\xi, z_1, \delta, H_p) = \frac{(\rho+a)(1+a\rho)^{(2-p)/p}}{h(\xi)(1+2a\rho+a^2)^{1/p}(1-|\xi|^2)^{1/p}}.$$

*Proof.* Put

$$g(z) = \frac{(W(z)+a)(1+aW(z))^{(2-p)/p}}{h(z)(1-\overline{\xi} z)^{2/p}}, \quad \alpha = \frac{\rho(1-|\xi|^2)^{(p-2)/p}}{h(\xi)(1+a\rho)^{2(p-1)/p}}.$$

By the residue theorem we have for every $f \in H_p$

$$\alpha \frac{1}{2\pi} \int_0^{2\pi} \overline{g(e^{i\theta})} |g(e^{i\theta})|^{p-2} f(e^{i\theta})\, d\theta$$
$$= \alpha \frac{1}{2\pi i} \int_{|z|=1} \frac{(1+aW(z))^{2(p-1)/p} h(z)}{W(z)(z-\xi)(1-\overline{\xi} z)^{(p-2)/p}} f(z)\, dz = f(\xi) - S_0 f(z_1).$$

In addition,

$$\|g\|_{H_p}^p = \frac{1}{2\pi} \int_0^{2\pi} \left|\frac{1+aW(e^{i\theta})}{1-\overline{\xi} e^{i\theta}}\right|^2 d\theta$$
$$= \frac{1}{2\pi i} \int_{|z|=1} \frac{(1+aW(z))(W(z)+a)}{W(z)(1-\overline{\xi} z)(z-\xi)}\, dz = \frac{1+2a\rho+a^2}{1-|\xi|^2}.$$

It follows from (7) that for $0 \le \delta < \delta_2$ $|g_0(z_1)| = \delta$ and for $\delta \ge \delta_2$ $|g_0(z_1)| = \delta_2 \le \delta$, $S_0 y \equiv 0$. Since $S_0 g_0(z_1) \ge 0$, we have $S_0 g_0(z_1) = \delta \|S_0\|$ for every $\delta \ge 0$. Now the proposition follows from Theorem 2. □

Note that in virtue of Theorem 2 the following generalization of the Schwartz Lemma can be obtained from Proposition 1:

$$(9)\quad \sup_{\substack{f\in BH_p\\ |f(0)|\le\delta}} |f(z)| = \begin{cases} \dfrac{(|z|+a)(1+a|z|)^{(2-p)/p}}{(1-|z|^2)^{1/p}(1+2a|z|+a^2)^{1/p}}, & 0 \le \delta \le \left(\dfrac{1-|z|}{2}\right)^{1/p}, \\ \dfrac{(1+a|z|)^{2/p}}{(1-|z|^2)^{1/p}(1+2a|z|+a^2)^{1/p}}, & \left(\dfrac{1-|z|}{2}\right)^{1/p} \le (1-|z|^2)^{1/p}, \\ (1-|z|^2)^{-1/p}, & \delta \ge (1-|z|^2)^{1/p}. \end{cases}$$

Here $a$ is defined by (7) for $z_1 = 0$.

Now consider the same problem for $X = A_p$. Put

$$\delta_1 = \frac{(2+\rho)^{2/p}(1-\rho)^{2/p}}{2^{1/p}(3-\rho^2)^{1/p}(1-|z_1|^2)^{2/p}}, \quad \delta_2 = \left(\frac{1-\rho^2}{1-|z_1|^2}\right)^{2/p},$$

$$b = \begin{cases} \dfrac{1}{1+a\rho}, & 0 \le \delta < \delta_1, \\ \dfrac{a}{a+\rho}, & \delta \ge \delta_1, \end{cases}$$

where $a \in [0,1]$ and satisfies the equations

(10)

$$\frac{a\rho^{2/p}((p/2-1)(1-a^2)b^2+b+b^2)^{2/p}(1-|z_1|^2)^{-2/p}}{\left[\dfrac{(p/2-1)(1-a^2)(1+2a\rho+a^2)\rho^2 b^4}{1-\rho^2} + \dfrac{\rho^2(2-\rho^2)}{(1-\rho^2)^2} + (1-b^2)^2\right]^{1/p}} = \delta$$

for $0 \le \delta < \delta_1$ and

$$\frac{(1-b^2)^{2/p}(1-\rho^2)^{2/p}}{(1-2(1-\rho^2)^2 b^2 + (1-\rho^2)^2 b^4)^{1/p}(1-|z_1|^2)^{2/p}} = \delta$$

for $\delta_1 \le \delta < \delta_2$. (The solution of the last equation may be given in direct form and the existence of solution (10) will be shown below.)

Put $a=0$ for $\delta\ge\delta_2$. Let

$$\varphi(z)=\begin{cases}(p/2-1)(1-a^2)(1-\rho W(z)) \\ \qquad\qquad +(1+aW(z))(2+a\rho-\rho W(z)), & 0\le\delta<\delta_1,\\ (1+aW(z))(2a+\rho-a\rho W(z)), & \delta\ge\delta_1,\end{cases}$$

$$g(z)=\begin{cases}\dfrac{W(z)+a}{1+aW(z)}\dfrac{(\varphi(z))^{2/p}}{(1-\overline{\xi}z)^{4/p}}, & 0\le\delta<\delta_1,\\ \dfrac{(\varphi(z))^{2/p}}{(1-\overline{\xi}z)^{4/p}}, & \delta\ge\delta_1.\end{cases}$$

**Proposition 2.** *Let $X=A_p$. Then for every $1\le p<\infty$ and $\delta\ge0$*

(i)

$$S_0y=b^2(1-\rho^2)^2\left(\frac{1-\overline{\xi}z_1}{1-|\xi|^2}\right)^{4/p}\left(\frac{\varphi(z_1)}{\varphi(\xi)}\right)^{(p-2)/p}y$$

*is an optimal algorithm,*

(ii) *$g_0=g/\|g\|_{A_p}$ is a worst function,*

(iii) *the intrinsic error is*

$$E(\xi,z_1,\delta,A_p)=\begin{cases}\dfrac{\rho((p/2)(1-\rho^2)+1)^{1/p}}{(1-|\xi|^2)^{2/p}}, & \delta=0,\\ \delta b\dfrac{a+\rho}{a}\left(\dfrac{1-|z_1|^2}{(1-|\xi|^2)(1-\rho^2)}\right)^{2/p}\left(\dfrac{\varphi(\xi)}{\varphi(z_1)}\right)^{2/p}, & \\ & 0<\delta<\delta_2,\\ \dfrac{1}{(1-|\xi|^2)^{2/p}}, & \delta\ge\delta_2.\end{cases}$$

*Proof.* Note that the functions

$$(p/2-1)(1-a^2)(1-\rho w)+(1+aw)(2+a\rho-\rho w)$$

and

$$(1+aw)(2a+\rho-a\rho w),$$

as the functions of $w$, have real zeros which are outside the interval $(-1,1)$. Therefore $\varphi(z)$ is zero free in $D$. Let $0\le\delta<\delta_1$. For $f\in H_\infty$ denote

$$Jf=\frac{1}{2\pi i}\int_{|z|=1}\frac{(1+aW(z))^2(\varphi(z))^{(p-2)/p}}{W(z)(W(z)-\rho)(1-\overline{\xi}z)^{2(p-2)/p}}f(z)\,dz.$$

Since $W(z)-\rho=e^{i\varphi}(z-\xi)(1-|z_1|^2)/((1-\overline{z}_1z)(1-\overline{z}_1\xi))$ we obtain by the residue theorem

$$e^{i\varphi}\frac{\rho(1-|z_1|^2)(1-|\xi|^2)^{2(p-2)/p}}{(1+a\rho)^2(1-\overline{z}_1\xi)(\varphi(\xi))^{(p-2)/p}}Jf=f(\xi)-S_0f(z_1).$$

On the other hand, in view of the equality $\overline{W(e^{i\theta})} = W^{-1}(e^{i\theta})$, we obtain by Stokes' formula

$$Jf = \frac{1}{2\pi i}\int_{|z|=1}\left(\frac{\overline{W(z)}+a}{1+a\overline{W(z)}}\right)^2 \frac{(1+a\overline{W(z)})^2(\varphi(z))^{(p-2)/p}}{(1-\rho\overline{W(z)})(1-\overline{\xi}z)^{2(p-2)/p}}f(z)\,dz$$
$$= \frac{1}{2\pi i}\int_{|z|=1}\left(\frac{\overline{W(z)}+a}{1+a\overline{W(z)}}\right)^{p/2+1} \frac{(1+a\overline{W(z)})^2}{1-\rho\overline{W(z)}}\left(\frac{W(z)+a}{1+aW(z)}\right)^{p/2-1}$$
$$\times\frac{(\varphi(z))^{(p-2)/p}}{(1-\overline{\xi}z)^{2(p-2)/p}}f(z)\,dz = e^{-i\varphi}\frac{(1-\overline{z}_1\xi)^2}{1-|z_1|^2}\frac{1}{\pi}\int_D\left(\frac{\overline{W(z)}+a}{1+a\overline{W(z)}}\right)^{p/2}$$
$$\times\frac{\overline{\varphi(z)}}{(1-\xi\overline{z})^2}\left(\frac{W(z)+a}{1+aW(z)}\right)^{p/2-1}\frac{(\varphi(z))^{(p-2)/p}}{(1-\overline{\xi}z)^{2(p-2)/p}}f(z)\,d\sigma(z)$$
$$= e^{-i\varphi}\frac{(1-\overline{z}_1\xi)^2}{1-|z_1|^2}\frac{1}{\pi}\int_D\overline{g(z)}|g(z)|^{p-2}f(z)\,d\sigma(z).$$

Thus for every $f\in H_\infty$ we have

(11)
$$\frac{\rho(1-|\xi|^2)^{2(p-2)/p}}{(1+a\rho)^2(\varphi(\xi))^{(p-2)/p}}\frac{1}{\pi}\int_D\overline{g(z)}|g(z)|^{p-2}f(z)\,d\sigma(z) = f(\xi)-S_0f(z_1).$$

As functions from $H_\infty$ are dense in $A_p$ for every $1\le p<\infty$, the equality (11) holds for every function from $A_p$. It is easily seen that $S_0g(z_1)\ge 0$. Therefore it follows from Theorem 2 that if $a\in[0,1]$ satisfies the condition $|g(z_1)|/\|g\|_{A_p}=\delta$, then $S_0$ is an optimal algorithm. For $f=g$, from (11) we have

$$\frac{\rho(1-|\xi|^2)^{2(p-2)/p}}{(1+a\rho)^2(\varphi(\xi))^{(p-2)/p}}\|g\|_{A_p}^p$$
$$= \frac{(\rho+a)(\varphi(\xi))^{2/p}}{(1+a\rho)(1-|\xi|^2)^{4/p}} - \left(\frac{1-\rho^2}{1+a\rho}\right)^2\frac{a\varphi(z_1)}{(1-|\xi|^2)^{4/p}(\varphi(\xi))^{(p-2)/p}}.$$

Hence

$$\|g\|_{A_p}^p = \frac{(\rho+a)(1+a\rho)}{\rho(1-|\xi|^2)^2}\varphi(\xi) - \frac{a(1-\rho^2)^2}{\rho(1-|\xi|^2)^2}\varphi(z_1).$$

We find by direct calculation that

$$\|g\|_{A_p}^p = \frac{1}{b^4\rho^2(1-|\xi|^2)^2}\left(\left(\frac{p}{2}-1\right)(1-a^2)(1-\rho^2)(1+2a\rho+a^2)\rho^2b^4\right.$$
$$\left.+1-2(1-\rho^2)^2b^2+(1-\rho^2)^2b^4\right).$$

Since $\|g\|_{A_p}>0$ for every $a\in[0,1]$, the function in the left-hand side of (10) is continuous as a function of $a$ ($a\in[0,1]$), and therefore the equation (10) has a solution for every $\delta \in [0, \delta_1)$. We have

$$|g(z_1)| = a\frac{|\varphi(z_1)|^{2/p}}{|1-\overline{\xi}z_1|^{4/p}} = a\frac{((p/2-1)(1-a^2)+2+a\rho)^{2/p}}{|1-\overline{\xi}z_1|^{4/p}}$$
$$= \frac{a((p/2-1)(1-a^2)+1+b^{-1})^{2/p}(1-\rho^2)^{2/p}}{(1-|\xi|^2)^{2/p}(1-|z_1|^2)^{2/p}}$$

and so the equation (10) means that $|g(z_1)|/\|g\|_{A_p} = \delta$.

The case $\delta \in [\delta_1, \delta_2]$ can be considered in the same way if we set

$$Jf = \frac{1}{2\pi i}\int_{|z|=1} \frac{(W(z)+a)^2(\varphi(z))^{(p-2)/p}}{W(z)(W(z)-\rho)(1-\overline{\xi}z)^{2(p-2)/p}} f(z)\,dz$$
$$= \frac{1}{2\pi i}\int_{|z|=1} \frac{(1+a\overline{W(z)})^2(\varphi(z))^{(p-2)/p}}{(1-\rho\overline{W(z)})(1-\overline{\xi}z)^{2(p-2)/p}} f(z)\,dz.$$

Let now $\delta \geq \delta_2$. Then $a = 0$, $g(z) = \rho^{2/p}(1-\overline{\xi}z)^{-4/p}$, and

$$\frac{1}{\pi}\int_D \overline{g(z)}|g(z)|^{p-2}f(z)\,d\sigma(z) = \frac{\rho^{2(p-1)/p}}{\pi}\int_D \frac{f(z)\,d\sigma(z)}{(1-\xi\overline{z})^2(1-\overline{\xi}z)^{2(p-2)/p}}$$
$$= \rho^{2(p-1)/p}\frac{f(\xi)}{(1-|\xi|^2)^{2(p-2)/p}}.$$

(Here we use the fact that the Bergman kernel $(1-\overline{\xi}z)^{-2}$ is the reproducing kernel on $A_p$.) Thus

$$\frac{(1-|\xi|^2)^{2(p-2)/p}}{\rho^{2(p-1)/p}}\frac{1}{\pi}\int_D \overline{g(z)}|g(z)|^{p-2}f(z)\,d\sigma(z) = f(\xi) - S_0 f(z_1). \tag{12}$$

Now let us verify that $|g(z_1)|/\|g\|_{A_p} \leq \delta$. Substituting $f = g$ in (12), we obtain

$$(1-|\xi|^2)^{2(p-2)/p}\|g\|_{A_p}^p = \frac{\rho^{2/p}}{(1-|\xi|^2)^{4/p}}$$

which yields

$$\frac{|g(z_1)|}{\|g\|_{A_p}} = \left(\frac{1-\rho^2}{1-|z_1|^2}\right)^{2/p} = \delta_2 \leq \delta.$$

The proposition is proved. □

Now we consider the same problem for $X = h_p$, $p > 1$. Put

$$\alpha(\lambda) = \frac{\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} P(z_1, e^{i\theta})(P(\xi, e^{i\theta}) - \lambda P(z_1, e^{i\theta}))_{(q)}\,d\theta}{\|P(\xi,\cdot) - \lambda P(z_1,\cdot)\|_q^{q-1}}, \quad \delta_1 = \alpha(0),$$

where $P(\xi, z) = (1-|\xi|^2)/|1-\overline{\xi}z|^2$ is the Poisson kernel, $1/p + 1/q = 1$, $(x)_{(q)} = |x|^{q-1}\operatorname{sign} x$, and

$$\|f\|_q = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^q\,d\theta\right)^{1/q}.$$

Let us show that for every $0 \le \delta \le \delta_1$ the equation

$$\alpha(\lambda) = \delta \tag{13}$$

has a solution $\lambda \in [0, (1+\rho)/(1-\rho)]$. For $z = e^{i\theta}$ and $\zeta = W(z) = e^{i\varphi}(z - z_1)/(1 - \overline{z}_1 z)$ we have

$$\frac{P(\xi, z)}{P(z_1, z)} = P(\rho, \zeta) = \frac{1-\rho^2}{|1-\rho\zeta|^2} \le \frac{1-\rho}{1+\rho} \tag{14}$$

($\rho$ and $\varphi$ are the same as in (6)). Thus for every $z = e^{i\theta}$

$$P(\xi, z) - \frac{1+\rho}{1-\rho} P(z_1, z) \le 0.$$

Therefore

$$\alpha\left(\frac{1+\rho}{1-\rho}\right) < 0.$$

Since $\alpha(\lambda)$ is continuous for $\lambda \in [0, (1+\rho)/(1-\rho)]$ and $\alpha(0) = \delta_1$, the equation (13) has a solution in this interval for every $0 \le \delta \le \delta_1$. We denote this solution $C_p(\xi, z_1, \delta)$. For $\delta > \delta_1$ we put $C_p(\xi, z_1, \delta) = 0$.

**Proposition 3.** *For $X = h_p$, $p > 1$,*

(i) *$S_0 y = C_p(\xi, z_1, \delta) y$ is an optimal algorithm,*

(ii)

$$u_0(\zeta) = \frac{\frac{1}{2\pi}\int_0^{2\pi} P(\zeta, e^{i\theta})(P(\xi, e^{i\theta}) - C_p(\xi, z_1, \delta)P(z_1, e^{i\theta}))_{(q)}\, d\theta}{\|P(\xi, \cdot) - C_p(\xi, z_1, \delta)P(z_1, \cdot)\|_q^{q-1}}$$

*is a worst function,*

(iii) *the intrinsic error is*

$$E(\xi, z_1, \delta, h_p) = u_0(\xi) = \|P(\xi, \cdot) - C_p(\xi, z_1, \delta)P(z_1, \cdot)\|_q + \delta C_p(\xi, z_1, \delta).$$

*Proof.* It is known (see [10]) that every function from $h_p$, $p > 1$, has boundary values almost everywhere. It is also known that boundary values reconstruct this function by Poisson transformation. So for every $u \in Bh_p$, $p > 1$, we have by the Hölder inequality

$$\begin{aligned} &|u(\xi) - C_p(\xi, z_1, \delta)u(z_1)| \\ &\quad = \left|\frac{1}{2\pi}\int_0^{2\pi} (P(\xi, e^{i\theta}) - C_p(\xi, z_1, \delta)P(z_1, e^{i\theta}))u(e^{i\theta})\, d\theta\right| \\ &\qquad \le \|P(\xi, \cdot) - C_p(\xi, z_1, \delta)P(z_1, \cdot)\|_q. \end{aligned}$$

On the other hand, the function

$$f(\theta) = \frac{(P(\xi, e^{i\theta}) - C_p(\xi, z_1, \delta)P(z_1, e^{i\theta}))_{(q)}}{\|P(\xi, \cdot) - C_p(\xi, z_1, \delta)P(z_1, \cdot)\|_q^{q-1}} \in BL_p(0, 2\pi)$$

and therefore (see [10]) $u_0 \in Bh_p$, and has almost everywhere boundary values which coincide with $f(\theta)$. Thus we obtain that

$$u_0(\xi) - C_p(\xi, z_1, \delta)u_0(z_1) = \|P(\xi, \cdot) - C_p(\xi, z_1, \delta)P(z_1, \cdot)\|_q.$$

Hence

$$\sup_{u\in Bh_p} |u(\xi) - C_p(\xi, z_1, \delta)u(z_1)| = u_0(\xi) - C_p(\xi, z_1, \delta)u_0(z_1). \tag{15}$$

Let $0 \le \delta \le \delta_1$. It follows from the definition of $C_p(\xi, z_1, \delta)$ that $u_0(z_1) = \delta$. Since $C_p(\xi, z_1, \delta) \ge 0$, we have from (15) that $u_0(\xi) \ge 0$ and

$$\sup_{u\in Bh_p} |u(\xi) - C_p(\xi, z_1, \delta)u(z_1)| = |u_0(\xi)| - \delta C_p(\xi, z_1, \delta).$$

For $\delta > \delta_1$ the same equality holds because $C_p(\xi, z_1, \delta) = 0$. Now the proposition follows from Corollary 1. □

We can easily find $C_\infty(\xi, z_1, \delta)$. In this case $q = 1$ and

$$\alpha(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} P(z_1, e^{i\theta})\operatorname{sign}(P(\xi, e^{i\theta}) - \lambda P(z_1, e^{i\theta}))\, d\theta.$$

In view of (14) the substitution $z = (e^{-i\varphi}\zeta + z_1)/(1 + \overline{z}_1 e^{-i\varphi}\zeta)$ yields

$$\begin{aligned}\alpha(\lambda) = \frac{1}{2\pi}&\int_0^{2\pi} \operatorname{sign}(P(\rho, e^{i\theta}) - \lambda)\, d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} \operatorname{sign}\left(\frac{1-\rho^2}{1 - 2\rho\cos\theta + \rho^2} - \lambda\right) d\theta \\ &= \begin{cases} 1, & \lambda \le \dfrac{1-\rho}{1+\rho}, \\ \dfrac{2}{\pi}\arccos\dfrac{\lambda(1+\rho^2) - (1-\rho^2)}{2\rho\lambda} - 1, & \dfrac{1-\rho}{1+\rho} \le \lambda \le \dfrac{1+\rho}{1-\rho}, \\ -1, & \lambda \ge \dfrac{1+\rho}{1-\rho}. \end{cases}\end{aligned}$$

Hence for $0 \le \delta < 1$ the solution of (13) is

$$C_\infty(\xi, z_1, \delta) = \frac{1-\rho^2}{1 + 2\rho\sin(\pi\delta/2) + \rho^2}.$$

If $\delta = 1$, every $\lambda \in [0, (1-\rho)/(1+\rho)]$ is a solution of (13).

For $0 \le \delta < 1$ and $z = e^{i\theta}$ we have

$$\begin{aligned}&\operatorname{sign}(P(\xi, z) - C_\infty(\xi, z_1, \delta)P(z_1, z)) \\ &= \operatorname{sign}\left(\frac{1-\rho^2}{|1 - \rho W(z)|^2} - C_\infty(\xi, z_1, \delta)\right) = \operatorname{sign}\left(\operatorname{Re} W(z) + \sin\frac{\pi}{2}\delta\right) \\ &= \operatorname{sign}\operatorname{Re}\frac{W(z) + \tan(\pi\delta/4)}{1 + W(z)\tan(\pi\delta/4)} = \frac{4}{\pi}\operatorname{Re}\arctan\frac{W(z) + \tan(\pi\delta/4)}{1 + W(z)\tan(\pi\delta/4)}.\end{aligned}$$

In the case $p = \infty$, $0 \le \delta < 1$, we obtain

$$u_0(\zeta) = \frac{4}{\pi}\operatorname{Re}\arctan\frac{W(\zeta) + \tan(\pi\delta/4)}{1 + W(\zeta)\tan(\pi\delta/4)}.$$

Thus the next corollary follows from Proposition 3.

**Corollary 3.** *For $X = h_\infty$*

(i)

$$S_0 y = \begin{cases} \dfrac{1-\rho^2}{1+2\rho\sin(\pi\delta/2)+\rho^2} y, & 0 \le \delta < 1, \\ c\dfrac{1-\rho}{1+\rho} y, & \delta = 1,\ c \in [0,1], \\ 0, & \delta > 1, \end{cases}$$

*is an optimal algorithm.* (*In the case $\delta = 1$, $c$ is an arbitrary value in* $[0,1]$.)

(ii)

$$u_0(z) = \frac{4}{\pi} \operatorname{Re} \arctan \frac{W(z)+\Delta}{1+\Delta W(z)}, \text{ where } \Delta = \begin{cases} \tan(\pi\delta/4), & 0 \le \delta < 1, \\ 1, & \delta \ge 1, \end{cases}$$

*is a worst function,*

(iii) *the intrinsic error is*

$$E(\xi, z_1, \delta, h_\infty) = u_0(\xi) = \frac{4}{\pi} \arctan \frac{\rho+\Delta}{1+\Delta\rho}.$$

The solution of the equation (13) may also be obtained in direct form for $p = 2$. Nevertheless, we prove a more general result for the Hilbert space.

Let $X$ be a complex (or real) Hilbert space. Consider the problem (1) in the case $W = BX$, $Y = Z = \mathbb{C}$ $(\mathbb{R})$, $Lx = (x, x_1)$, $Ix = (x, x_2)$, $x_1, x_2 \in X$. The intrinsic error will be denoted by $E(x_1, x_2, \delta, X)$.

**Proposition 4.** *Let $x_1$ and $x_2$ be linear independent elements from the Hilbert space $X$. Put*

$$\varepsilon = \min\left\{\delta, \frac{|(x_1,x_2)|}{\|x_1\|}\right\}.$$

*Then*

(i)

$$S_0 y = \lambda \frac{(x_2, x_1)}{\|x_2\|^2} y,$$

*where*

$$\lambda = 1 - \frac{\varepsilon}{|(x_1,x_2)|}\sqrt{\frac{\|x_1\|^2\|x_2\|^2 - |(x_1,x_2)|^2}{\|x_2\|^2 - \varepsilon^2}},$$

*is an optimal algorithm,*

(ii)

$$x_0 = \sqrt{\frac{\|x_2\|^2 - \varepsilon^2}{\|x_1\|^2\|x_2\|^2 - |(x_1,x_2)|^2}} \left(x_1 - \lambda \frac{(x_1,x_2)}{\|x_2\|^2} x_2\right)$$

*is a worst element,*

(iii) *the intrinsic error is*

$$E(x_1,x_2,\delta,X)=\sqrt{1-\frac{\varepsilon^2}{\|x_2\|^2}}\sqrt{\|x_1\|^2-\frac{|(x_1,x_2)|^2}{\|x_2\|^2}}+\varepsilon\frac{|(x_1,x_2)|}{\|x_2\|^2}.$$

*Proof.* We have

$$\sup_{\|x\|\le 1}|(x,x_1)-S_0(x,x_2)|=\sup_{\|x\|\le 1}\left|\left(x,x_1-\lambda\frac{(x_1,x_2)}{\|x_2\|^2}x_2\right)\right|$$
$$=\left\|x_1-\lambda\frac{(x_1,x_2)}{\|x_2\|^2}x_2\right\|=\sqrt{\frac{\|x_1\|^2\|x_2\|^2-|(x_1,x_2)|^2}{\|x_2\|^2-\varepsilon^2}}.$$

Moreover

$$(x_0,x_1)-S_0(x_0,x_2)=\left(x_0,x_1-\lambda\frac{(x_1,x_2)}{\|x_2\|^2}x_2\right)=\left\|x_1-\lambda\frac{(x_1,x_2)}{\|x_2\|^2}x_2\right\|.$$

Thus $\|x_0\|=1$ and $\sup_{\|x\|\le 1}|(x,x_1)-S_0(x,x_2)|=(x_0,x_1)-S_0(x_0,x_2)$. Since

$$S_0(x_0,x_2)=\frac{\lambda(1-\lambda)|(x_1,x_2)|^2}{\|x_2\|^2\left\|x_1-\lambda\frac{(x_1,x_2)}{\|x_2\|^2}x_2\right\|}\ge 0$$

and

$$|(x_0,x_2)|=\sqrt{\frac{\|x_2\|^2-\varepsilon^2}{\|x_1\|^2\|x_2\|^2-|(x_1,x_2)|^2}}|(x_1,x_2)|(1-\lambda)=\varepsilon,$$

we obtain $S_0(x_0,x_2)=\delta\|S_0\|$. To finish the proof of the proposition we need only apply Corollary 1. □

The problems of optimal recovery in Hilbert spaces from inaccurate data were investigated in [11]. (See also [2] for a more general situation.)

Let $X$ be a Hilbert space of functions $f\colon\Omega\to\mathbb{C}$ $(\mathbb{R})$ with the reproducing kernel $K\colon\Omega\times\Omega\to\mathbb{C}$ $(\mathbb{R})$, i.e.,

$$f(z)=(f(\cdot),K(\cdot,z))$$

for every $f\in X$ and $z\in\Omega$. Consider the problem (1) for $W=BX$, $Lf=f(\xi)$, $If=f(z_1)$. If $K(\cdot,\xi)$ and $K(\cdot,z_1)$ are linearly independent (i.e., the class $BX$ distinguishes the points $\xi$ and $z_1$), then from Proposition 4 we get Corollary 4.

**Corollary 4.** *Put*

$$\varepsilon=\min\left\{\delta,\frac{|K(z_1,\xi)|}{\sqrt{K(\xi,\xi)}}\right\}.$$

*Then*

(i)

$$S_0 y = \lambda \frac{K(\xi, z_1)}{K(z_1, z_1)} y,$$

*where*

$$\lambda = 1 - \frac{\varepsilon}{|K(z_1, \xi)|} \sqrt{\frac{K(\xi, \xi) K(z_1, z_1) - |K(z_1, \xi)|^2}{K(z_1, z_1) - \varepsilon^2}}$$

*is an optimal algorithm,*

(ii)

$$f_0(z) = \sqrt{\frac{K(z_1, z_1) - \varepsilon^2}{K(\xi, \xi) K(z_1, z_1) - |K(z_1, \xi)|^2}} \times \left( K(z, \xi) - \lambda \frac{K(z_1, \xi)}{K(z_1, z_1)} K(z, z_1) \right)$$

*is a worst function, and*

(iii) *the intrinsic error is*

$$E(\xi, z_1, \delta, X) = \sqrt{1 - \frac{\varepsilon^2}{K(z_1, z_1)}} \sqrt{K(\xi, \xi) - \frac{|K(z_1, \xi)|^2}{K(z_1, z_1)}} + \varepsilon \frac{|K(z_1, \xi)|}{K(z_1, z_1)}.$$

We list some examples of Hilbert spaces with reproducing kernels:

1) $H_2$, $\quad K(\xi, z) = (1 - \xi \overline{z})^{-1}$, $\quad (f, g) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})}\, d\theta$,

2) $A_2$, $\quad K(\xi, z) = (1 - \xi \overline{z})^{-2}$, $\quad (f, g) = \dfrac{1}{\pi} \displaystyle\int_D f(z) \overline{g(z)}\, d\sigma(z)$,

3) $h_2$, $\quad K(\xi, z) = 2\,\mathrm{Re}(1 - \xi \overline{z})^{-1} - 1$,
$$(u, v) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) v(e^{i\theta})\, d\theta,$$

4) $a_2$, $\quad K(\xi, z) = 2\,\mathrm{Re}(1 - \xi \overline{z})^{-2} - 1$,
$$(u, v) = \frac{1}{\pi} \int_D u(z) v(z)\, d\sigma(z).$$

Note that we can obtain the generalization of Schwarz's Lemma in the same way as (9):

$$\sup_{\substack{f \in BX \\ |f(z_1)| \le \delta}} |f(\xi)| = E(\xi, z_1, \delta, X),$$

where $E(\xi, z_1, \delta, X)$ can be found from the corresponding proposition for $X = H_p, A_p, h_p$, and $a_2$.

Put

$$D(\xi, z_1, \delta, X) = \{z \in D : |g_0(z)| \le \delta\},$$

where $X = H_p, A_p, h_p$, or $a_2$ and $g_0(z)$ is a worst function for the appropriate recovery problem. Consider the information operator $\widetilde{I} f =$

$f_{|D(\xi,z_1,\delta,X)}$ instead of $If = f(z_1)$ and let $Y$ be the space of functions which are continuous in $D(\xi, z_1, \delta, X)$ with the norm

$$\|y\| = \sup_{z \in D(\xi,z_1,\delta,X)} |y(z)|.$$

It follows from Corollary 1 that the optimal algorithm, the worst function, and the intrinsic error will stay the same. Thus the additional information (with the same inaccuracy) about the behaviour of the function $f$ in $D(\xi, z_1, \delta, X)$ will not decrease the intrinsic error. In other words, the point $z_1$ forms some "shadow" set in which any additional information is useless.

## 3. Optimal Recovery of the Derivative from Inaccurate Data

We turn now to the problem (1) for $X = H_p$, $Z = \mathbb{C}$, $Y = l_q^2$, $Lf = f'(0)$, $If = (f(-h), f(h))$, $h \in (0,1)$. The intrinsic error will be denoted by $E_q'(h, \delta, H_p)$.

There is the well-known algorithm

$$f'(0) \approx \frac{f(h) - f(-h)}{2h}$$

which is not optimal even in the case $\delta = 0$ (see [12]). It was shown in [2] that

$$f'(0) \approx (1 - h^4)\frac{f(h) - f(-h)}{2h}$$

is an optimal algorithm for $\delta = 0$ and $p = \infty$. It follows from [8] that this algorithm is optimal for $\delta = 0$ and every $1 \le p \le \infty$. Moreover it is also optimal for $\delta = 0$ and $X = h_\infty$ (see [9]).

Now we consider the case when the value of functions in the points $-h$ and $h$ are known with an error $\le \delta$ in the norm of $l_q^2$, that is, we know $y_1$ and $y_2$ such that

$$|f(-h) - y_1|^q + |f(h) - y_2|^q \le \delta^q, \quad 1 \le q < \infty,$$
$$\max\{|f(-h) - y_1|, |f(h) - y_2|\} \le \delta, \quad q = \infty.$$

Put

$$\varepsilon_p = \begin{cases} 1/p, & 1 \le p < \infty, \\ 0, & p = \infty, \end{cases} \qquad \delta_1 = h2^{\varepsilon_q - \varepsilon_p}(1 + h^2)^{-\varepsilon_p}, \quad \delta_2 = h2^{\varepsilon_q},$$

$$\alpha(z) = \begin{cases} 1, & 0 \le \delta < \delta_1, \\ \dfrac{a^2 - z^2}{1 - a^2 z^2}, & \delta \ge \delta_1 \end{cases}$$

Let $a \in [h, 1]$ be a solution of the equation

$$\frac{h(a^2 - h^2)(1 - a^2h^2)^{2\varepsilon_p - 1}}{\alpha(h)(1 - h^4)^{\varepsilon_p}(1 - 2a^2h^2 + a^4)^{\varepsilon_p}} = \delta 2^{-\varepsilon_q}, \tag{16}$$

where $0 \le \delta \le \delta_2$. (The existence of the solution follows from the continuity of the function in the left hand side of this equation). Put $a = h$ for $\delta > \delta_2$.

**Proposition 5.** *For every* $\delta \ge 0$, $1 \le p, q \le \infty$

(i)

$$f'(0) \approx S_0 y = \frac{\alpha(h)(1-a^2h^2)^{2(1-\varepsilon_p)}}{\alpha(0)(1-h^4)^{1-2\varepsilon_p}} \frac{y_2 - y_1}{2h}$$

*is an optimal algorithm,*

(ii)

$$g_0(z) = \left(\frac{1-h^4}{1-2a^2h^2+a^4}\right)^{\varepsilon_p} \frac{z(a^2-z^2)(1-a^2z^2)^{2\varepsilon_p-1}}{\alpha(z)(1-h^2z^2)^{2\varepsilon_p}}$$

*is a worst function,*

(iii) *the intrinsic error is*

$$E'_q(h, \delta, H_p) = \frac{a^2}{\alpha(0)} \left(\frac{1-h^4}{1-2a^2h^2+a^4}\right)^{\varepsilon_p}.$$

*Proof.* Put

$$g(z) = \frac{z(a^2-z^2)(1-a^2z^2)^{2\varepsilon_p-1}}{\alpha(z)(1-h^2z^2)^{2\varepsilon_p}}, \quad \varphi(z) = \left(\frac{1-a^2z^2}{1-h^2z^2}\right)^2.$$

For every $f \in H_p$ and $z = e^{i\theta}$ we obtain by the residue theorem

$$f'(0) - S_0 I f = -\frac{h^2}{\alpha(0)} \frac{1}{2\pi i} \int_{|z|=1} \frac{\alpha(z)(1-a^2z^2)^{2(1-\varepsilon_p)}}{z^2(z^2-h^2)(1-h^2z^2)^{1-2\varepsilon_p}} f(z)\, dz$$

$$= \begin{cases} \dfrac{h^2}{\alpha(0)} \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \overline{g(z)} |g(z)|^{p-2} f(z)\, d\theta, & 1 \le p < \infty, \\ \dfrac{h^2}{\alpha(0)} \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \overline{g(z)} |\varphi(z)| f(z)\, d\theta, & p = \infty. \end{cases}$$

For $f = g$ we have from these equations

$$\|g\|_{H_p}^p = \|\varphi\|_{H_1} = \frac{1-2a^2h^2+a^4}{1-h^4}.$$

Note that if $p = \infty$, $|g(e^{i\theta})| \equiv 1$. Now to use Corollary 2 we must prove that

$$I g_0 = \delta a^* = \frac{\delta}{2^{\varepsilon_q}}(-1, 1)$$

if $0 \le \delta < \delta_2$ and $\|I g_0\|_{l_q^2} \le \delta$ if $\delta \ge \delta_2$. Let $0 \le \delta < \delta_2$. Since $g_0(-h) = -g_0(h)$ it is sufficient to prove the equation

$$g_0(h) = \delta 2^{-\varepsilon_q}$$

which coincides with (16). If $\delta \ge \delta_2$, $g_0(z) = z$ and $\|I g_0\|_{l_q^2} = h 2^{\varepsilon_q} = \delta_2 \le \delta$. This completes the proof of the proposition. □

Note that $S_0 y \equiv 0$ for $\delta \geq 2^{\varepsilon_q}$. If $\delta < 2^{\varepsilon_q}$ we can consider the problem of finding such a value $h_0$, that

$$E_q'(h_0, \delta, H_p) = \min_{h\in(0,1)} E_q'(h, \delta, H_p).$$

The value $h_0$ is called an optimal value of $h$. We give the solution of this problem for $p = \infty$.

**Proposition 6.** *For $p = \infty$, $1 \leq q \leq \infty$, and $0 \leq \delta < 2^{\varepsilon_q}$ the optimal value $h_0$ satisfies the equation*

$$\delta h_0^4 + 2^{1+\varepsilon_q} h_0^3 - \delta^2 2^{1-\varepsilon_q} h_0 - \delta = 0. \tag{17}$$

*The equality*

$$\min_{h\in(0,1)} E_q'(h, \delta, H_\infty) = h_0^2$$

*holds. The optimal value $h_0$ can also be found from the equality*

$$h_0 = \sqrt{k}\, \mathrm{sn}(K/3, k),$$

*where $k$ is determined by the equation*

$$\sqrt{k} = 2h_1^{1/4} \frac{\sum_{m=0}^{\infty} h_1^{m(m+1)}}{1 + 2\sum_{m=1}^{\infty} h_1^{m^2}}, \quad h_1 = e^{-\pi\Lambda'/(3\Lambda)} \tag{18}$$

*or*

$$\frac{K'}{K} = \frac{\Lambda'}{3\Lambda}; \tag{19}$$

*here $K$, $\Lambda$ denote the complete elliptic integrals of the first kind with respective moduli $k$, $\lambda = \delta^2 4^{-\varepsilon_q}$ and $K'$, $\Lambda'$ denote the ones with complementary moduli.*

*Proof.* From Proposition 5 we have

$$E_q'(h_0, \delta, H_\infty) = \begin{cases} a^2, & 0 \leq \delta < h 2^{\varepsilon_q}, \\ 1, & \delta \geq h 2^{\varepsilon_q}, \end{cases}$$

where $a \in [h, 1]$ and is determined by the equation

$$h\frac{a^2 - h^2}{1 - a^2 h^2} = \frac{\delta}{2^{\varepsilon_q}}. \tag{20}$$

Extracting $a^2$ from this equation and minimizing it as a function of $h \in (0, 1)$ we obtain that the minimum $h_0$ is unique and satisfies the equation (17). Taking a derivative from (20), we have

$$\frac{a^2 - h^2}{1 - a^2 h^2} - 2h^2 \frac{1 - a^4}{(1 - a^2 h^2)^2} + 2aa'h \frac{1 - h^4}{(1 - a^2 h^2)^2} = 0.$$

Thus if $h_0$ is minimum then $g_0'(h_0) = 0$, where

$$g_0(z) = z\frac{a^2 - z^2}{1 - a^2 z^2}. \tag{21}$$

Now it is sufficient to find a function $g_0(z)$ like (21) such that for some $h_0 \in (0, 1)$, $g_0(h_0) = \delta 2^{-\varepsilon_q}$ and $g_0'(h_0) = 0$. It follows from Lemma 2.2

of [7] that this function is a Blaschke product of order 3 with minimal norm

$$\|g_0\| = \max_{z\in[-\sqrt{k},\sqrt{k}]} |g_0(z)| = \delta 2^{-\varepsilon_q},$$

where $k$ is determined by the conditions $|g_0(-\sqrt{k})| = |g_0(\sqrt{k})| = \delta 2^{-\varepsilon_q}$. From [13] this function can be written in the form

$$g_0(z) = z\frac{k\,\mathrm{sn}^2(2K/3,k) - z^2}{1 - k\,\mathrm{sn}^2(2K/3,k)z^2}.$$

This function can be rewritten by using the first fundamental transformation of degree 3 (see [14])

$$g_0(z) = \sqrt{\lambda}\,\mathrm{sn}\left(3\Lambda u/K,\lambda\right), \quad z = \sqrt{k}\,\mathrm{sn}(u,k),$$

where $\lambda = \delta^2 4^{-\varepsilon_q}$ and $k$ satisfies (18), (19). If we put $h_0 = \sqrt{k}\,\mathrm{sn}(K/3,k)$ then $g_0(h_0) = \delta 2^{-\varepsilon_q}$ and $g_0'(h_0) = 0$. This completes the proof of the proposition. □

It is easily shown from (17) that

$$h_0 = 2^{-(1+\varepsilon_q)/3}\delta^{1/3} + O\left(\delta^{5/3}\right),$$

and consequently

$$\min_{h\in(0,1)} E_q'(h_0,\delta,H_\infty) = 4^{-(1+\varepsilon_q)/3}\delta^{2/3} + O\left(\delta^2\right).$$

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