ON THE LEBESGUE CONSTANTS FOR INTERPOLATION OF ANALYTIC FUNCTIONS

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Let $\tau := (t_1, \dots, t_n)$ be a system of distinct points in the interval $[-1, 1]$ and let $k \in [0, 1)$. We put

$$L_n(\tau, p, k) = \max_{t \in [-1,1]} \|D(t)\|_p,$$

where $D(t) = (D_1(t), \dots, D_n(t))$,

$$D_j(t) = \frac{\omega_j(t)}{\omega_j(t_j)} \left(1 - kW_j^2(t)\right), \quad \omega_j(t) = \prod_{m \neq j} W_m(t),$$

$$W_j(t) = \frac{t - t_j}{1 - kt_j t},$$

$$\|a\|_p = \begin{cases} \left(\displaystyle\sum_{j=1}^n |a_j|^p \right)^{1/p}, & 1 \le p < \infty, \\ \displaystyle\max_{1 \le j \le n} |a_j|, & p = \infty. \end{cases}$$

For $k = 0$

$$L_n(\tau, p, 0) = \Lambda_n(\tau, p),$$

where

$$\Lambda_n(\tau, p) = \max_{t \in [-1,1]} \|l(t)\|_p,$$

$$l(t) = (l_1(t), \dots, l_n(t)), \quad l_j(t) = \prod_{\substack{m=1 \\ m \neq j}}^n \frac{t - t_m}{t_j - t_m}.$$

Thus, $\Lambda_n(\tau, p)$ is the Lebesgue constant occurring in the theory of approximation by Lagrange interpolating polynomials. Lebesgue constants (mainly for $p = 1$) are studied in a rather great number of papers (see, for example, [2–5], [10–12] and the references cited there).

The quantity $L_n(\tau, p, k)$ is linked up with the problems of interpolation of analytic functions. We denote by B_k the class of functions analytic in the circle

$$D_k = \{\, z \in \mathbb{C} : |z| < 1/\sqrt{k} \,\}, \quad k \in)0, 1),$$

and bounded there in absolute value by 1. We call a method S_0 the best method of reconstruction of the function value $f \in B_k$ at a point

$t \in D_k$ on the basis of the information $I(f) = (f(t_1), \dots, f(t_n))$ if

$$E(t,\tau,k) \equiv \inf_S \sup_{f \in B_k} |f(t) - S(t, I(f))| = \sup_{f \in B_k} |f(t) - S_0(t, I(f))|,$$

where the infimum is taken with respect to all possible functions $S(t, I(f))$. It follows from [8] that for any point $t \in [-l, 1]$ the best method is given by

$$S_0(t, I(f)) = \sum_{j=1}^n D_j(t) f(t_j),$$

where

$$E(t,\tau,k) = k^{n/2}|W(t)|, \quad W(t) = \prod_{j=1}^n \frac{t - t_j}{1 - k t_j t}.$$

Now let us use, instead of $I(f)$, an information on approximate values $\widetilde{I} = (f_1, \dots, f_n)$ such that for every $f \in B_k$

$$\|I(f) - \widetilde{I}\|_q \le \delta.$$

We put

$$R_q(t,\tau,k,\delta) = \sup_{f \in B_k} \sup_{\widetilde{I}: \|I(f) - \widetilde{I}\|_q \le \delta} |f(t) - S_0(t, \widetilde{I})|,$$

$$F_q(\tau,k,\delta) = \sup_{t \in [-1,1]} R_q(t,\tau,k,\delta).$$

Theorem 1. *For all $t \in [-1, 1]$ we have*

$$R_q(t,\tau,k,\delta) = E(t,\tau,k) + \delta \|D(t)\|_p \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

Proof. Let

$$f \in B_k \quad \text{and} \quad \|I(f) - \widetilde{I}\|_q \le \delta.$$

Then

$$|f(t) - S_0(t, \widetilde{I})| \le |f(t) - S_0(t, I(f))| + \sum_{j=1}^n |D_j(t)|f_j - f(t_j)| \\ \le E(t,\tau,k) + \delta \|D(t)\|_p.$$

Hence

$$R_q(t,\tau,k,\delta) \le E(t,\tau,k) + \delta \|D(t)\|_p.$$

For $1 \le p < \infty$, we set

$$f_j = -\delta \frac{|D_j(t)|^{p-1} \operatorname{sign} D_j(t)}{\|D(t)\|_p^{p-1}}, \quad j = 1, \dots, n;$$

while for $p = \infty$ and $\max_{1 \le j \le n} |D_j(t)| = |D_m(t)|$ let

$$f_j = \begin{cases} -\delta \operatorname{sign} D_m(t), & j = m, \\ 0, & j \ne m. \end{cases}$$

We consider the function

$$f(z) = k^{n/2} W(z) \operatorname{sign} W(t).$$

By virtue of the facts that $f \in B_k$, $I(f) = 0$, and $\|\widetilde{I}\|_q = \delta$, we have

$$R_q(t, \tau, k, \delta) \geq |f(t) - S_0(t, \widetilde{I})| = E(t, \tau, k) + \delta \|D(t)\|_p.$$

The theorem is proved. $\square$

Thus,

$$F_q(\tau, k, \delta) \leq \max_{t \in [-1,1]} E(t, \tau, k) + \delta L_n(\tau, p, k). \tag{1}$$

It follows from [7] that

$$\inf_{\tau} \max_{t \in [-1,1]} E(t, \tau, k) = \max_{t \in [-1,1]} E(t, Z, k),$$

where

$$Z = \left\{ \operatorname{sn} \left[\left(\frac{2j-1}{n} - 1 \right) K, k \right] \right\}_1^n;$$

here and in the sequel, K, L and M, denote the complete elliptic integrals of the first kind for the moduli, k, l and μ_n, respectively, while K', L' and M_n' denote those for the complemented moduli; concerning the notations $\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$, etc. we refer to [13, p. 240]. The point system Z plays a role which is analogous to that played by the Chebyshev system

$$T = \left\{ \cos \frac{2j-1}{2n} \pi \right\}_1^n$$

in the case of interpolation by Lagrange polynomials. In addition, for $k = 0$ the system Z coincides with the Chebyshev system, due to the equalities

$$K = \pi/2 \quad \text{and} \quad \operatorname{sn}(x, 0) = \sin x.$$

By this token,

$$L_n(Z, p, 0) = \Lambda_n(T, p). \tag{2}$$

We consider some estimates of the quantity $L_n(Z, p, k)$.

Theorem 2. *For all $n \geq 1$, $2 \leq p \leq \infty$, and $k \in [0, 1)$ we have*

$$L_n(Z, p, k) \leq \sqrt{1 + \operatorname{sn}\left(\frac{n-1}{n} K, k\right)} < \sqrt{2}. \tag{3}$$

Proof. Let $0 < k < 1$. Substituting

$$\nu_1 = \ldots = \nu_n = 2, \quad g(\xi) = 1, \quad x = \sqrt{k} t,$$
$$z_j = \sqrt{k} t_j, \quad j = 1, \ldots, n,$$

in [9, Lemma 2.1], we have

$$1-\sum_{j=1}^{n} D_j^2(t)h_j(t)=k^{2n}W^4(t), \tag{4}$$

where

$$h_j(t)=\frac{1+kW_j^2(t)}{1-kW_j^2(t)}-\frac{2\omega_j'(t_j)W_j(t)}{\omega_j(t_j)W_j'(t_j)(1-kW_j^2(t))}.$$

In the case $k=0$, equality (4) is of the form

$$1-\sum_{j=1}^{n} l_j^2(t)v_j(t)=0,$$

$$v_j(t)=1-\frac{\omega''(t_j)}{\omega'(t_j)}(t-t_j),\quad \omega(t)=\prod_{j=1}^{n}(t-t_j),$$

and remains valid, since it is the expansion of the unity by the interpolating Hermite polynomial (see [4], [11]). For the system Z of nodes the function $W(t)$ can be written by means of the first basic transformation of nth degree of elliptic functions (see, for example [1, p. 284]) in the parametric form

$$W(t)=(-1)^{n+1}a_n\operatorname{sn}\left[(nx-1)M_n,\mu_n\right],\quad t=\operatorname{sn}[(x-1)K,k], \tag{5}$$

where

$$a_n=\prod_{j=1}^{[n/2]}\operatorname{sn}^2\left(\frac{2j-1}{n}K,k\right),\quad \mu_n=k^n a_n^2.$$

Since $W(t)=W_j(t)\omega_j(t)$, we have

$$\omega_j(t_j)=\frac{W'(t_j)}{W_j'(t_j)},\quad \omega_j'(t_j)=\frac{W''(t_j)W_j'(t_j)-W'(t_j)W_j''(t_j)}{2\left[W_j'(t_j)\right]^2}.$$

Computing the values $W'(t_j)$ and $W''(t_j)$ from representation (5), we get

$$h_j(t)=\frac{1+kW_j^2(t)}{1-kW_j^2(t)}-\frac{(1-k)^2(1+kt_j^2)(t-t_j)(1-kt_jt)t_j}{(1-t_j)^2(1-k^2t_j^2)(1-kt_j^2)(1-kt^2)}.$$

It is not hard to see that for all $t\in[-1,1]$ and $k\in[0,1)$ the inequalities

$$(1-k)^2\le(1-kt^2)(1-kt_j^2),\quad (1+kt_j^2)\le 1+k|t_j|,$$
$$t_j(t-t_j)(1-kt_jt)\le|t_j|(1-|t_j|)(1-k|t_j|)$$

are satisfied. Hence

$$h_j(t)\ge 1-\frac{|t_j|}{1+|t_j|}\ge\left[1+\operatorname{sn}\left(\frac{n-1}{n}K,k\right)\right]^{-1}.$$

Thus,

$$\left[1+\operatorname{sn}\left(\frac{n-1}{n}K,k\right)\right]^{-1}\sum_{j=1}^{n}D_j^2(t)\le\sum_{j=1}^{n}D_j^2(t)h_j(t)$$
$$=1-k^{2n}W^4(t)\le 1.$$

Consequently,

$$\|D(t)\|_2\le\sqrt{1+\operatorname{sn}\left(\frac{n-1}{n}K,k\right)}<\sqrt{2}.$$

Now inequality (3) follows from the fact that $\|\cdot\|_p\le\|\cdot\|_2$ for $2\le p\le\infty$. The theorem is proved. $\square$

By virtue of (2) and (3), for $2\le p\le\infty$ we obtain the inequality

$$\Lambda_n(T,p)\le\sqrt{1+\cos\frac{\pi}{2n}}<\sqrt{2},$$

which was obtained in [4] for the case $p=2$.

Taking (1) and (5) into account, Theorem 2 implies the following

Corollary 1. *For all $1\le q\le 2$, $k\in(0,1)$, and $\delta\ge 0$ we have*

$$F_q(Z,k,\delta)\le\sqrt{\mu_n}+\delta\sqrt{2}.$$

We consider the case $p=1$.

Theorem 3. *For all $n\ge 1$ and $k\in[0,1)$ we have*

$$L_n(Z,1,k)=[1+\alpha_n(k)]\frac{L}{nM_n}\sum_{j=1}^{n}\operatorname{ctn}\frac{2j-1}{2n}L, \tag{6}$$

where

$$0\le\alpha_n(k)\le\frac{2kK^2}{(1-k)^2n^2},\qquad l=\frac{2\sqrt{k}}{1+k},\qquad \operatorname{ctn}x=\frac{\operatorname{cn}x\operatorname{dn}x}{\operatorname{sn}x}$$

(*here and in the sequel, we do not indicate the dependence of the elliptic functions upon the modulus if the latter equals l*). *In addition, $\alpha_n(k)=0$ for*

$$n\ge\frac{8k}{(1-k)^2}+1. \tag{7}$$

Proof. We put

$$t(u)=\left(\frac{2}{1+k}u-1\right)\left(1-\frac{2k}{1+k}u\right)^{-1}.$$

Changing variable $t=t(u)$, we get that

$$L_n(Z,1,k)=\max_{u\in[0,1]}\sum_{j=1}^{n}|\widetilde{D}_j(u)|,$$

where

$$\widetilde{D}_j(u) = \frac{\widetilde{W}(u)(1-l^2u^2)}{\widetilde{W}'(u_j)(u-u_j)(1-l^2u_ju)}, \quad \widetilde{W}(u) = \prod_{j=1}^n \frac{u-u_j}{1-i^2u_ju},$$

while $t(u_j) = t_j$. By the Gauss transformation of elliptic functions [1, p. 134], one can show that

$$u_j = \operatorname{sn}^2 \frac{2j-1}{2n} L, \quad j = 1, \dots, n.$$

Applying the first basic transformation of $2n$-th degree, we can write the function $\widetilde{W}(u)$ in the form

$$\widetilde{W}(u) = (-1)^n \widetilde{a}_n \operatorname{sn}\left[\left(\frac{2nt}{L}+1\right) M_n, \mu_m\right], \quad u = \operatorname{sn}^2 t,$$

$$\widetilde{a}_n = \prod_{j=1}^n \operatorname{sn}^2 \frac{2j-1}{2n} L.$$

Hence, finding $\widetilde{W}'(u_j)$ and using identities well-known in the theory of elliptic functions, we get that

$$\widetilde{D}_j(\operatorname{sn}^2 t) = (-1)^{j+1} \frac{L}{2nM_n} \operatorname{sn}\left[\left(\frac{2nt}{L}+1\right) M_n, \mu_m\right] \times \left[\operatorname{ctn}(t_j+t) + \operatorname{ctn}(t_j-t)\right]. \tag{8}$$

From the inequality

$$\sum_{j=1}^n |\widetilde{D}_j(\operatorname{sn}^2 t)| \le \frac{L}{2nM_n} \left|\operatorname{sn}\left[\left(\frac{2nt}{L}+1\right) M_n, \mu_m\right]\right| \times \sum_{j=1}^n \left[|\operatorname{ctn}(t_j+t)| + |\operatorname{ctn}(t_j-t)|\right],$$

which turns into equality for $t \in [-L/(2n), L/(2n)]$, furthermore, from the evenness and periodicity, with period L/n, of the function occurring on the right-hand side of this inequality it follows that

$$\max_{u\in[0,1]} \sum_{j=1}^n |\widetilde{D}_j(u)| = \max_{t\in[0,L/(2n)]} \sum_{j=1}^n |\widetilde{D}_j(\operatorname{sn}^2 t)| = \max_{u\in[0,u_1]} \sum_{j=1}^n |\widetilde{D}_j(u)|.$$

Since for $u \in [0, u_1]$

$$|\widetilde{D}_j(u)| = \frac{\widetilde{\omega}_j(u)}{|\widetilde{W}'(u_j)|} \frac{1-l^2u^2}{(1-l^2u_ju)^2}, \quad \widetilde{\omega}_j(u) = \prod_{\substack{m=1\\ m\ne j}}^n \frac{u_m-u}{1-i^2u_mu},$$

and since each of the factors $(u_m-u)/(1-l^2u_ju)$ is monotone decreasing, while the function $(1-l^2u^2)/(1-l^2u_ju)^2$ is increasing, we can infer that

$$\max_{u\in[0,u_1]}\sum_{j=1}^{n}|\widetilde{D}_j(u)| \le \frac{\widetilde{\omega}_j(0)}{|\widetilde{W}'(u_j)|}\frac{1-l^2u_1^2}{(1-l^2u_ju_1)^2} = |\widetilde{D}_j(0)|\frac{1-l^2u_1^2}{(1-l^2u_ju_1)^2}.$$

By virtue of the inequalities

$$\frac{1-l^2u_1^2}{(1-l^2u_ju_1)^2} \le 1+\frac{2l^2u_nu_1}{(1-l^2u_nu_1)^2},$$

$$l^2u_nu_1 = l^2\operatorname{sn}^2\left(L-\frac{L}{2n}\right)\operatorname{sn}^2\frac{L}{2n} \le l^2\operatorname{sn}^4\frac{L}{2} = \frac{l^2}{(1+\sqrt{1-l^2})^2} = k,$$

as well as

$$l^2u_nu_1 \le l^2\operatorname{sn}^2\frac{L}{2n} \le \frac{l^2L^2}{4n^2} = \frac{kK^2}{n^2},$$

we have

$$\sum_{j=1}^{n}|\widetilde{D}_j(0)| \le \max_{u\in[0,u_1]}\sum_{j=1}^{n}|\widetilde{D}_j(u)| \le \left[1+\frac{2kK^2}{(1-k)^2n^2}\right]\sum_{j=1}^{n}|\widetilde{D}_j(0)|.$$

Now equality (6) follows from (8) for $t=0$.

Let inequality (7) be satisfied. In order to prove that $\alpha_n(k)=0$ it is enough to prove the decrease of the functions

$$\varphi_j(u) = \widetilde{\omega}_j(u)\frac{1-l^2u^2}{(1-l^2u_ju)^2}$$

for $u\in[0,u_1]$. From the relations

$$\begin{aligned}(1-l^2u^2)\frac{\varphi_j'(u)}{\varphi_j(u)} &= 2l^2\frac{u_j-u}{1-l^2u_ju} - (1-l^2u^2)\sum_{\substack{m=1\\m\neq j}}^{n}\frac{1-l^2u_m^2}{(u_m-u)(1-l^2u_mu)}\\ &\le 2l^2u_j - \sum_{\substack{m=1\\m\neq j}}^{n}\frac{1-l^2u_m^2}{u_m} \le 2l^2-(n-1)(1-l^2)\end{aligned}$$

it follows that $\varphi_j'(u)\le 0$ for $u\in[0,u_1]$ and

$$n \ge \frac{2l^2}{1-l^2}+1 = \frac{8k}{(1-k)^2}+1.$$

The theorem is proved. □

Corollary 2. *If* (7) *is satisfied, then we have*

$$F_\infty(Z,k,\delta) = \sqrt{\mu_n}+\delta L_n(Z,1,k).$$

Proof. It follows from the proof of Theorem 3 that if condition (7) is satisfied, then

$$L_n(Z,1,k) = \sum_{j=1}^{n}|D_j(1)| = \frac{L}{nM_n}\sum_{j=1}^{n}\operatorname{ctn}\frac{2j-1}{2n}L. \tag{9}$$

By virtue of representation (5),

$$\max_{t\in[-1,1]} E(t,Z,k)=E(1,Z,k)=\sqrt{\mu_n}.$$

Hence and by Theorem 1,

$$\begin{aligned}F_\infty(Z,k,\delta)\ge R_\infty(1,Z,k,\delta)&=E(1,Z,k)+\delta\|D(1)\|_1\\&=\max_{t\in[-1,1]} E(t,Z,k)+\delta L_n(Z,1,k).\end{aligned}$$

The lower estimate follows from inequality (1). The corollary is proved. □

We note that for $l=0$ we have $\operatorname{ctn} x=\cot x$. Therefore, taking (2) into account for $k=0$, equalities (9) turn into the following:

$$\Lambda_n(T,1)=\sum_{j=1}^n |l_j(1)|=\frac{1}{n}\sum_{j=1}^n \cot\frac{2j-1}{4n}\pi,$$

which were proved in [3], [10].

Theorem 4. *If* (7) *is satisfied, then we have*

$$\begin{aligned}L_n(Z,1,k)=\frac{2}{\pi}\ln n+A(k)+\sum_{s=1}^{m-1}\frac{a_s}{n^{2s}}\left[B_{2s}\left(2^{2s-1}-1\right)-4sH_{2s-1}(k)\right]\\+\frac{a_m}{n^{2m}}\rho_m(n,k)-h^{2n}\left[\frac{8}{\pi}\ln n+B(k)\right]\theta_n(k),\end{aligned}$$

where

$$A(k)=\frac{2}{\pi}\left(\gamma+\ln\frac{4}{(1+k)K}\right),$$

$\gamma=0.5772\ldots$ *is the Euler constant,* B_{2s} *are the Bernoulli numbers,*

$$a_s=4\frac{|B_{2s}|\left(2^{2s-1}-1\right)\pi^{2s-1}}{2^{2s}s(2s)!},$$

$$H_s(k)=\sum_{r=1}^\infty \frac{h^{2r-1}}{1+h^{2r-1}}(2r-1)^s,\quad h=\exp\left(-\frac{\pi K'}{2K}\right),$$

$$-2\pi m H_{2m}(k)\le(-1)^{m+1}\rho_m(n,k)\le 2\pi m H_{2m}(k)+|B_{2m}|\left(2^{2m-1}-1\right),$$

$$B(k)=4A(k)+\frac{2\pi^2}{3}H_2(k)+\frac{\pi}{18},\quad 0\le\theta_n(k)\le 1.$$

Proof. It was proved in [12] (see also [5]) that for any function $f\in C^{2m}_{[0,2n]}$ we have

$$\begin{aligned}(10)\quad \sum_{j=1}^n f(2j-1)=\frac{1}{2}\int_0^{2n} f(x)\,dx\\+\sum_{s=1}^{m-1}\frac{B_{2s}}{(2s)!}\left(1-2^{2s-1}\right)\left(f^{(2s-1)}(2n)-f^{(2s-1)}(0)\right)+r_m(n),\end{aligned}$$

where

$$r_m(n) = -\frac{2^{2m}}{(2m)!}\int_0^n f^{(2m)}(2x)y_m(x)\,dx,$$

$$y_m(x) = B^*_{2m}\left(x+\frac{1}{2}\right) - B^*_{2m}\left(\frac{1}{2}\right),$$

while $B^*_{2m}(x)$ are the periodic Bernoulli functions with period 1. In addition, if $f \in C^{2m+2}_{[0,2n]}$ and

$$f^{(2m+2)}(x)f^{(2m)}(x) \geq 0 \quad \text{for all} \quad x \in [0,2n],$$

then

(11)

$$r_m(n) = \frac{B_{2m}}{(2m)!}\left(1-2^{2m-1}\right)\left[f^{(2m-1)}(2n) - f^{(2m-1)}(0)\right]\theta, \quad 0 \leq \theta \leq 1.$$

Using the representation of the function $\operatorname{sn} u$ by means of the theta functions and their expansions into infinite products [6, pp. 214, 205, 209], after simple computations we arrive at the equality

(12)

$$\operatorname{ctn} u = \frac{d}{du}\ln \operatorname{sn} u = \frac{\pi}{2L}\cot\frac{\pi u}{2L} - \frac{2\pi}{L}\sum_{r=1}^{\infty}\frac{h^r}{1+h^r}\sin\frac{\pi r u}{L}, \quad |\operatorname{Im} u| < L',$$

where

$$h = \exp\left(-\frac{\pi L'}{L}\right) = \exp\left(-\frac{\pi K'}{2K}\right).$$

We set

$$f(x) = \operatorname{ctn}\frac{xL}{2n} - \frac{2n}{xL}.$$

Then $f(x) = f_1(x) + f_2(x)$, where

(13)

$$f_1(x) = \frac{\pi}{2L}\cot\frac{\pi x}{4n} - \frac{2n}{xL} = -\frac{1}{L}\sum_{r=1}^{\infty}\frac{|B_{2r}|\pi^{2r}}{2^{2r-1}(2r)!n^{2r-1}}x^{2r-1}, \quad |x| < 4n,$$

$$f_2(x) = -\frac{2\pi}{L}\sum_{r=1}^{\infty}\frac{h^r}{1+h^r}\sin\frac{\pi r x}{2n}.$$

Consequently,

$$f^{(2s-1)}(0) = -\frac{|B_{2s}|\pi^{2s}}{L2^{2s}sn^{2s-1}} - (-1)^{s-1}\frac{4\pi^{2s}}{L2^{2s}n^{2s-1}}\sum_{r=1}^{\infty}\frac{h^r}{1+h^r}r^{2s-1}.$$

We set

$$g(x) = f_1(x+2n) = -\frac{\pi}{2L}\tan\frac{\pi x}{4n} - \frac{2n}{L(x+2n)}.$$

By the known expansion of $\tan z$, we obtain that

$$f_1^{(2s-1)}(2n) = g^{(2s-1)}(0) = -\frac{|B_{2s}|\pi^{2s}(2^{2s}-1)}{L2^{2s}sn^{2s-1}} + \frac{(2s-1)!}{L2^{2s-1}n^{2s-1}}.$$

In this way,

$$f^{(2s-1)}(2n) = -\frac{|B_{2s}|\pi^{2s}(2^{2s}-1)}{L2^{2s}sn^{2s-1}} + \frac{(2s-1)!}{L2^{2s-1}n^{2s-1}} \\ - (-1)^{s+1}\frac{4\pi^{2s}}{L2^{2s}n^{2s-1}}\sum_{r=1}^{\infty}(-1)^r\frac{h^r}{1+h^r}r^{2s-1}.$$

Now from (10) and the fact that

$$\frac{1}{2}\int_0^{2n}\left(\operatorname{ctn}\frac{xL}{2n} - \frac{2n}{xL}\right)dx = \frac{n}{L}\int_0^L\left(\operatorname{ctn}u - \frac{1}{u}\right)du = \frac{n}{L}\log\frac{\operatorname{sn}u}{u}\bigg|_0^L \\ = \frac{n}{L}\ln\frac{1}{L},$$

it follows that

$$\frac{L}{n}\sum_{j=1}^{n}\left[\operatorname{ctn}\frac{2j-1}{2n}L - \frac{2n}{(2j-1)L}\right] = \ln\frac{1}{L} \\ + \frac{\pi}{2}\sum_{s=1}^{m-1}\frac{a_s}{n^{2s}}\left[B_{2s}\left(2^{2s-1}-1\right) - 4sH_{2s-1}(k)\right] - \sum_{s=1}^{m-1}\frac{B_{2s}\left(2^{2s-1}-1\right)}{2^{2s}sn^{2s}} \\ + r_m^{(1)}(n) + r_m^{(2)}(n),$$

where

$$r_m^{(j)}(n) = -\frac{L2^{2m}}{n(2m)!}\int_0^n f_j^{(2m)}(2x)y_m(x)\,dx, \quad j = 1,2.$$

Since

$$f_1^{(s)}(x) \le 0 \quad \text{for} \quad x \in [0,2n]$$

holds for an arbitrary s (cf. (13)), by (11) we have

$$0 \le (-1)^{m+1}r_m^{(1)}(n) \le \frac{\pi}{2}\frac{a_m}{n^{2m}}|B_{2m}|\left(2^{2m-1}-1\right) - \frac{|B_{2m}|\left(2^{2m-1}-1\right)}{2^{2m}mn^{2m}}.$$

To estimate the quantity $r_m^{(2)}(n)$, first we note that for any natural number r

$$\int_0^n y_m(x)\sin\frac{2\pi rx}{n}\,dx = 0. \tag{14}$$

This equality follows from the periodicity of the integrand with period n and its oddness. Thus,

$$f_2^{(2m)}(2x) = (-1)^{m+1}\frac{\pi^{2m+1}}{L2^{2m-1}n^{2m}}\sum_{r=1}^{\infty}\frac{h^r}{1+h^r}r^{2m}\sin\frac{\pi rx}{n},$$

which together with (14) gives

$$
\begin{aligned}
&|r_m^{(2)}(n)| \\
&= \frac{2\pi^{2m+1}}{(2m)!n^{2m+1}}\left|\sum_{r=1}^{\infty}\frac{h^{2r-1}}{1+h^{2r-1}}(2r-1)^{2m}\int_0^n y_m(x)\sin\frac{\pi(2r-1)x}{n}\,dx\right| \\
&\qquad\qquad \le \frac{2\pi^{2m+1}}{(2m)!n^{2m+1}}H_{2m}(k)\int_0^n |y_m(x)|\,dx.
\end{aligned}
$$

From well-known properties of the Bernoulli polynomials it follows that

$$(-1)^{m+1}y_m(x) \ge 0 \quad \text{and} \quad \int_0^n y_m(x)\,dx = n\frac{B_{2m}\left(2^{2m-1}-1\right)}{2^{2m-1}}.$$

A fortiori,

$$|r_m^{(2)}(n)| \le \pi^2 m \frac{a_m}{n^{2m}} H_{2m}(k).$$

We apply the equality

$$\sum_{j=1}^{n}\frac{2}{2j-1} = \ln n + 2\ln 2 + \gamma + \sum_{s=1}^{m-1}\frac{B_{2s}\left(2^{2s-1}-1\right)}{s2^{2s}n^{2s}} + \delta_m(n),$$

(see [12]), where

$$0 \le (-1)^{m+1}\delta_m(n) \le \frac{|B_{2m}|\left(2^{2m-1}-1\right)}{m2^{2m}n^{2m}}.$$

We have

(15)

$$
\begin{aligned}
\frac{L}{n}\sum_{j=1}^{n}\operatorname{ctn}\frac{2j-1}{2n}L &= \frac{L}{n}\sum_{j=1}^{n}\left[\operatorname{ctn}\frac{2j-1}{2n}L - \frac{2n}{(2j-1)L}\right] + \sum_{j=1}^{n}\frac{2}{2j-1} \\
&= \ln n + \frac{\pi}{2}A(k) + \frac{\pi}{2}\sum_{s=1}^{m-1}\frac{a_s}{n^{2s}}\left[B_{2s}\left(2^{2s-1}-1\right) - 4sH_{2s-1}(k)\right] \\
&\qquad + \frac{\pi}{2}\frac{a_m}{n^{2m}}\rho_m(n,k).
\end{aligned}
$$

Denoting by $A_n(k)$ the left part in (15), for $m=1$ we get that

(16) $$A_n(k) = \ln n + \frac{\pi}{2}A(k) + \frac{\pi}{2}\frac{a_1}{n^2}\rho_1(n,k) \le \ln n + \frac{\pi}{8}B(k).$$

Theorem 3 implies that, under (7),

(17) $$L_n(Z,1,k) = M_n^{-1}A_n(k).$$

For the complete elliptic integral M_n of the first kind the following equalities are valid [1, pp. 277, 284] :

$$\sqrt{\frac{2M_n}{\pi}} = 1 + 2\sum_{r=1}^{m}h_1^{r^2}, \quad h_1 = \exp\left(-\frac{\pi M_n'}{M_n}\right) = h^{2n}.$$

Consequently,

$$1 \le \sqrt{\frac{2M_n}{\pi}} \le 1 + \frac{2h_1}{1-h_1}.$$

Hence it follows easily that

$$M_n^{-1} = \frac{2}{\pi} - \frac{8}{\pi} h^{2n} \sigma_n(k), \quad 0 \le \sigma_n(k) \le 1. \tag{18}$$

Now (15)–(18) imply the statement of the theorem. The theorem is proved. □

For $k = 0$, from Theorem 4 a representation of $\Lambda_n(T, 1)$ follows, which was obtained in [12] (see also [2]):

$$\Lambda_n(T,1) = \frac{2}{\pi} \ln n + \frac{2}{\pi} \left(\gamma + \ln \frac{8}{\pi} \right) + \sum_{s=1}^{m-1} \frac{a_s}{n^{2s}} B_{2s} \left(2^{2s-1} - 1 \right)$$
$$+ \frac{a_m}{n^{2m}} \rho_m(n, 0), \quad 0 \le (-1)^{m+1} \rho_m(n, 0) \le |B_{2m}| \left(2^{2m-1} - 1 \right).$$

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О константах Лебега при интерполяции аналитических функций

К. Ю. ОСИПЕНКО

Для системы различных точек $\tau = (t_1, \dots, t_n)$ из отрезка $[-1, 1]$ и $k \in [0, 1)$ вводится величина

$$L_n(\tau, p, k) = \max_{t \in [-1,1]} \left(\sum_{j=1}^{n} |D_j(t)|^p \right)^{1/p},$$

где

$$D_j(t) = \frac{\omega_j(t)}{\omega_j(t_j)} \left(1 - kW_j^2(t)\right), \quad \omega_j(t) = \prod_{m \neq j} W_m(t),$$

$$W_m(t) = \frac{t - t_m}{1 - kt_m t}.$$

При $k = 0$ она совпадает с константой Лебега, связанной с интерполяцией многочленом Лагранжа. Показана связь величины $L_n(\tau, p, k)$ с задачами интерполяции аналитических функций. Для системы

$$Z = \left\{ \operatorname{sn} \left[\left(\frac{2j-1}{n} - 1 \right) K, k \right] \right\}_1^n,$$

являющейся аналогом чебышевской системы, получены оценки $L_n(Z, p, k)$ при $p \geq 2$ и $p = 1$.

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