

RECOVERY OF DIFFERENTIAL OPERATORS FROM A NOISY FOURIER TRANSFORM

K. YU. OSIPENKO

ABSTRACT. The paper concerns problems of the recovery of differential operators from a noisy Fourier transform. In particular, optimal methods are obtained for the recovery of powers of generalized Laplace operators from a noisy Fourier transform in the L_2 -metric.

1. INTRODUCTION

Let X be a linear space, Y, Z be normed linear spaces. The problem of optimal recovery of the linear operator $\Lambda: X \rightarrow Z$ by inaccurately given values of the linear operator $I: X \rightarrow Y$ on the set $W \subset X$ is posed as a problem of finding the value

$$E(\Lambda, W, I, \delta) = \inf_{\varphi: Y \rightarrow Z} \sup_{\substack{x \in W, y \in Y \\ \|Ix - y\|_Y \leq \delta}} \|\Lambda x - \varphi(y)\|_Z,$$

called the *error of optimal recovery*, and the mapping φ on which the lower bound is attained, called the *optimal recovery method* (here $\delta \geq 0$ is a parameter that characterizes the error of setting the values of the operator I). Initially, this problem was posed for the case when Λ is a linear functional, Y is a finite-dimensional space and the information is known exactly ($\delta = 0$), by S. A. Smolyak [1]. In fact, this statement was a generalization of A. N. Kolmogorov's problem about the best quadrature formula on the class of functions [2], in which the integral and the values of the functions are replaced by arbitrary linear functionals and there is no condition for the linearity of the recovery method. Subsequently, much research has been devoted to extensions of this problem (see [3]–[11], and the references given therein).

One of the first papers in which the problem of constructing an optimal recovery method for a linear operator was considered was the paper [4]. This topic was further developed in the papers [12]–[20]. It turned out that in some cases it is possible to construct a whole family of optimal recovery methods for a linear operator. The study of such families began in [21] and continued in [22], [23], [15], [20]. Some general approach to constructing of family of optimal recovery methods was proposed in [24].

The aim of this paper is to construct families of optimal recovery methods for powers of generalized Laplace operators and the Weil derivative from a noisy Fourier transform in the L_2 -metric.

Date: 08.02.2024.

2010 Mathematics Subject Classification. 41A65, 42B10, 49N30.

Key words and phrases. Optimal recovery, differential operators, Fourier transform.

2. OPTIMAL RECOVERY METHODS FROM A NOISY FOURIER TRANSFORM

Let S be the Schwartz space of rapidly decreasing C^∞ -functions on $\mathbb{R}^d$, S' be the corresponding space of distributions, and let $F: S' \rightarrow S'$ be the Fourier transform. Set

$$X_p = \{ x(\cdot) \in S' : \varphi(\cdot)Fx(\cdot) \in L_2(\mathbb{R}^d), Fx(\cdot) \in L_p(\mathbb{R}^d) \}.$$

We define the operator D as follows

$$Dx(\cdot) = F^{-1}(\varphi(\cdot)Fx(\cdot))(\cdot).$$

Put

$$(1) \quad \Lambda x(\cdot) = F^{-1}(\psi(\cdot)Fx(\cdot))(\cdot).$$

Consider the problem of the optimal recovery of values of the operator Λ on the class

$$W_p = \{ x(\cdot) \in X_p : \|Dx(\cdot)\|_{L_2(\mathbb{R}^d)} \leq 1 \}$$

from the noisy Fourier transform of the function $x(\cdot)$. Assume that $\Lambda x(\cdot) \in L_2(\mathbb{R}^d)$ for all $x(\cdot) \in X_p$. As recovery methods we consider all possible mappings $m: L_p(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)$. The error of a method m is defined by

$$e_p(\Lambda, D, m) = \sup_{\substack{x(\cdot) \in W_p, y(\cdot) \in L_p(\mathbb{R}^d) \\ \|Fx(\cdot) - y(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta}} \|\Lambda x(\cdot) - m(y)(\cdot)\|_{L_2(\mathbb{R}^d)}.$$

The quantity

$$(2) \quad E_p(\Lambda, D) = \inf_{m: L_p(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} e_p(\Lambda, D, m)$$

is called the error of optimal recovery, and the method on which the infimum is attained, an optimal method.

It is easily checked that

$$(3) \quad E_p(\Lambda, D) \geq \sup_{\substack{x(\cdot) \in W_p \\ \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta}} \|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)}.$$

Indeed, let $x(\cdot) \in W_p$, $\|Fx(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta$, and let $m: L_p(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)$ be an arbitrary recovery method. Since $x(\cdot) \in W_p$ and $-x(\cdot) \in W_p$, we have

$$2\|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \|\Lambda x(\cdot) - m(0)(\cdot)\|_{L_2(\mathbb{R}^d)} + \|-\Lambda x(\cdot) - m(0)(\cdot)\|_{L_2(\mathbb{R}^d)} \leq 2e_p(\Lambda, D, m).$$

It follows that, for any method m ,

$$e_p(\Lambda, D, m) \geq \sup_{\substack{x(\cdot) \in W_p \\ \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta}} \|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)}.$$

Now the required inequality follows by taking the lower bound on the left over all methods.

3. OPTIMAL RECOVERY METHODS FOR $\Lambda_{\theta}^{\eta/2}$

Consider the polar transformation in $\mathbb{R}^d$

$$\begin{aligned} t_1 &= \rho \cos \omega_1, \\ t_2 &= \rho \sin \omega_1 \cos \omega_2, \\ &\dots\dots\dots \\ t_{d-1} &= \rho \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \cos \omega_{d-1}, \\ t_d &= \rho \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \sin \omega_{d-1}. \end{aligned}$$

Set $\omega = (\omega_1, \dots, \omega_{d-1})$. For any function $f(\cdot)$ we put

$$\tilde{f}(\omega) = |f(\cos \omega_1, \dots, \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \sin \omega_{d-1})|.$$

Note that if $|f(\cdot)|$ is a homogenous function of degree κ , then $\tilde{f}(\omega) = \rho^{-\kappa}|f(t)|$.

Let $|\psi(\cdot)|$ be homogenous function of degree η and $|\varphi(\cdot)|$ be homogenous functions of degrees ν , $\psi(t) \neq 0$ and $\varphi_i(t) \neq 0$ for almost all $t \in \mathbb{R}^d$. Set

$$\gamma = \frac{\nu - \eta}{\nu + d(1/2 - 1/p)}, \quad q^* = \frac{1}{\gamma(1/2 - 1/p)},$$

$$C_p(\nu, \eta) = \gamma^{-\frac{\gamma}{p}} (1 - \gamma)^{-\frac{1-\gamma}{2}} \left(\frac{B(q^* \gamma/p + 1, q^*(1 - \gamma)/2)}{2|\nu - \eta|} \right)^{1/q^*},$$

where $B(\cdot, \cdot)$ is the Euler beta-function.

It follows from [25, Theorem 6] (see also [20, Theorem 3]) the following result

Theorem 1. *Let $2 < p \leq \infty$, $\gamma \in (0, 1)$. Assume that*

$$(4) \quad I = \int_{\Pi^{d-1}} \frac{\tilde{\psi}^{q^*}(\omega)}{\tilde{\varphi}^{q^*(1-\gamma)}(\omega)} J(\omega) d\omega < \infty, \quad \Pi^{d-1} = [0, \pi]^{d-2} \times [0, 2\pi].$$

Then

$$E_p(\Lambda, D) = \frac{1}{(2\pi)^{d\gamma/2}} C_p(\nu, \eta) I^{1/q^*} \delta^\gamma.$$

The method

$$\hat{m}(y)(t) = F^{-1} \left(\left(1 - \beta \frac{|\varphi(t)|^2}{|\psi(t)|^2} \right)_+ \psi(t)y(t) \right),$$

where

$$\beta = \frac{1-\gamma}{(2\pi)^{d\gamma}} C_p^2(\nu, \eta) \left(\delta I^{1/2-1/p} \right)^{2\gamma},$$

is optimal.

Moreover, the sharp inequality

$$\|\Lambda x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \frac{C_p(\nu, \eta) I^{1/q^*}}{(2\pi)^{d\gamma/2}} \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^\gamma \|Dx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\gamma}$$

holds.

Put

$$\psi_\theta(\xi) = (|\xi_1|^\theta + \dots + |\xi_d|^\theta)^{2/\theta}, \quad \theta > 0.$$

We denote by $\Lambda_\theta^{\eta/2}$ the operator Λ which is defined by (1) for $\psi(\cdot) = \psi_\theta^{\eta/2}(\cdot)$. In particular, $\Lambda_2 = -\Delta$, where Δ is the Laplace operator.

Consider problem (2) for $\Lambda = \Lambda_\theta^{\eta/2}$ and $D = \Lambda_\mu^{\nu/2}$. Then for I from (4) we have

$$(5) \quad I = \int_{\Pi^{d-1}} \frac{\left(\sum_{k=1}^d \tilde{t}_k^\theta(\omega)\right)^{\eta q^*/\theta}}{\left(\sum_{k=1}^d \tilde{t}_k^\mu(\omega)\right)^{\nu q^*(1-\gamma)/\mu}} J(\omega) d\omega,$$

where

$$\begin{aligned} \tilde{t}_1(\omega) &= \cos \omega_1, \\ \tilde{t}_2(\omega) &= \sin \omega_1 \cos \omega_2, \\ &\dots\dots\dots \\ \tilde{t}_{d-1}(\omega) &= \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \cos \omega_{d-1}, \\ \tilde{t}_d(\omega) &= \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \sin \omega_{d-1}. \end{aligned}$$

Note that

$$\sum_{k=1}^d \tilde{t}_k^2(\omega) = 1.$$

If $\mu \leq 2$, then

$$(6) \quad \sum_{k=1}^d \tilde{t}_k^\mu(\omega) \geq \sum_{k=1}^d \tilde{t}_k^2(\omega) = 1.$$

For $\mu > 2$ by Hölder's inequality

$$1 = \sum_{k=1}^d \tilde{t}_k^2(\omega) \leq \left(\sum_{k=1}^d \tilde{t}_k^\mu(\omega)\right)^{\frac{2}{\mu}} d^{1-\frac{2}{\mu}}.$$

Thus,

$$(7) \quad \sum_{k=1}^d \tilde{t}_k^\mu(\omega) \geq d^{1-\frac{\mu}{2}}.$$

It follows by (6) and (7) that $I < \infty$.

Corollary 1. *Let $2 < p \leq \infty$, $\nu > \eta \geq 0$, and $\theta, \mu > 0$. Then*

$$E_p(\Lambda_\theta^{\eta/2}, \Lambda_\mu^{\nu/2}) = \frac{1}{(2\pi)^{d\gamma/2}} C_p(\nu, \eta) I^{1/q^*} \delta^\gamma,$$

where I is defined by (5). The method

$$\hat{m}(y)(t) = F^{-1} \left(\left(1 - \beta \frac{\psi_\mu^\nu(t)}{\psi_\theta^\eta(t)} \right)_+ \psi_\theta^{\eta/2}(t) y(t) \right),$$

where

$$\beta = \frac{1-\gamma}{(2\pi)^{d\gamma}} C_p^2(\nu, \eta) \left(\delta I^{1/2-1/p} \right)^{2\gamma},$$

is optimal.

Moreover, the sharp inequality

$$\|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \frac{C_p(\nu, \eta) I^{1/q^*}}{(2\pi)^{d\gamma/2}} \|F x(\cdot)\|_{L_p(\mathbb{R}^d)}^\gamma \|\Lambda_\mu^{\nu/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\gamma}$$

holds.

Now we consider the case when $p = 2$.

Theorem 2. *Let $\nu > \eta > 0$ and $0 < \theta \leq \mu$. Then*

$$(8) \quad E_2(\Lambda_\theta^{\eta/2}, \Lambda_\mu^{\nu/2}) = d^{\eta(1/\theta-1/\mu)} \left(\frac{\delta}{(2\pi)^{d/2}} \right)^{1-\eta/\nu},$$

and all methods

$$(9) \quad \widehat{m}(y)(t) = F^{-1} \left(a(t) \psi_\theta^{\eta/2}(t) y(t) \right),$$

where $a(\cdot)$ are measurable functions satisfying the condition

$$(10) \quad \psi_\theta^\eta(\xi) \left(\frac{|1 - a(\xi)|^2}{\lambda_2 \psi_\mu^\nu(\xi)} + \frac{|a(\xi)|^2}{(2\pi)^d \lambda_1} \right) \leq 1,$$

in which

$$\lambda_1 = \frac{d^{2\eta(1/\theta-1/\mu)}}{(2\pi)^d} \left(1 - \frac{\eta}{\nu} \right) \left(\frac{(2\pi)^d}{\delta^2} \right)^{\eta/\nu}, \quad \lambda_2 = d^{2\eta(1/\theta-1/\mu)} \frac{\eta}{\nu} \left(\frac{(2\pi)^d}{\delta^2} \right)^{\eta/\nu-1},$$

are optimal.

The sharp inequality

$$(11) \quad \|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \frac{d^{\eta(1/\theta-1/\mu)}}{(2\pi)^{d(1-\eta/\nu)/2}} \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\eta/\nu} \|\Lambda_\mu^{\nu/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\eta/\nu}$$

holds.

Proof. It follows from (3) that

$$(12) \quad E_2(\Lambda_\theta^{\eta/2}, \Lambda_\mu^{\nu/2}) \geq \sup_{\substack{x(\cdot) \in W_2 \\ \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \delta}} \|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}.$$

Consider the extremal problem

$$\|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \rightarrow \max, \quad \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \leq \delta^2, \quad \|\Lambda_\mu^{\nu/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \leq 1.$$

Given $0 < \varepsilon < d^{-1/\mu} (2\pi)^{d/\nu} \delta^{-2/\nu}$, we set

$$\widehat{\xi}_\varepsilon = \frac{1}{d^{1/\mu}} \left(\frac{(2\pi)^d}{\delta^2} \right)^{\frac{1}{2\nu}} (1, \dots, 1) - (\varepsilon, \dots, \varepsilon), \quad B_\varepsilon = \{\xi \in \mathbb{R}^d : |\xi - \widehat{\xi}_\varepsilon| < \varepsilon\}.$$

Consider a function $x_\varepsilon(\cdot)$ such that

$$Fx_\varepsilon(\xi) = \begin{cases} \frac{\delta}{\sqrt{\text{mes } B_\varepsilon}}, & \xi \in B_\varepsilon, \\ 0, & \xi \notin B_\varepsilon. \end{cases}$$

Then $\|Fx_\varepsilon(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \delta^2$ and

$$\|\Lambda_\mu^{\nu/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{\delta^2}{(2\pi)^d \text{mes } B_\varepsilon} \int_{B_\varepsilon} (|\xi_1|^\mu + \dots + |\xi_d|^\mu)^{2\nu/\mu} d\xi \leq 1.$$

By virtue of (12) we have

$$\begin{aligned} E_2^2(\Lambda_\theta^{\eta/2}, \Lambda_\mu^{\nu/2}) &\geq \|\Lambda_\theta^{\eta/2} x_\varepsilon(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \\ &= \frac{\delta^2}{(2\pi)^d \text{mes } B_\varepsilon} \int_{B_\varepsilon} \psi_\theta^\eta(\xi) d\xi = \frac{\delta^2}{(2\pi)^d} \psi_\theta^\eta(\widetilde{\xi}_\varepsilon), \quad \widetilde{\xi}_\varepsilon \in B_\varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we obtain the estimate

$$(13) \quad E_2^2(\Lambda_\theta^{\eta/2}, \Lambda_\mu^{\nu/2}) \geq d^{2\eta(1/\theta-1/\mu)} \left(\frac{\delta^2}{(2\pi)^d} \right)^{1-\eta/\nu}.$$

We will find optimal methods among methods (9). Passing to the Fourier transform we have

$$\|\Lambda_\theta^{\eta/2} x(\cdot) - \widehat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi_\theta^\eta(\xi) |Fx(\xi) - a(\xi)y(\xi)|^2 d\xi.$$

We set $z(\cdot) = Fx(\cdot) - y(\cdot)$ and note that

$$\int_{\mathbb{R}^d} |z(\xi)|^2 d\xi \leq \delta^2, \quad \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi_\mu^\nu(\xi) |Fx(\xi)|^2 d\xi \leq 1.$$

Then

$$\|\Lambda_\theta^{\eta/2} x(\cdot) - \widehat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi_\theta^\eta(\xi) |(1-a(\xi))Fx(\xi) + a(\xi)z(\xi)|^2 d\xi.$$

We write the integrand as

$$\psi_\theta^\eta(\xi) \left| \frac{(1-a(\xi))\sqrt{\lambda_2}\psi_\mu^{\nu/2}(\xi)Fx(\xi)}{\sqrt{\lambda_2}\psi_\mu^{\nu/2}(\xi)} + \frac{a(\xi)}{(2\pi)^{d/2}\sqrt{\lambda_1}}(2\pi)^{d/2}\sqrt{\lambda_1}z(\xi) \right|^2.$$

Applying the Cauchy-Bunyakovskii-Schwarz inequality we obtain the estimate

$$\begin{aligned} \|\Lambda_\theta^{\eta/2} x(\cdot) - \widehat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 &\leq \sup_{\xi \in \mathbb{R}^d} S(\xi) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (\lambda_2 \psi_\mu^\nu(\xi) |Fx(\xi)|^2 + (2\pi)^d \lambda_1 |z(\xi)|^2) d\xi, \end{aligned}$$

where

$$S(\xi) = \psi_\theta^\eta(\xi) \left(\frac{|1-a(\xi)|^2}{\lambda_2 \psi_\mu^\nu(\xi)} + \frac{|a(\xi)|^2}{(2\pi)^d \lambda_1} \right).$$

If we assume that $S(\xi) \leq 1$ for almost all ξ , then taking into account (13), we get

$$\begin{aligned} (14) \quad e_2^2(\Lambda_\theta^{\eta/2}, \Lambda_\mu^{\nu/2}, \widehat{m}) &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (\lambda_2 \psi_\mu^\nu(\xi) |Fx(\xi)|^2 + (2\pi)^d \lambda_1 |z(\xi)|^2) d\xi \leq \lambda_2 + \lambda_1 \delta^2 \\ &= d^{2\eta(1/\theta-1/\mu)} \left(\frac{\delta^2}{(2\pi)^d} \right)^{1-\eta/\nu} \leq E_2^2(\Lambda_\theta^{\eta/2}, \Lambda_\mu^{\nu/2}). \end{aligned}$$

This proves (8) and shows that the methods under consideration are optimal.

It remains to verify that the set of functions $a(\cdot)$ satisfying (10) is nonempty. Put

$$a(\xi) = \frac{(2\pi)^d \lambda_1}{(2\pi)^d \lambda_1 + \lambda_2 \psi_\mu^\nu(\xi)}.$$

Then

$$S(\xi) = \frac{\psi_\theta^\eta(\xi)}{(2\pi)^d \lambda_1 + \lambda_2 \psi_\mu^\nu(\xi)}.$$

Since $\theta \leq \mu$ by Hölder's inequality

$$\sum_{j=1}^d |\xi_j|^\theta \leq \left(\sum_{j=1}^d |\xi_j|^\mu \right)^{\theta/\mu} d^{1-\theta/\mu}.$$

Putting $\rho = (|\xi_1|^\theta + \dots + |\xi_d|^\theta)^{1/\theta}$, we obtain

$$\sum_{j=1}^d |\xi_j|^\mu \geq \rho^\mu d^{1-\mu/\theta}.$$

Thus,

$$S(\xi) \leq \frac{\rho^{2\eta}}{(2\pi)^d \lambda_1 + \lambda_2 \rho^{2\nu} d^{2\nu(1/\mu-1/\theta)}}.$$

It is easily checked that the function $f(\rho) = (2\pi)^d \lambda_1 + \lambda_2 \rho^{2\nu} d^{2\nu(1/\mu-1/\theta)} - \rho^{2\eta}$ reaches a minimum on $[0, +\infty)$ at

$$\rho_0 = d^{1/\theta-1/\mu} \left(\frac{(2\pi)^d}{\delta^2} \right)^{1/(2\nu)}.$$

Moreover, $f(\rho_0) = 0$. Consequently, $f(\rho) \geq 0$ for all $\rho \geq 0$. Hence $S(\xi) \leq 1$ for all ξ .

Let $x(\cdot) \in X_2$ for $\varphi(\cdot) = \psi_\mu^{\nu/2}(\cdot)$. Put $A = \|\Lambda_\mu^{\nu/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} + \varepsilon$, $\varepsilon > 0$. Consider $\widehat{x}(\cdot) = x(\cdot)/A$. Put $\delta = \|F\widehat{x}(\cdot)\|_{L_2(\mathbb{R}^d)}$. It follows from (14) that

$$\sup_{\substack{x(\cdot) \in W_2 \\ \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \delta}} \|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} = E_2(\Lambda_\theta^{\eta/2}, \Lambda_\mu^{\nu/2}) = d^{\eta(1/\theta-1/\mu)} \left(\frac{\delta}{(2\pi)^{d/2}} \right)^{1-\eta/\nu}. \quad (15)$$

Thus,

$$\|\Lambda_\theta^{\eta/2} \widehat{x}(\cdot)\|_{L_2(\mathbb{R}^d)} \leq d^{\eta(1/\theta-1/\mu)} \left(\frac{\delta}{(2\pi)^{d/2}} \right)^{1-\eta/\nu}.$$

Consequently,

$$\|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \frac{d^{\eta(1/\theta-1/\mu)}}{(2\pi)^{d(1-\eta/\nu)/2}} \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\eta/\nu} \left(\|\Lambda_\mu^{\nu/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} + \varepsilon \right)^{\eta/\nu}.$$

Letting $\varepsilon \rightarrow 0$ we obtain (11).

If there exists a

$$C < \frac{d^{\eta(1/\theta-1/\mu)}}{(2\pi)^{d(1-\eta/\nu)/2}}$$

for which

$$\|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq C \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\eta/\nu} \|\Lambda_\mu^{\nu/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\eta/\nu},$$

then

$$\sup_{\substack{x(\cdot) \in W_2 \\ \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \delta}} \|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq C \delta^{1-\eta/\nu} < \frac{d^{\eta(1/\theta-1/\mu)}}{(2\pi)^{d(1-\eta/\nu)/2}} \delta^{1-\eta/\nu}.$$

This contradicts with (15). $\square$

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}_+^d$. We define the operator D^α (the derivative of order α) by

$$D^\alpha x(\cdot) = F^{-1}((i\xi)^\alpha Fx(\xi))(\cdot),$$

where $(i\xi)^\alpha = (i\xi_1)^{\alpha_1} \dots (i\xi_d)^{\alpha_d}$. For $\mu = 2\nu$ we have

$$\begin{aligned} \|\Lambda_{2\nu}^{\nu/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^2 &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (|\xi_1|^{2\nu} + \dots + |\xi_d|^{2\nu}) |Fx(\xi)|^2 d\xi \\ &= \sum_{j=1}^d \|D^{\nu e_j} x(\cdot)\|_{L_2(\mathbb{R}^d)}^2, \end{aligned}$$

where $e_j, j = 1 \dots, d$, is a standard basis in $\mathbb{R}^d$.

From Theorem 2 we obtain the following result.

Corollary 2. *Let $\nu > \eta > 0$ and $0 < \theta \leq 2\nu$. Then*

$$E_2(\Lambda_\theta^{\eta/2}, \Lambda_{2\nu}^{\nu/2}) = d^{\eta(1/\theta - 1/(2\nu))} \left(\frac{\delta}{(2\pi)^{d/2}} \right)^{1-\eta/\nu},$$

and all methods

$$\widehat{m}(y)(t) = F^{-1} \left(a(t) \psi_\theta^{\eta/2}(t) y(t) \right),$$

where $a(\cdot)$ are measurable functions satisfying the condition

$$\psi_\theta^\eta(\xi) \left(\frac{|1 - a(\xi)|^2}{\lambda_2 \psi_{2\nu}^\nu(\xi)} + \frac{|a(\xi)|^2}{(2\pi)^d \lambda_1} \right) \leq 1,$$

in which

$$\begin{aligned} \lambda_1 &= \frac{d^{2\eta(1/\theta - 1/(2\nu))}}{(2\pi)^d} \left(1 - \frac{\eta}{\nu} \right) \left(\frac{(2\pi)^d}{\delta^2} \right)^{\eta/\nu}, \\ \lambda_2 &= d^{2\eta(1/\theta - 1/(2\nu))} \frac{\eta}{\nu} \left(\frac{(2\pi)^d}{\delta^2} \right)^{\eta/\nu - 1}, \end{aligned}$$

are optimal.

The sharp inequality

(16)

$$\|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \frac{d^{\eta(1/\theta - 1/(2\nu))}}{(2\pi)^{d(1-\eta/\nu)/2}} \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\eta/\nu} \left(\sum_{j=1}^d \|D^{\nu e_j} x(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \right)^{\eta/(2\nu)}$$

holds.

For integer ν inequality (16) can be rewritten in the form

$$\|\Lambda_\theta^{\eta/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \frac{d^{\eta(1/\theta - 1/(2\nu))}}{(2\pi)^{d(1-\eta/\nu)/2}} \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\eta/\nu} \left(\sum_{j=1}^d \left\| \frac{\partial^\nu x}{\partial t_j^\nu}(\cdot) \right\|_{L_2(\mathbb{R}^d)}^2 \right)^{\eta/(2\nu)}.$$

4. OPTIMAL RECOVERY METHODS FOR D^α

Now we consider problem (2) for $\Lambda = D^\alpha$ and $D = \Lambda_\mu^{\nu/2}$. Then for I from (4) we have

$$(17) \quad I = \int_{\Pi^{d-1}} \frac{(\tilde{t}_1^{\alpha_1}(\omega) \dots \tilde{t}_d^{\alpha_d}(\omega))^{q_1}}{\left(\sum_{k=1}^d \tilde{t}_k^\mu(\omega) \right)^{\nu q_1(1-\gamma_1)/\mu}} J(\omega) d\omega,$$

where

$$\gamma_1 = \frac{\nu - |\alpha|}{\nu + d(1/2 - 1/p)}, \quad q_1 = \frac{1}{\gamma_1(1/2 - 1/p)}, \quad |\alpha| = \alpha_1 + \dots + \alpha_d.$$

It follows by (6) and (7) that $I < \infty$.

Corollary 3. *Let $2 < p \leq \infty$, $\nu > |\alpha| \geq 0$, and $\mu > 0$. Then*

$$E_p(D^\alpha, \Lambda_\mu^{\nu/2}) = \frac{1}{(2\pi)^{d\gamma_1/2}} C_p(\nu, |\alpha|) I^{1/q_1} \delta^{\gamma_1},$$

where I is defined by (17). The method

$$\widehat{m}(y)(t) = F^{-1} \left(\left(1 - \beta \frac{\psi_\mu^\nu(t)}{t_1^{2\alpha_1} \dots t_d^{2\alpha_d}} \right)_+ (it)^\alpha y(t) \right),$$

where

$$\beta = \frac{1 - \gamma_1}{(2\pi)^{d\gamma_1}} C_p^2(\nu, |\alpha|) \left(\delta I^{1/2-1/p} \right)^{2\gamma_1},$$

is optimal.

Moreover, the sharp inequality

$$\|D^\alpha x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \frac{C_p(\nu, |\alpha|) I^{1/q_1}}{(2\pi)^{d\gamma_1/2}} \|Fx(\cdot)\|_{L_p(\mathbb{R}^d)}^{\gamma_1} \|\Lambda_\mu^{\nu/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\gamma_1}$$

holds.

Consider the case when $p = 2$.

Theorem 3. *Let $2\nu \geq \mu > 0$ and $0 < |\alpha| < \nu$. Then*

$$(18) \quad E_2(D^\alpha, \Lambda_\mu^{\nu/2}) = \left(\frac{\delta}{(2\pi)^{d/2}} \right)^{1-|\alpha|/\nu} |\alpha|^{-|\alpha|/\mu} \prod_{\substack{j=1 \\ \alpha_j \neq 0}}^d \alpha_j^{\alpha_j/\mu},$$

and all methods

$$(19) \quad \widehat{m}(y)(t) = F^{-1} (a(t)(it)^\alpha y(t)),$$

where $a(\cdot)$ are measurable functions satisfying the condition

$$(20) \quad \psi_\theta^\eta(\xi) \left(\frac{|1 - a(\xi)|^2}{\lambda_2 \psi_\mu^\nu(\xi)} + \frac{|a(\xi)|^2}{(2\pi)^d \lambda_1} \right) \leq 1,$$

in which

$$\lambda_1 = \frac{|\alpha|^{-2|\alpha|/\mu}}{(2\pi)^d} \left(1 - \frac{|\alpha|}{\nu} \right) \left(\frac{(2\pi)^d}{\delta^2} \right)^{|\alpha|/\nu} \prod_{\substack{j=1 \\ \alpha_j \neq 0}}^d \alpha_j^{2\alpha_j/\mu},$$

$$\lambda_2 = \frac{|\alpha|^{-2|\alpha|/\mu+1}}{\nu} \left(\frac{(2\pi)^d}{\delta^2} \right)^{|\alpha|/\nu-1} \prod_{\substack{j=1 \\ \alpha_j \neq 0}}^d \alpha_j^{2\alpha_j/\mu},$$

are optimal.

The sharp inequality

$$(21) \quad \|D^\alpha x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \frac{|\alpha|^{-|\alpha|/\mu}}{(2\pi)^{d(1-|\alpha|/\nu)/2}} \prod_{\substack{j=1 \\ \alpha_j \neq 0}}^d \alpha_j^{\alpha_j/\mu} \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-|\alpha|/\nu} \|\Lambda_\mu^{\nu/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{|\alpha|/\nu}$$

holds.

Proof. It follows from (3) that

$$(22) \quad E_2(D^\alpha, \Lambda_\mu^{\nu/2}) \geq \sup_{\substack{x(\cdot) \in W_2 \\ \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \delta}} \|D^\alpha x(\cdot)\|_{L_2(\mathbb{R}^d)}.$$

Consider the extremal problem

$$\|D^\alpha x(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \rightarrow \max, \quad \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \leq \delta^2, \quad \|\Lambda_\mu^{\nu/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \leq 1.$$

Given

$$0 < \varepsilon < \min \left\{ |\alpha|^{-1/\mu} \left(\frac{(2\pi)^d}{\delta^2} \right)^{1/(2\nu)} \alpha_j^{1/\mu} : \alpha_j > 0, j = 1, \dots, d \right\},$$

we set

$$\begin{aligned} \widehat{\xi}_\varepsilon &= |\alpha|^{-1/\mu} \left(\frac{(2\pi)^d}{\delta^2} \right)^{1/(2\nu)} (\alpha_1^{1/\mu}, \dots, \alpha_d^{1/\mu}) - (\varepsilon_1, \dots, \varepsilon_d), \quad \varepsilon_j = \begin{cases} \varepsilon, & \alpha_j > 0, \\ 0, & \alpha_j = 0, \end{cases} \\ B_\varepsilon &= \{\xi \in \mathbb{R}^d : |\xi - \widehat{\xi}_\varepsilon| < \varepsilon\}. \end{aligned}$$

Consider a function $x_\varepsilon(\cdot)$ such that

$$Fx_\varepsilon(\xi) = \begin{cases} \frac{\delta}{\sqrt{\text{mes } B_\varepsilon}} \left(1 + d\varepsilon^\nu \left(\frac{\delta^2}{(2\pi)^d} \right)^{\mu/(2\nu)} \right)^{-1/2}, & \xi \in B_\varepsilon, \\ 0, & \xi \notin B_\varepsilon. \end{cases}$$

Then

$$\|Fx_\varepsilon(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \delta^2 \left(1 + d\varepsilon^\nu \left(\frac{\delta^2}{(2\pi)^d} \right)^{\mu/(2\nu)} \right)^{-1} \leq \delta^2$$

and

$$\begin{aligned} & \|\Lambda_\mu^{\nu/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \\ &= \frac{\delta^2}{(2\pi)^d \text{mes } B_\varepsilon} \left(1 + d\varepsilon^\nu \left(\frac{\delta^2}{(2\pi)^d} \right)^{\mu/(2\nu)} \right)^{-1} \int_{B_\varepsilon} (|\xi_1|^\mu + \dots + |\xi_d|^\mu)^{2\nu/\mu} d\xi \\ &\leq \frac{\delta^2}{(2\pi)^d} \left(1 + d\varepsilon^\nu \left(\frac{\delta^2}{(2\pi)^d} \right)^{\mu/(2\nu)} \right)^{-1} \left(\left(\frac{(2\pi)^d}{\delta^2} \right)^{\mu/(2\nu)} |\alpha|^{-1} \sum_{j=1}^d \alpha_j + d\varepsilon^\mu \right)^{2\nu/\mu} \\ &= 1. \end{aligned}$$

By virtue of (22) we have

$$\begin{aligned} E_2^2(D^\alpha, \Lambda_\mu^{\nu/2}) &\geq \|D^\alpha x_\varepsilon(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \\ &= \frac{\delta^2}{(2\pi)^d \text{mes } B_\varepsilon} \left(1 + d\varepsilon^\nu \left(\frac{\delta^2}{(2\pi)^d} \right)^{\mu/(2\nu)} \right)^{-1} \int_{B_\varepsilon} |\xi_1|^{2\alpha_1} \dots |\xi_d|^{2\alpha_d} d\xi \\ &= \frac{\delta^2}{(2\pi)^d} \left(1 + d\varepsilon^\nu \left(\frac{\delta^2}{(2\pi)^d} \right)^{\mu/(2\nu)} \right)^{-1} |\widetilde{\xi}_1|^{2\alpha_1} \dots |\widetilde{\xi}_d|^{2\alpha_d}, \quad (\widetilde{\xi}_1, \dots, \widetilde{\xi}_d) \in B_\varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we obtain the estimate

$$(23) \quad E_2^2(D^\alpha, \Lambda_\mu^{\nu/2}) \geq \left(\frac{\delta^2}{(2\pi)^d} \right)^{1-|\alpha|/\nu} |\alpha|^{-2|\alpha|/\mu} \prod_{\substack{j=1 \\ \alpha_j \neq 0}}^d \alpha_j^{2\alpha_j/\mu}.$$

We will find optimal methods among methods (19). Passing to the Fourier transform we have

$$\|D^\alpha x(\cdot) - \widehat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi_1|^{2\alpha_1} \dots |\xi_d|^{2\alpha_d} |Fx(\xi) - a(\xi)y(\xi)|^2 d\xi.$$

We set $z(\cdot) = Fx(\cdot) - y(\cdot)$ and note that

$$\int_{\mathbb{R}^d} |z(\xi)|^2 d\xi \leq \delta^2, \quad \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi_\mu^\nu(\xi) |Fx(\xi)|^2 d\xi \leq 1.$$

Then

$$\begin{aligned} \|D^\alpha x(\cdot) - \widehat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi_1|^{2\alpha_1} \dots |\xi_d|^{2\alpha_d} |(1 - a(\xi)) Fx(\xi) + a(\xi)z(\xi)|^2 d\xi. \end{aligned}$$

We write the integrand as

$$|\xi_1|^{2\alpha_1} \dots |\xi_d|^{2\alpha_d} \left| \frac{(1 - a(\xi))\sqrt{\lambda_2}\psi_\mu^{\nu/2}(\xi) Fx(\xi)}{\sqrt{\lambda_2}\psi_\mu^{\nu/2}(\xi)} + \frac{a(\xi)}{(2\pi)^{d/2}\sqrt{\lambda_1}} (2\pi)^{d/2}\sqrt{\lambda_1}z(\xi) \right|^2.$$

Applying the Cauchy-Bunyakovskii-Schwarz inequality we obtain the estimate

$$\begin{aligned} \|D^\alpha x(\cdot) - \widehat{m}(y)(\cdot)\|_{L_2(\mathbb{R}^d)}^2 &\leq \text{vraisup}_{\xi \in \mathbb{R}^d} S(\xi) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (\lambda_2 \psi_\mu^\nu(\xi) |Fx(\xi)|^2 + (2\pi)^d \lambda_1 |z(\xi)|^2) d\xi, \end{aligned}$$

where

$$S(\xi) = |\xi_1|^{2\alpha_1} \dots |\xi_d|^{2\alpha_d} \left(\frac{|1 - a(\xi)|^2}{\lambda_2 \psi_\mu^\nu(\xi)} + \frac{|a(\xi)|^2}{(2\pi)^d \lambda_1} \right).$$

If we assume that $S(\xi) \leq 1$ for almost all ξ , then taking into account (23), we get

$$\begin{aligned} (24) \quad e_2^2(D^\alpha, \Lambda_\mu^{\nu/2}, \widehat{m}) &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (\lambda_2 \psi_\mu^\nu(\xi) |Fx(\xi)|^2 + (2\pi)^d \lambda_1 |z(\xi)|^2) d\xi \leq \lambda_2 + \lambda_1 \delta^2 \\ &= \left(\frac{\delta^2}{(2\pi)^d} \right)^{1-|\alpha|/\nu} |\alpha|^{-2|\alpha|/\mu} \prod_{\substack{j=1 \\ \alpha_j \neq 0}}^d \alpha_j^{2\alpha_j/\mu} \leq E_2^2(D^\alpha, \Lambda_\mu^{\nu/2}). \end{aligned}$$

This proves (18) and shows that the methods under consideration are optimal.

It remains to verify that the set of functions $a(\cdot)$ satisfying (20) is nonempty. Put

$$a(\xi) = \frac{(2\pi)^d \lambda_1}{(2\pi)^d \lambda_1 + \lambda_2 \psi_\mu^\nu(\xi)}.$$

Then

$$S(\xi) = \frac{|\xi_1|^{2\alpha_1} \dots |\xi_d|^{2\alpha_d}}{(2\pi)^d \lambda_1 + \lambda_2 \psi_\mu^\nu(\xi)}.$$

Consider the function

$$H(t) = -1 + (2\pi)^d \lambda_1 e^{-2(\alpha, t)} + \lambda_2 \left(\sum_{j=1}^d e^{\mu t_j - \frac{\mu}{\nu}(\alpha, t)} \right)^{2\nu/\mu},$$

where $(\alpha, t) = \alpha_1 t_1 + \dots + \alpha_d t_d$. It is easy to prove that $H(\cdot)$ is a convex function. Moreover, $H(\hat{t}) = 0$ and the derivative of $H(\cdot)$ at the point $\hat{t}$ is also zero, where

$$\hat{t} = \left(\frac{1}{\mu} \ln \frac{\alpha_1}{|\alpha|} + \frac{1}{2\nu} \ln \frac{(2\pi)^d}{\delta^2}, \dots, \frac{1}{\mu} \ln \frac{\alpha_d}{|\alpha|} + \frac{1}{2\nu} \ln \frac{(2\pi)^d}{\delta^2} \right).$$

Consequently, $H(t) \geq 0$ for all $t \in \mathbb{R}^d$. It means that

$$e^{-2(\alpha, t)} \leq (2\pi)^d \lambda_1 + \lambda_2 \left(\sum_{j=1}^d e^{\mu t_j} \right)^{2\nu/\mu}.$$

Put $|\xi_j| = e^{t_j}$, $j = 1, \dots, d$. Then we obtain

$$|\xi_1|^{2\alpha_1} \dots |\xi_d|^{2\alpha_d} \leq (2\pi)^d \lambda_1 + \lambda_2 \psi_\mu^\nu(\xi).$$

Thus, $S(\xi) \leq 1$.

The proof of (21) is similar to the proof of (11). □

For $\mu = 2\nu$ we obtain

Corollary 4. *Let $0 < |\alpha| < \nu$. Then*

$$E_2(D^\alpha, \Lambda_{2\nu}^{\nu/2}) = \left(\frac{\delta}{(2\pi)^{d/2}} \right)^{1-|\alpha|/\nu} |\alpha|^{-|\alpha|/(2\nu)} \prod_{\substack{j=1 \\ \alpha_j \neq 0}}^d \alpha_j^{\alpha_j/(2\nu)},$$

and all methods

$$\widehat{m}(y)(t) = F^{-1}(a(t)(it)^\alpha y(t)),$$

where $a(\cdot)$ are measurable functions satisfying the condition

$$\psi_\theta^\eta(\xi) \left(\frac{|1 - a(\xi)|^2}{\lambda_2 \psi_{2\nu}^\nu(\xi)} + \frac{|a(\xi)|^2}{(2\pi)^d \lambda_1} \right) \leq 1,$$

in which

$$\lambda_1 = \frac{1}{(2\pi)^d} \left(1 - \frac{|\alpha|}{\nu} \right) \left(\frac{(2\pi)^d}{|\alpha| \delta^2} \right)^{|\alpha|/\nu} \prod_{\substack{j=1 \\ \alpha_j \neq 0}}^d \alpha_j^{\alpha_j/\nu},$$

$$\lambda_2 = \frac{|\alpha|}{\nu} \left(\frac{(2\pi)^d}{|\alpha| \delta^2} \right)^{|\alpha|/\nu - 1} \prod_{\substack{j=1 \\ \alpha_j \neq 0}}^d \alpha_j^{\alpha_j/\nu},$$

are optimal.

The sharp inequality

$$(25) \quad \|D^\alpha x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \frac{|\alpha|^{-|\alpha|/(2\nu)}}{(2\pi)^{d(1-|\alpha|/\nu)/2}} \prod_{\substack{j=1 \\ \alpha_j \neq 0}}^d \alpha_j^{\alpha_j/(2\nu)} \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-|\alpha|/\nu} \left(\sum_{j=1}^d \|D^{\nu e_j} x(\cdot)\|_{L_2(\mathbb{R}^d)}^2 \right)^{|\alpha|/(2\nu)}$$

holds.

For integer ν inequality (25) can be rewritten in the form

$$\begin{aligned} & \|D^\alpha x(\cdot)\|_{L_2(\mathbb{R}^d)} \\ & \leq \frac{|\alpha|^{-|\alpha|/(2\nu)}}{(2\pi)^{d(1-|\alpha|/\nu)/2}} \prod_{\substack{j=1 \\ \alpha_j \neq 0}}^d \alpha_j^{\alpha_j/(2\nu)} \|Fx(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-|\alpha|/\nu} \left(\sum_{j=1}^d \left\| \frac{\partial^\nu x}{\partial t_j^\nu}(\cdot) \right\|_{L_2(\mathbb{R}^d)}^2 \right)^{|\alpha|/(2\nu)}. \end{aligned}$$

REFERENCES

- [1] S. A. Smolyak, On optimal recovery of functions and functionals of them, Diss... Cand. Phys.-Math. Sci., Moscow State Univ., Moscow, 1965. (Russian)
- [2] S. M. Nikol'skii, "On the question of estimates for approximation by quadrature formulae", *Uspekhi Mat. Nauk* **5**:2(36) (1950), 165–177. (Russian)
- [3] C. A. Micchelli and T. J. Rivlin, "A survey of optimal recovery", *Optimal estimation in approximation theory*, Proc. Internat. Sympos. (Freudenstadt 1976), Plenum, New York 1977, pp. 1–54.
- [4] A. A. Melkman and C. A. Micchelli, "Optimal estimation of linear operators in Hilbert spaces from inaccurate data" *SIAM J. Numer. Anal.*, **16** (1979), 87–105.
- [5] J. F. Traub and H. Woźniakowski, *A general theory of optimal algorithms*, ACM Monograph Series, Academic Press, New York–London 1980; Russian transl., Mir, Moscow 1983.
- [6] V. V. Arestov, "Optimal recovery of operators and related problems", *Proc. All-Union School in Function Theory* (Dushanbe, August 1986), Trudy Mat. Inst. Steklov., vol. 189, Nauka, Moscow 1989, pp. 3–20; English transl., *Proc. Steklov Inst. Math.* **189**:4 (1990), 1–20.
- [7] G. G. Magaril-Il'yaev and K. Yu. Osipenko, "Optimal recovery of functionals based on inaccurate data", *Mat. Zametki* **50**:6 (1991), 85–93; English transl., *Math. Notes* **50**:6 (1991), 1274–1279.
- [8] L. Plaskota, *Noisy information and computational complexity*, Cambridge University Press, Cambridge 1996.
- [9] K. Yu. Osipenko, *Optimal recovery of analytic functions*, Nova Science Publ., Inc., Huntington, New York 2000.
- [10] G. G. Magaril-Il'yaev and V. M. Tikhomirov, *Convex analysis: theory and applications*, Editorial URSS, Moscow 2011; English transl. of 1st ed., Transl. Math. Monogr., vol. 222, Amer. Math. Soc., Providence, RI 2003.
- [11] K. Yu. Osipenko, *Introduction to optimal recovery theory*, Lan Publishing House, St. Petersburg. 2022 (Russian).
- [12] G. G. Magaril-Il'yaev and K. Yu. Osipenko, "Optimal recovery of functions and their derivatives from Fourier coefficients prescribed with an error", *Mat. Sb.* **193**:3 (2002), 79–100; English transl., *Sb. Math.* **193**:3 (2002), 387–407.
- [13] G. G. Magaril-Il'yaev and K. Yu. Osipenko, "Optimal recovery of functions and their derivatives from inaccurate information about a spectrum and inequalities for derivatives", *Funktsional. Anal. i Prilozhen.* **37**:3 (2003), 51–64; English transl., *Funct. Anal. Appl.* **37**:3 (2003), 203–214.
- [14] K. Yu. Osipenko, "The Hardy–Littlewood–Pólya inequality for analytic functions from Hardy–Sobolev spaces", *Mat. Sb.* **197**:3 (2006), 15–34; English transl., *Sb. Math.* **197**:3 (2006), 315–334.
- [15] K. Yu. Osipenko, "Optimal recovery of linear operators in non-Euclidean metrics", *Mat. Sb.* **205**:10 (2014), 77–106; English transl., *Sb. Math.* **205**:10 (2014), 1442–1472.
- [16] K. Yu. Osipenko, "Optimal recovery of operators and multidimensional Carlson type inequalities", *J. Complexity* **32**:1 (2016), 53–73.
- [17] V. V. Arestov, "Best uniform approximation of the differentiation operator by operators bounded in the space L_2 ", *Tr. Inst. Mat. Mekh.* **24**:4 (2018), 34–56; English transl., *Proc. Steklov Inst. Math.* **308**:1 (2020), 9–30.
- [18] V. V. Arestov, "Best approximation of a differentiation operator on the set of smooth functions with exactly or approximately given Fourier transform", *Mathematical Optimization Theory and Operation Research. MOTOR 2019*. Lecture Notes in Comput. Sci., vol. 11548, Springer, Cham 2019, pp. 434–448.

- [19] V. V. Arestov, “Uniform approximation of differentiation operators by bounded linear operators in the space L_r ”, *Anal. Math.*, **46**:3 (2020), 425–445.
- [20] K. Yu. Osipenko, “Optimal recovery in weighted spaces with homogeneous weights”, *Mat. Sb.* **213**:3 (2022), 111–138; English transl., *Sb. Math.* **213**:3 (2022), 385–411.
- [21] G. G. Magaril-Il’yaev and K. Yu. Osipenko, “On optimal harmonic synthesis from inaccurate spectral data”, *Funktsional. Anal. i Prilozhen.* **44**:3 (2010), 76–79; English transl., *Funct. Anal. Appl.* **44**:3 (2010), 223–225.
- [22] G. G. Magaril-Il’yaev and K. Yu. Osipenko, “The Hardy–Littlewood–Pólya inequality and the reconstruction of derivatives from inaccurate data”, *Dokl. Ross. Akad. Nauk* **438**:3 (2011), 300–302; English transl., *Dokl. Math.* **83**:3 (2011), 337–339.
- [23] G. G. Magaril-Il’yaev and K. Yu. Osipenko, “On optimal recovery of solutions to difference equations from inaccurate data”, *Probl. Mat. Anal.* **69** (2013), 47–54; English transl., *J. Math. Sci., New York* **189**:4 (2013), 596–603.
- [24] K. Yu. Osipenko, “On the construction of families of optimal recovery methods for linear operators”, *Izv. RAN. Ser. Mat.* **88**:1 (2024), 98–120.
- [25] K. Yu. Osipenko, “Optimal recovery and generalized Carlson inequality for weights with symmetry properties”, *J. Complexity* **81** 101807 (2024), pp.35.

MOSCOW STATE UNIVERSITY, INSTITUTE OF INFORMATION TRANSMISSION PROBLEMS (KHARKEVICH INSTITUTE), RAS, MOSCOW

E-mail address: `kosipenko@yahoo.com`