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On N-Width of One Class of Holomorphic Functions

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1. Introduction. Let

$$B_n = \left\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z|^2 = \sum_{l=1}^{n} |z_l|^2 < 1 \right\}$$

be the unit ball of $\mathbb{C}^n$. Remind that $H_2(B_n)$ is a space of holomorphic in B_n functions which satisfy the inequality

$$\|f\|_2 = \sup_{0<r<1} \left(\int_{S^{2n-1}} |f(rz)|^2 \, d\sigma_n(z) \right)^{1/2} < \infty$$

where σ_n is a positive normalized measure on $S^{2n-1} = \partial B_n$ which is invariant with respect to orthogonal group O^{2n} (see Rudin [1]). The unit ball of the Hilbert space $H_2(B_n)$ we denote $BH_2(B_n)$. Let $L_{\infty,r}(B_n)$ $(0 < r < 1)$ be the space $H_2(B_n)$ supplied with the following norm

$$\|f\|_{\infty,r} = \max_{|z|=r} |f(z)|.$$

The purpose of this work is finding the precise values of Gelfand and linear N-width of the class $BH_2(B_n)$ in $L_{\infty,r}(B_n)$ metrics for some values of N. In the case $n = 1$ we shall find N-width for arbitrary $N \in \mathbb{Z}_+$. There are a number of works devoted to calculating of precise values of N-width of Hardy classes BH_p in integral q-metrics (see, for example, [2–5]). All of them considered the case $p \geq q$ only and in all these works $n = 1$. In our case the situation is opposite.

2. Main results. Remind definitions of Gelfand and linear N-width. Let A be the subset of Banach space X.

$$d^N(A, X) = \inf_{L^N} \sup_{x \in A \cap L^N} \|x\|$$

where L^N is a subspace of X, $\operatorname{codim} L^N = N$, is the Gelfand N-width of A.

$$\lambda_N(A, X) = \inf_{\Lambda_N} \sup_{x \in A} \|x - \Lambda_N x\|,$$

where Λ_N is a linear continuous operator mapping X into it's N-dimensional subspace, is the linear N-width of A. It is easy to verify that

$$d^N(A, X) \leq \lambda_N(A, X). \tag{1}$$

For $m \in \mathbb{Z}_+$ put

$$N_m = \begin{cases} 0, & m = 0, \\ \displaystyle\sum_{l=0}^{m-1} \binom{n+l-1}{n-1}, & m > 0. \end{cases}$$

Note that N_m is equal to dimension of the space of polynomials of n variables which degree is no more than $m - 1$.

Theorem.

$$d^{N_m}(BH_2(B_n), L_{\infty,r}(B_n)) = \lambda_{N_m}(BH_2(B_n), L_{\infty,r}(B_n))$$
$$= \begin{cases} \dfrac{1}{(1-r^2)^{n/2}}, & m=0, \\ \sqrt{\dfrac{1}{(1-r^2)^n} - \sum_{l=0}^{m-1} \binom{n+l-1}{n-1} r^{2l}}, & m>0. \end{cases}$$

Before proving this theorem we shall prove two lemmas.

Lemma 1. *Let $f \in BH_2(B_n)$, $w \in B_n$, $f = \sum_{l=0}^{\infty} F_l$ is a homogeneous expansion of f. Then*

$$\left| f(w) - \sum_{l=0}^{s} F_l(w) \right| \le \sqrt{\frac{1}{(1-|w|^2)^n} - \sum_{l=0}^{s} \binom{n+l-1}{n-1} |w|^{2l}}.$$

Proof. Recall that that the Caushy's kernel for B_n is

$$K(w,z) = \frac{1}{(1-\langle w,z\rangle)^n}$$

where $\langle w,z\rangle = \sum_{l=1}^{n} w_l \overline{z}_l$ is the hermitian binary product. Put

$$K_s(w,z) = \frac{1}{(1-\langle w,z\rangle)^n} - \sum_{|p|\le s} \frac{(n+|p|-1)!}{(n-1)!p!} w^p \overline{z}^p$$

(for multyindex $p = (p_1, \dots, p_n)$ $|p| = p_1 + \dots + p_n$, $a^p = a_1^{p_1} \dots a_n^{p_n}$, $p! = p_1! \dots p_n!$).

For $f \in BH_2(B_n)$ we have

$$\int_{S^{2n-1}} K_s(w,z) f(z)\, d\sigma_n(z) = f(w) - \sum_{l=0}^{s} F_l(w). \tag{2}$$

We use that monomials are orthogonal in $H_2(B_n)$ and

$$\|z^p\|_2^2 = \int_{S^{2n-1}} |z_1|^{p_1} \dots |z_n|^{p_n}\, d\sigma_n(z) = \frac{(n-1)!p!}{(n+|p|-1)!}$$

(see Rudin [1]). Now it follows from (2) and Hölder inequality that for $f \in BH_2(B_n)$

$$\left| f(w) - \sum_{l=0}^{s} F_l(w) \right| \le \left(\int_{S^{2n-1}} |K_s(w,z)|^2 \, d\sigma_n(z) \right)^2$$
$$= \sqrt{\frac{1}{(1-|w|^2)^n} - \sum_{|p|\le s} \frac{(n+|p|-1)!}{(n-1)!p!} |w_1|^{2p_1} \dots |w_n|^{2p_n}}$$
$$= \sqrt{\frac{1}{(1-|w|^2)^n} - \sum_{|p|\le s} \binom{n+|p|-1}{n-1} |w|^{2|p|}}.$$

The lemma is proved. □

Remark. In fact, it follows from (2)and Theorem 1 from [6] that the Taylor's series is an optimal recovery algorithm for functions from $BH_2(B_n)$ if the Taylor's information in the origin is at the disposal. In case $n = 1$ this fact is well-known (see [7]).

Lemma 2. *Let* $s \in \mathbb{N}$, $N_m \le s < N_{m+1}$, $f_1, \dots, f_s$ *be the orthonormal set of functions of* $H_2(B_n)$, $0 < r < 1$. *Then*

$$\int_{S^{2n-1}} \sum_{l=1}^{s} |f_l(rz)|^2 \, d\sigma_n(z) \le \sum_{l=0}^{m-1} \binom{n+l-1}{n-1} r^{2l} + (s - N_m) r^{2m}. \tag{3}$$

Proof. Let F_l be the homogeneous polynomial of degree l. It is obvious that

$$\int_{S^{2n-1}} |F_l(rz)|^2 \, d\sigma_n(z) = r^{2l} \|F_l\|_2^2. \tag{4}$$

If $f \in H_2(B_n)$ and $f(z) = \sum_{l=k}^{\infty} F_l(z)$ is a homogeneous decomposition of f, then it follows from (4) and the orthogonality of monomials that

$$\int_{S^{2n-1}} |f(rz)|^2 \, d\sigma_n(z) \le r^{2k} \|f\|_2^2. \tag{5}$$

Let U be a unitary transformation of $\mathbb{C}^s$ and

$$\begin{pmatrix} u_{11} & \dots & u_{1s} \\ \dots & \dots & \dots \\ u_{s1} & \dots & u_{ss} \end{pmatrix}$$

is the matrix of this transformation. The mapping U transforms the orthonormal set $\{f_1, \dots, f_s\}$ into the set $\{g_1, \dots, g_s\}$, $g_l = \sum_{k=1}^{s} u_{lk} f_k$ which is the orthonormal one (in $H_2(B_n)$) too. Besides that for every

$w \in B_n$ we have $\displaystyle\sum_{l=1}^{s} |g_l(w)|^2 = \sum_{l=1}^{s} |f_l(w)|^2$ because U is the unitary transformation. Hence

$$\int_{S^{2n-1}} \sum_{l=1}^{s} |f_l(rz)|^2 \, d\sigma_n(z) = \int_{S^{2n-1}} \sum_{l=1}^{s} |g_l(rz)|^2 \, d\sigma_n(z).$$

It is easy to check that there exists the unitary transformation U such that for $N_q < k \le N_{q+1}$ the homogeneous expansion of g_k has no polynomials of degree less than $q+1$. Indeed, let U_1 be the unitary transformation of $\mathbb{C}^s$ which transforms $(f_1(0), \dots, f_s(0))$ into $(a_1, 0, \dots, 0)$ where $a_1 = \left(\displaystyle\sum_{l=1}^{s} |f_l(0)|^2 \right)^{1/2}$. Transformation U_1 maps $(f_1, \dots, f_s)$ into $(g_1^1, \dots, g_s^1)$ and $g_2^1(0) = \ldots = g_s^1(0) = 0$. Let $\widetilde{U}_2$ be the unitary transformation of $\mathbb{C}^{s-1}$ which maps $\left\{ \dfrac{\partial g_2^1}{\partial z_1}\Big|_0, \dots, \dfrac{\partial g_s^1}{\partial z_1}\Big|_0 \right\}$ into $(a_2, 0, \dots, 0)$, and

$$U_2 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \widetilde{U}_2 & \\ 0 & & & \end{pmatrix}, \qquad U_2 \colon \{g_1^1, \dots, g_s^1\} \to \{g_1^1, g_2^2 \dots, g_s^2\}.$$

Then $\dfrac{\partial g_3^2}{\partial z_1}\Big|_0 = \ldots = \dfrac{\partial g_s^2}{\partial z_1}\Big|_0 = 0$. Iterating this process we shall construct the sequence of transformations $U_3, \dots, U_s$. The transformation $U = U_s \circ U_{s-1} \circ \ldots \circ U_1$ satisfies our requirement. We have $U \colon \{f_1, \dots, f_s\} \to \{g_1^1, \dots, g_s^s\}$, for every $N_q < k \le N_{q+1}$

$$\int_{S^{2n-1}} |g_k^k(rz)|^2 \, d\sigma_n(z) \le r^{2q}.$$

The lemma is proved. □

Proof of the Theorem. In view of (1) it is sufficient to prove that

$$(6) \quad \lambda_{N_k}(BH_2(B_n), L_{\infty,r}(B_n)) \le \sqrt{\frac{1}{(1-r^2)^n} - \sum_{l=0}^{k-1} \binom{n+l-1}{n-1} r^{2l}},$$

$$(7) \quad d^{N_k}(BH_2(B_n), L_{\infty,r}(B_n)) \ge \sqrt{\frac{1}{(1-r^2)^n} - \sum_{l=0}^{k-1} \binom{n+l-1}{n-1} r^{2l}}.$$

The upper evaluation (6) follows from Lemma 1. Operator Λ_{N_k} is the one which maps a function of $H_2(B_n)$ into its Taylor's series of order N_k.

The lower bound (7) will be proved by using Lemma 2. Consider some subspace L^{N_k} of $H_2(B_n)$ which codimension is equal to N_k. Let

$f_1, \dots, f_{N_k}$ be an orthonormal basis in $(L^{N_k})^{\perp}$. For $w \in B_n$ define the following function

$$h_w(z) = \frac{1}{(1 - \langle z, w\rangle)^n} - \sum_{l=1}^{N_k} \overline{f_l(w)} f_l(z).$$

It is easy to check that for every $w \in B_n$

$$h_w \in L^{N_k}.$$

Therefore

$$\sup_{g \in BH_2(B_n) \cap L^{N_k}} \|g\|_{\infty,r} \ge \sup_{|w|=r} \frac{\|h_w(\cdot)\|_{\infty,r}}{\|h_w(\cdot)\|_2} \ge \sup_{|w|=r} \frac{|h_w(w)|}{\|h_w(\cdot)\|_2}$$
$$= \sup_{|w|=r} \frac{1}{\|h_w(\cdot)\|_2} \left(\frac{1}{(1-r^2)^n} - \sum_{l=1}^{N_k} |f_l(w)|^2 \right).$$

The direct computation shows that

$$\|h_w(\cdot)\|_2 = \left(\frac{1}{(1-|w|^2)^n} - \sum_{l=1}^{N_k} |f_l(w)|^2 \right)^{1/2}.$$

Hence

$$\sup_{g \in BH_2(B_n) \cap L^{N_k}} \|g\|_{\infty,r} \ge \sup_{|w|=r} \sqrt{\frac{1}{(1-r^2)^n} - \sum_{l=1}^{N_k} |f_l(w)|^2}$$
$$= \sqrt{\frac{1}{(1-r^2)^n} - \inf_{|w|=r} \sum_{l=1}^{N_k} |f_l(w)|^2}.$$

In accordance with Lemma 2

$$\inf_{|w|=r} \sum_{l=1}^{N_k} |f_l(w)|^2 \le \int_{S^{2n-1}} \sum_{l=1}^{N_k} |f_l(rw)|^2 \, d\sigma_n(w) \le \sum_{l=0}^{k-1} \binom{n+l-1}{n-1} r^{2l}.$$

Theorem is proved. □

If $n = 1$ then $N_k = k$, the series $\sum_{l=0}^{k-1} \binom{n+l-1}{n-1} r^{2l}$ converts into $\sum_{l=0}^{k-1} r^{2l}$ and we obtain the following corollary.

Corollary 1.

$$d^k(BH_2, L_{\infty,r}) = \lambda_k(BH_2, L_{\infty,r}) = \frac{r^k}{\sqrt{1-r^2}}.$$

References

[1] W. Rudin, Function Theory in the Unit Ball of $\mathbb{C}^n$. Springer-Verlag. Berlin, Heidelberg, New York, 1980.

[2] V.M. Tikhomirov, Approximation Theory, Itogi Nauki i Techniki, Ser. Sovr. Probl. Math., Fund. Napravlenia, 1987 (in Russian).

[3] A. Pinkus, N-Width in Approximation Theory, Berlin, Springer, 1985.

[4] O.G. Parfenov, Gelfand's n-width of the unit ball of Hardy class in the weighted space, Math. Zametki, V. 39 (1985), No. 2, 171–175.

[5] J.F. Fisher and C.A. Micchelli, The n-width of sets of analytic functions, Duke Math. J., 47 (1980), 789–801.

[6] K.Yu. Osipenko and M.I. Stessin, On recovery problems in the Hardy and Bergman spaces, Math. Zametki, to appear (in Russian).

[7] T.J. Rivlin, The optimal recovery of functions, Contemp. Math., 9 (1982), 121–151.