

ON OPTIMAL EXTRAPOLATION AND INTERPOLATION OF FUZZY ANALYTIC FUNCTIONS

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§1. INTRODUCTION

Let \mathcal{B} denote the class of analytic functions on the unit disc $K = \{z : |z| < 1\}$ with modulus at most 1. We consider the problem of finding the best approximation of a function $f \in \mathcal{B}$ at some point $z_0 \in K$, knowing its values at the points $z_1, \dots, z_n \in K$ with an error $\leq \delta$, i.e. under the condition of knowing some values f_1, \dots, f_n which satisfy the inequalities

$$|f(z_j) - f_j| \leq \delta, \quad j = 1, \dots, n.$$

We call

$$(1.1) \quad r(z_0, z_1, \dots, z_n, \delta) = \inf_S \sup_{f \in \mathcal{B}} \sup_{\substack{f_1, \dots, f_n \\ |f(z_j) - f_j| \leq \delta, j=1, \dots, n}} |f(z_0) - S(f_1, \dots, f_n)|$$

the error of the best approximation, where the infimum is taken on the set of all functions (called methods) $S: \mathbb{C}^n \rightarrow \mathbb{C}$. A method for which the infimum indicated in (1.1) is attained, is called optimal.

Let E be a subset of K . We call

$$(1.2) \quad r_n(\delta, E) = \inf_{z_1, \dots, z_n \in E} \sup_{z_0 \in E} r(z_0, z_1, \dots, z_n, \delta)$$

the error of the optimal interpolation on E for n points. Those knots for which the infimum in (1.2) is attained are called optimal knots of interpolation.

Problems (1.1), (1.2) and similar ones were investigated for $\delta = 0$ in [1], [2], [5–7], [10–12]. The case $\delta \geq 0$ of (1.1) was dealt with in [10] (for $n = 1$) and in [8] (for $z_0, z_1, \dots, z_n \in (-1, 1)$). In these papers it was shown that

$$(1.3) \quad r(z_0, z_1, \dots, z_n, \delta) = \sup_{\substack{f \in \mathcal{B} \\ |f(z_j)| \leq \delta, j=1, \dots, n}} |f(z_0)|$$

(cf. also [5]).

Set

$$(1.4) \quad R_n(z_0, \delta, E) = \inf_{z_1, \dots, z_n \in E} r(z_0, z_1, \dots, z_n, \delta),$$

$$(1.5) \quad R(z_0, \delta, E) = \inf_n R_n(z_0, \delta, E).$$

It is worth while considering the problems (1.4) and (1.5) only for $z_0 \notin E$ and $0 < \delta < 1$, since else their solution is obvious. We call $R_n(z_0, \delta, E)$ the error of the optimal extrapolation at z_0 for n points from E , and the points where the infimum (1.4) is attained will be called optimal knots of extrapolation for the given n .

If there exist numbers n for which the infimum (1.5) is attained, then the smallest of them is called the informativity order of E , and will be denoted by $\text{In}(z_0, \delta, E)$. In the opposite case we say that E has infinite informativity order. The optimal knots for $n = \text{In}(z_0, \delta, E)$ will be called optimal knots of extrapolation on E (for given z_0 and δ). Thus, the informativity order of E is the minimal number of points from E giving the best possible extrapolation of a function in a point outside E provided the function is known on E with error δ .

Problems (1.4) and (1.5) for $E = [-l, 0] \subset (-1, 1)$, $E = (-1, 0)$ and $z_0 \in (0, 1)$ were investigated in [9]. In the present paper we deal with the same problems for arbitrary closed sets $E \subset (-1, 1)$ and $z_0 \in (-1, 1) \setminus E$, as well as with (1.2) for $E = [\alpha, \beta] \subset (-1, 1)$ and for δ near to 1.

§2. OPTIMAL EXTRAPOLATION

Let E be a closed subset of K . Put

$$\|f\|_E = \max_{z \in E} |f(z)|.$$

First we discuss the problem of Blaschke products

$$(2.1) \quad B_n(z) = \lambda \prod_{j=1}^n \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}, \quad |\alpha_j| < 1, \quad |\lambda| = 1,$$

of order n with minimal norm

$$(2.2) \quad \delta_n(E) = \inf_{B_n} \|B_n\|_E,$$

on E .

The quantity $\delta_n(E)$ was dealt with in [4], [6] for $E = [a, b] \subset (-1, 1)$. In [7] it was proved that for every closed set $E \subset K$ the inequalities

$$(2.3) \quad \delta_n(E) \geq \exp \left[-\frac{n}{c(E)} \right], \quad n \geq 1,$$

$$(2.4) \quad \delta_n(E) < \exp \left[-\frac{n}{c(E) + \varepsilon} \right]$$

hold for arbitrary $\varepsilon > 0$ and for sufficiently large n , where $c(E)$ is the capacity of the condenser $(E, \mathbb{C} \setminus K)$. Let us also mention that problem (2.2) is closely connected with the problem of widths of classes of analytic functions (cf. [2]).

We assume that $E \subset (-1, 1)$. Let

$$\rho = \max_{z \in E} |z|, \quad \alpha^* = \min\{\rho, |\text{Re } \alpha|\} \text{ sign } \text{Re } \alpha.$$

It is easy to see that

$$(2.5) \quad \left| \frac{z - \alpha}{1 - \bar{\alpha}z} \right| \geq \left| \frac{z - \alpha^*}{1 - \alpha^*z} \right|$$

for all $z \in [-\rho, \rho] \supset E$ and all $\alpha \in K$. Hence, in determining the number (2.2), we can confine ourselves to Blaschke products (2.1) with $\alpha_j \in [-\rho, \rho]$.

From the inequality

$$\left| \frac{z - \alpha}{1 - \alpha z} - \frac{z - \beta}{1 - \beta z} \right| \leq \frac{1}{1 - \rho^2} |\alpha - \beta|,$$

which holds for every $\alpha, \beta, z \in [-\rho, \rho]$, we obtain

$$\begin{aligned} |B_n^1 - B_n^2| &= \left| B_{n-1}^1 \left(\frac{z - \alpha_n}{1 - \alpha_n z} - \frac{z - \beta_n}{1 - \beta_n z} \right) + (B_{n-1}^1 - B_{n-1}^2) \frac{z - \beta_n}{1 - \beta_n z} \right| \\ &\leq \frac{1}{1 - \rho^2} |\alpha_n - \beta_n| + |B_{n-1}^1 - B_{n-1}^2| \leq \dots \leq \frac{1}{1 - \rho^2} \sum_{j=1}^n |\alpha_j - \beta_j|. \end{aligned}$$

here

$$B_k^1 = \prod_{j=1}^k \frac{z - \alpha_j}{1 - \alpha_j z}, \quad B_k^2 = \prod_{j=1}^k \frac{z - \beta_j}{1 - \beta_j z}, \quad k = 1, \dots, n.$$

Hence it follows that the function

$$\varphi(\alpha_1, \dots, \alpha_n) := \max_{z \in E} \left| \prod_{j=1}^n \frac{z - \alpha_j}{1 - \alpha_j z} \right|$$

is continuous for $(\alpha_1, \dots, \alpha_n) \in [-\rho, \rho]^n$ and, consequently, there exists a Blaschke product having real zeros, for which the infimum (2.2) is attained. As the inequality (2.5) is strict for $\alpha^* \neq \alpha$, every Blaschke product of order n for which this infimum is attained has its real zeros in the interval $[-\rho, \rho]$.

Lemma 2.1. *Suppose that the closed set $E \subset (-1, 1)$ consists of more than n elements and let $B_n^*(z)$ be a Blaschke product, normed by the condition $B_n^*(1) = 1$, such that*

$$\delta_n(E) = \|B_n^*\|_E.$$

If $f \in \mathcal{B}$ is a function real on the interval $(-1, 1)$, such that there exist $n + 1$ points $z_1 < z_2 < \dots < z_{n+1}$ in E with the property

$$f(z_j)f(z_{j+1}) < 0, \quad j = 1, \dots, n,$$

then

$$(2.6) \quad \min_{1 \leq j \leq n+1} |f(z_j)| \leq \delta_n(E).$$

Besides that, (2.6) turns into equality only if $f(z) \equiv \lambda B_n^$ where $\lambda = \pm 1$.*

Proof. The results of [13, §10.2] concerning the interpolation of functions from the class \mathcal{B} imply the existence of a Blaschke product $B_m(z)$ of order $m \leq n + 1$ which is real on the real axis and satisfies

$$B_m(z_j) = f(z_j), \quad j = 1, \dots, n + 1;$$

moreover, $f(z) \equiv B_m(z)$ if $m \leq n$, and for $m = n + 1$ one can choose $B_m(z)$ normed either by $B_m(1) = 1$ or by $B_m(1) = -1$. Suppose $|f(z_j)| \geq \delta_n(E)$, $j = 1, \dots, n + 1$. Consider the function

$$\varphi(z) = B_m(z) - \lambda B_n^*(z) \operatorname{sign} B_m(1),$$

where $\lambda = 1$ if $m \leq n$, and $\lambda = -1$, $B_m(1) = -\operatorname{sign} f(z_{n+1})$ if $m = n + 1$. In virtue of its normalization, $B_n^*(z)$ is real in $(-1, 1)$ whence

$$(2.7) \quad (-1)^{j+n+1} \varphi(z_j) \operatorname{sign} f(z_{n+1}) \geq 0, \quad j = 1, \dots, n + 1.$$

Thus, $\varphi(z)$ has at least n zeros (with multiplicity) in the interval $(-1, 1)$. On the other hand, by [8], Lemma 1.1, we have for the number k of roots of the function $\varphi(z) \not\equiv 0$ in the disc K

$$(2.8) \quad k \leq (m + n - l)/2,$$

where l is the number of roots of $\varphi(z)$ in the unit disc. If $m < n$ or $m = n$ (in this latter case $\varphi(-1) = \varphi(1) = 0$ whence $l \geq 2$), then (2.8) contradicts the fact that $k \geq n$. Hence in these cases the only possibility is $\varphi(z) \equiv 0$ which means that $f(z)$ coincides with $B_n^*(z)$ up to signs. If $m = n + 1$, then the normalization of $B_m(z)$ implies

$$\varphi(1) = -2 \operatorname{sign} f(z_{n+1}).$$

Taking into account (2.7), we obtain that the number of zeros of $\varphi(z)$ in the interval $(-1, 1)$ is at least $n + 1$. This contradicts (2.8), which proves the lemma. \square

Lemma 2.2. *Suppose that the closed set $E \subset (-1, 1)$ consists of more than n points. Then the solution of problem (2.2) is unique up to a factor λ , $|\lambda| = 1$. The Blaschke product B_n^* normed by $B_n^*(1) = 1$ is solution of this problem iff there exist $n + 1$ points $z_1 < \dots < z_{n+1}$ in E such that*

$$(2.9) \quad B_n^*(z_j) = (-1)^{n+j+1} \|B_n^*\|_E, \quad j = 1, \dots, n + 1.$$

Proof. Let B_n^* be an extremal function for the problem (2.2), normed by $B_n^*(1) = 1$. It has been proved that the zeros of B_n^* are real. Let $\alpha_1 \leq \dots \leq \alpha_n$ be these zeros. We are going to show that they are all different and that every interval (α_{j-1}, α_j) , $j = 1, \dots, n + 1$ ($\alpha_0 = -1$, $\alpha_{n+1} = 1$), contains some point of E in which $|B_n^*| = \|B_n^*\|_E$. Suppose this is not the case, and let

$$\max_{z \in [\alpha_{k-1}, \alpha_k] \cap E} |B_n^*(z)| = \delta < \|B_n^*\|_E$$

for some k , $1 \leq k \leq n+1$ (putting $\delta = 0$ if $[\alpha_{k-1}, \alpha_k] \cap E = \emptyset$). Set

$$\varphi(z) = \prod_{j=k-1}^k \frac{z - \alpha_j}{1 - \alpha_j z}.$$

Choose $\varepsilon > 0$ subject to $\delta + \varepsilon < \|B_n^*\|_E$ and consider the function

$$B_n(z) = g(z) \prod_{\substack{j=1 \\ j \neq k-1, k}}^n \frac{z - \alpha_j}{1 - \alpha_j z} \quad \text{where} \quad g(z) = \frac{\varphi(z) - \varepsilon}{1 - \varepsilon \varphi(z)}.$$

It is not difficult to show that

$$g(z) = \prod_{j=k-1}^k \frac{z - \beta_j}{1 - \beta_j z}$$

where $[\alpha_{k-1}, \alpha_k] \subset [\beta_{k-1}, \beta_k]$. From the inequalities

$$\begin{aligned} |g(z)| &\leq |\varphi(z)| + \varepsilon, & z \in [\alpha_{k-1}, \alpha_k], \\ |g(z)| &< |\varphi(z)|, & z \in (-1, 1) \setminus (\beta_{k-1}, \beta_k), \\ |g(z)| &< \varepsilon < \|B_n^*\|_E, & z \in (\beta_{k-1}, \alpha_{k-1}) \cup (\alpha_k, \beta_k), \end{aligned}$$

it follows that $\|B_n\|_E < \|B_n^*\|_E$. The contradiction obtained proves the existence of points $z_j \in (\alpha_{j-1}, \alpha_j) \cap E$, $j = 1, \dots, n+1$, which satisfy (2.9). The converse statement and unicity both follow from Lemma 2.1. This completes the proof of the lemma. \square

Theorem 2.1. *Let E be a closed subset of $(-1, 1)$. The following assertions hold:*

1) for $\delta_n(E) \leq \delta < \delta_{n-1}(E)$ ($\delta_0(E) = 1$) and for arbitrary $z_0 \in (-1, 1) \setminus E$ the inequality

$$(2.10) \quad n \leq \text{In}(z_0, E, \delta) \leq n+1$$

holds. Besides that, if $z_0 \in (-1, 1) \setminus \text{conv } E$ — where $\text{conv } E$ denotes the convex hull of E —, then $\text{In}(z_0, E, \delta) = n$;

2) for every $\delta \in (0, 1)$ and $z_0 \in (-1, 1) \setminus E$ the inequality

$$\text{In}(z_0, E, \delta) \geq c(E) \ln 1/\delta$$

holds, and

$$\lim_{\delta \rightarrow 0} \frac{\text{In}(z_0, E, \delta)}{\ln 1/\delta} = c(E);$$

here $c(E)$ is the capacity of the condenser $(E, \mathbb{C} \setminus K)$.

Proof. Consider the generalized Heins problem of finding

$$(2.11) \quad \sup_{\substack{f \in \mathcal{B} \\ |f(z)| \leq \delta, z \in E}} |f(z_0)|$$

(cf. [3], [4], [8], [9]). From [3] it follows that there exists a Blaschke product of finite order m which solves this problem. In [8] it was proved that for the extremality of the Blaschke product $B_m(z)$ normed by the

condition $B_m(z_0) > 0$ it is necessary and sufficient that the inequality $|B_m(z)| \leq \delta$, $z \in E$ is fulfilled, and that elements $z_1 < \dots < z_m$ of E exist such that

$$(2.12) \quad B_m(z_j) = \begin{cases} (-1)^{p+j}\delta, & j = 1, \dots, p, \\ (-1)^{p+j+1}\delta, & j = p+1, \dots, m, \end{cases}$$

where p is defined by $z_0 \in (z_p, z_{p+1})$ and $p = 0$ if $z_0 \in (-1, z_1)$, $p = m$ if $z_0 \in (z_m, 1)$. Thus, we have for arbitrary $y_1, \dots, y_n \in E$

$$(2.13) \quad \sup_{\substack{f \in \mathcal{B} \\ |f(y_j)| \leq \delta, j=1, \dots, n}} |f(z_0)| \geq \sup_{\substack{f \in \mathcal{B} \\ |f(z)| \leq \delta, z \in E}} |f(z_0)| \\ = \sup_{\substack{f \in \mathcal{B} \\ |f(z_j)| \leq \delta, j=1, \dots, m}} |f(z_0)| = B_m(z_0),$$

Hence and from (1.3) follows

$$(2.14) \quad R(z_0, \delta, E) = R_m(z_0, \delta, E).$$

In [9] the continuity of the left hand side of (2.13) as a function of y_1, \dots, y_n was proved. Since E is closed, the existence of optimal knots of extrapolation for arbitrary n follows.

Let $n < m$, and let y_1, \dots, y_n be optimal knots of extrapolation for n . From the uniqueness of the extremal function f normed by the condition $f(z_0) > 0$ in the problem of finding the supremum on the left side of (2.13), and from the fact that the extremal function is a Blaschke product of order not exceeding n , we obtain

$$R_n(z_0, \delta, E) > R_m(z_0, \delta, E).$$

Thus, m is the smallest number which satisfies (2.14), i.e. $\text{In}(z_0, E, \delta) = m$.

Suppose $\delta_n(E) \leq \delta < \delta_{n-1}(E)$. The obvious relationship

$$(2.15) \quad \delta_m(E) \leq \|B_m\|_E = \delta$$

yields $m \geq n$. By (2.12), for B_m there is an $(m-1)$ -point alternation on E . Using Lemma 2.1, we obtain

$$(2.16) \quad \delta < \delta_{m-2}(E).$$

If $z_0 \in (-1, 1) \setminus \text{conv } E$, then the cardinality of the alternation is m and we have

$$(2.17) \quad \delta < \delta_{m-1}(E).$$

From (2.16) it follows that $m \geq n+1$, and from (2.17) $m \geq n$. This proves the first assertion.

Now choose an arbitrary $\varepsilon > 0$. From (2.15) and (2.16), taking into account (2.3) and (2.4), we obtain for sufficiently small δ

$$\exp \left[-\frac{m}{c(E)} \right] \leq \delta < \exp \left[-\frac{m-2}{c(E) + \varepsilon} \right].$$

These inequalities yield assertion 2), which completes the proof. \square

§3. OPTIMAL INTERPOLATION

Consider problem (1.2) with $E = [\alpha, \beta] \subset (-1, 1)$. If $\delta \geq 1$, then $r_n(\delta, E) = 1$; therefore we shall take for granted $\delta < 1$. Define the pseudohyperbolic distance of points of the unit disc K by

$$\rho(u, v) = \left| \frac{u - v}{1 - \bar{v}u} \right|.$$

If $W(z)$ is any conform transformation of the disc K , then we have $r_n(\delta, E) = r_n(\delta, W(E))$. There is a conform transformation of K which maps the interval $[\alpha, \beta]$ onto the symmetric interval $[-k, k]$. As conform transformations of the disc do not change pseudohyperbolic distances, we have

$$\rho(\alpha, \beta) = \rho(-k, k) = \frac{2k}{1 + k^2}.$$

Thus,

$$(3.1) \quad r_n(\delta, [\alpha, \beta]) = r_n(\delta, [-k, k]),$$

where

$$k = \frac{\rho(\alpha, \beta)}{1 + \sqrt{1 - \rho^2(\alpha, \beta)}}.$$

Thereby we have proved that, for fixed n and δ , $r_n(\delta, [\alpha, \beta])$ depends only on the pseudohyperbolic distance $\rho(\alpha, \beta)$.

From [9] the continuity of the function

$$\varphi(z_1, \dots, z_n) = \sup_{z_0 \in [\alpha, \beta]} r(z_0, z_1, \dots, z_n, \delta)$$

follows for $z_1, \dots, z_n \in [\alpha, \beta]$. This implies the existence of optimal knots of interpolation.

Denote the quantity (1.2) for $n \geq 2$ by $r_n^*(\delta, [\alpha, \beta])$ provided $z_1 = \alpha$, $z_n = \beta$.

Lemma 3.1. *Let $0 \leq \delta < 1$ and $\rho(\alpha_1, \beta_1) > \rho(\alpha, \beta)$. Then*

$$(3.2) \quad r_n(\delta, [\alpha_1, \beta_1]) > r_n(\delta, [\alpha, \beta]),$$

$$(3.3) \quad r_n^*(\delta, [\alpha_1, \beta_1]) > r_n^*(\delta, [\alpha, \beta]).$$

Proof. First we prove (3.2). In virtue of (3.1) it suffices to prove

$$r_n(\delta, [-k_1, k_1]) > r_n(\delta, [-k, k])$$

for $k_1 > k$. Let u_1, \dots, u_n be optimal knots of interpolation for the problem (1.2) with $E = [-k_1, k_1]$, and let $u_0 \in E$ be such that

$$(3.4) \quad r(u_0, u_1, \dots, u_n, \delta) = r_n(\delta, [-k_1, k_1]).$$

Consider the points $z_j = ku_j/k_1$, $j = 0, \dots, n$. Obviously, $z_j \in [-k, k]$. Let $f^*(z)$ be an extremal function for the problem (1.3). Then $|f^*(z)| \leq \delta$, $j = 1, \dots, n$, and $|f^*(z_0)| \geq r_n(\delta, [-k, k])$. Now consider the function

$g(z) = f^*(kz/k_1)$. Note that $g(u_j) = f(z_j)$, $j = 0, \dots, n$, furthermore, since $f^*(z)$ is a Blaschke product, $g(z)$ is not. Hence

$$(3.5) \quad \sup_{\substack{f \in \mathcal{B} \\ |f(u_j)| \leq \delta, j=1, \dots, n}} |f(z_0)| > |g(u_0)| \geq r_n(\delta, [-k, k]).$$

Here the left hand side coincides with $r_n(\delta, [-k_1, k_1])$ in virtue of (1.3) and (3.4). The inequality (3.3) can be proved analogously. The lemma is proved \square

Theorem 3.1. *Let $[\alpha, \beta] \subset (-1, 1)$. Set $a = \operatorname{arth} \alpha$, $b = \operatorname{arth} \beta$, $d = b - a$. For*

$$(3.6) \quad \frac{\operatorname{th} d \operatorname{th} \frac{n-2}{n-1} d}{1 + \sqrt{1 - \operatorname{th}^2 d \operatorname{th}^2 \frac{n-2}{n-1} d}} \leq \delta < 1$$

the identity

$$(3.7) \quad r_n^*(\delta, [\alpha, \beta]) = \frac{\operatorname{th}^2 \frac{d}{2(n-1)} + \delta}{1 + \delta \operatorname{th}^2 \frac{d}{2(n-1)}}$$

holds, and the only system of optimal knots is

$$(3.8) \quad z_j^0 = \operatorname{th} \left[\frac{a+b}{2} - (n+1-2j) \frac{d}{2(n-1)} \right], \quad j = 1, \dots, n.$$

Proof. The knots (3.8) are uniformly distributed on the segment $[\alpha, \beta]$ with respect to the pseudohyperbolic distance, i.e.

$$\rho(z_j^0, z_{j+1}^0) = \operatorname{th} \frac{d}{n-1}, \quad j = 1, \dots, n-1.$$

Consequently, in every system of knots $\alpha = z_1 \leq \dots \leq z_n = \beta$ one can find a pair z_k, z_{k+1} such that

$$(3.9) \quad \rho(z_k, z_{k+1}) \geq \operatorname{th} \frac{d}{n-1}.$$

Consider the function

$$\varphi(z) = \frac{\delta - W(z)}{1 - \delta W(z)}, \quad \text{where} \quad W(z) = \prod_{j=k}^{k+1} \frac{z - z_j}{1 - z_j z}.$$

We want to prove that $r(z, z_1, \dots, z_n, \delta) = \varphi(z)$ for $z \in [z_k, z_{k+1}]$. As $\varphi(z)$ is a Blaschke product of order 2, it is sufficient to prove that $|\varphi(z_j)| \leq \delta$, $j = 1, \dots, n$ (cf. (2.12)). By the monotonicity of $\varphi(z)$ on the segments $[\alpha, z_k]$, $[z_{k+1}, \beta]$ we have to check only $\varphi(\alpha) \geq -\delta$, $\varphi(\beta) \geq -\delta$. We have

$$\varphi(\alpha) = \frac{\delta - W(\alpha)}{1 - \delta W(\alpha)} = \frac{\delta - \rho(\alpha, z_k) \rho(\alpha, z_{k+1})}{1 - \delta \rho(\alpha, z_k) \rho(\alpha, z_{k+1})}.$$

As $\rho(\alpha, z_{k+1}) \leq \operatorname{th} d$, and

$$\begin{aligned} \rho(\alpha, z_k) &= \frac{\rho(\alpha, z_{k+1}) - \rho(z_k, z_{k+1})}{1 - \rho(z_k, z_{k+1})\rho(\alpha, z_{k+1})} \leq \frac{\operatorname{th} d - \operatorname{th} d/(n-1)}{1 - \operatorname{th} d/(n-1)\operatorname{th} d} \\ &= \operatorname{th} \frac{n-2}{n-1}d, \end{aligned}$$

it follows

$$\varphi(\alpha) \geq \left(\delta - \operatorname{th} d \operatorname{th} \frac{n-2}{n-1}d \right) / \left(1 - \delta \operatorname{th} d \operatorname{th} \frac{n-2}{n-1}d \right).$$

Taking into account (3.6), we obtain $\varphi(\alpha) \geq -\delta$. Analogously, $\varphi(\beta) \geq -\delta$. It is not difficult to see that

$$\max_{z \in [z_k, z_{k+1}]} \varphi(z) = \frac{p^2 + \delta}{1 + \delta p^2},$$

where

$$p = \frac{\rho(z_k, z_{k+1})}{1 + \sqrt{1 - \rho^2(z_k, z_{k+1})}} = \operatorname{th} \frac{\operatorname{arth} \rho(z_k, z_{k+1})}{2}.$$

In virtue of (3.9), the inequalities

$$(3.10) \quad \sup_{z \in [\alpha, \beta]} r(z, z_1, \dots, z_n, \delta) \geq \max_{z \in [z_k, z_{k+1}]} \varphi(z) \geq \frac{\operatorname{th}^2 \frac{d}{2(n-1)} + \delta}{1 + \delta \operatorname{th}^2 \frac{d}{2(n-1)}}$$

hold. They turn into equalities (for every k) if $z_j = z_j^0$, $j = 1, \dots, n$. Thus, we have proved (3.7) and the optimality of the knots (3.8). Their uniqueness follows from the observation that for any other system one can find a pair z_k, z_{k+1} for which inequality (3.9) and, consequently, (3.10) are strict. This completes the proof. \square

Theorem 3.2. *With the notations of Theorem 3.1, let θ_n satisfy the equation*

$$(3.11) \quad \theta_n + 2 \operatorname{arth} \operatorname{th}^2 \frac{\theta_n}{2(n-1)} = d.$$

Then

$$(3.12) \quad r_n(\delta, [\alpha, \beta]) = \left(\operatorname{th}^2 \frac{\theta_n}{2(n-1)} + \delta \right) / \left(1 + \delta \operatorname{th}^2 \frac{\theta_n}{2(n-1)} \right)$$

holds for $\operatorname{th} \theta_n/2 \leq \delta < 1$, and the only system of optimal knots is

$$(3.13) \quad z_j^0 = \operatorname{th} \left[\frac{a+b}{2} - (n+1-2j) \frac{\theta_n}{2(n-1)} \right], \quad j = 1, \dots, n.$$

Proof. First of all note that for $\theta_n \in [0, +\infty)$ the left side of (3.11) increases monotonically from 0 to $+\infty$. Therefore (3.11) has a single

solution. Let $\alpha \leq z_1 \leq \dots \leq z_n \leq \beta$ be an arbitrary system of knots. Let us prove that for

$$(3.14) \quad \rho(z_1, z_n) \leq 2\delta/(1 + \delta^2)$$

the identity

$$(3.15) \quad \sup_{z \in [\alpha, z_1]} r(z, z_1, \dots, z_n, \delta) = \frac{\rho(\alpha, z_1) + \delta}{1 + \delta\rho(\alpha, z_1)}$$

holds. Put

$$\varphi(z) = \frac{\delta - W(z)}{1 - \delta W(z)}, \quad \text{where} \quad W(z) = \frac{z - z_1}{1 - z_1 z}.$$

By the monotonicity of $\varphi(z)$ we have for $z \in [z_1, z_n]$

$$\delta \geq \varphi(z) \geq \varphi(z_n) = \frac{\delta - \rho(z_1, z_n)}{1 - \delta\rho(z_1, z_n)} \geq -\delta.$$

Hence it follows that for $z \in [\alpha, z_1]$

$$r(z, z_1, \dots, z_n, \delta) = \varphi(z).$$

This obviously implies (3.15). Analogously, taking for granted (3.14), one can prove

$$(3.16) \quad \sup_{z \in [z_n, \beta]} r(z, z_1, \dots, z_n, \delta) = \frac{\rho(z_n, \beta) + \delta}{1 + \delta\rho(z_n, \beta)}.$$

From Theorem 3.1 we obtain

$$(3.17) \quad r_n^*(\delta, [z_1^0, z_n^0]) = \sup_{z \in [z_1^0, z_n^0]} r(z, z_1^0, \dots, z_n^0, \delta) = e_n(\delta),$$

where $e_n(\delta)$ denotes the right side of (3.12). In virtue of (3.13) and (3.11) we have

$$(3.18) \quad \rho(\alpha, z_1^0) = \rho(z_n^0, \beta) = \text{th} \frac{d - \theta_n}{2} = \text{th}^2 \frac{\theta_n}{2(n-1)}.$$

As $\rho(z_1^0, z_n^0) = \text{th} \theta_n$, the inequality (3.14) holds by the condition $\text{th} \theta_n/2 \leq \delta$. Therefore (3.15)–(3.17) imply

$$(3.19) \quad \sup_{z \in [\alpha, \beta]} r(z, z_1^0, \dots, z_n^0, \delta) = e_n(\delta).$$

Let $\alpha \leq z_1 \leq \dots \leq z_n \leq \beta$ be an arbitrary system of knots different from z_1^0, \dots, z_n^0 . If $\rho(z_1, z_n) > \rho(z_1^0, z_n^0) = \text{th} \theta_n$, then Lemma 3.1 yields

$$r_n^*(\delta, [z_1, z_n]) > r_n^*(\delta, [z_1^0, z_n^0]).$$

Hence and from (3.17),

$$(3.20) \quad \sup_{z \in [\alpha, \beta]} r(z, z_1, \dots, z_n, \delta) > e_n(\delta).$$

If

$$(3.21) \quad z_j = z_j^0, \quad j = 1, n$$

then, in consequence of the uniqueness of the system of optimal knots for $r_n^*(\delta, [z_1^0, z_n^0])$ proved in Theorem 3.1, the inequality (3.20) still holds.

Suppose $\rho(z_1, z_n) \leq \text{th } \theta_n$ and at least one of the relations (3.21) does not hold. Then either $\rho(\alpha, z_1) > \rho(\alpha, z_1^0)$ or $\rho(z_n, \beta) > \rho(z_n^0, \beta)$. Let e.g. $\rho(\alpha, z_1) > \rho(\alpha, z_1^0)$. The condition $\rho(z_1, z_n) \leq \text{th } \theta_n$ guarantees (3.14). Therefore (3.15) and (3.18) imply

$$\sup_{z \in [\alpha, z_1]} r(z, z_1, \dots, z_n, \delta) > e_n(\delta).$$

Hence (3.20) follows. The case $\rho(z_n, \beta) > \rho(z_n^0, \beta)$ is analogous. Thus, (3.20) holds for any system of knots other than z_1^0, \dots, z_n^0 whence taking into consideration (3.19), the identity (3.12) follows, as well as the uniqueness of the system (3.13). The theorem is proved. \square

If

$$\frac{\rho(\alpha, \beta)}{1 + \sqrt{1 - \rho^2(\alpha, \beta)}} \leq \delta < 1,$$

the conditions of Theorems 3.1 and 3.2 are fulfilled for every n . From (3.7) and (3.12) it follows that in this case the asymptotic equations

$$r_{n+1}^*(\delta, [\alpha, \beta]) = \delta + \frac{(1 - \delta^2)d^2}{4n^2} - \frac{(1 - \delta^2)(2 + 3\delta)d^4}{48n^4} + O\left(\frac{1}{n^6}\right),$$

$$r_{n+1}^*(\delta, [\alpha, \beta]) - r_{n+1}(\delta, [\alpha, \beta]) = \frac{(1 - \delta^2)d^3}{4n^2} + O\left(\frac{1}{n^6}\right),$$

hold, where $d = \text{arth } \rho(\alpha, \beta)$.

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Об оптимальной экстраполяции и интерполяции неточно заданных аналитических функций

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Для класса \mathcal{B} аналитических в единичном круге функций, ограниченных по модулю единицей, погрешностью наилучшего приближения в точке z_0 по значениям в точках z_1, \dots, z_n , заданным с погрешностью δ , называется величина

$$r(z_0, z_1, \dots, z_n, \delta) = \inf_S \sup_{f \in \mathcal{B}} \sup_{\substack{f_1, \dots, f_n \\ |f(z_j) - f_j| \leq \delta \\ j=1, \dots, n}} |f(z_0) - S(f_1, \dots, f_n)|,$$

где нижняя грань берется по всевозможным функциям $S: \mathbb{C}^n \rightarrow \mathbb{C}$. Для $E \subset (-1, 1)$ и $z_0 \in (-1, 1) \setminus E$ рассматривается задача о нахождении порядка информативности множества E , т.е. минимального n , на котором достигается нижняя грань в равенстве

$$R(z_0, \delta, E) = \inf_n \inf_{z_1, \dots, z_n \in E} r(z_0, z_1, \dots, z_n, \delta).$$

Кроме того, при δ , близких к 1, решена задача о нахождении величины

$$r_n(\delta, E) = \inf_{z_1, \dots, z_n \in E} \sup_{z_0 \in E} r(z_0, z_1, \dots, z_n, \delta)$$

и найдены узлы, на которых достигается нижняя грань.

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ПЕТРОВКА 27

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