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# ON OPTIMAL RECOVERY OF HOLOMORPHIC FUNCTIONS AND SCHWARTZ LEMMA IN THE UNIT BALL OF $\mathbb{C}^n$

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Introduction. Let $\Omega$ be a subset of $\mathbb{C}^n$ and $\mu$ is a nonnegative measure on $\Omega$. For $1 \le p \le \infty$ let $L_p(\Omega, \mu)$ be the Lebesgue space of complex-valued functions on $\Omega$ with the usual norm

$$\|f\|_p = \left( \int_\Omega |f(z)|^p \, d\mu(z) \right)^{1/p}, \quad 1 \le p < \infty,$$
$$\|f\|_\infty = \operatorname*{vrai\,sup}_{z \in \Omega} |f(z)|.$$

Denote $BX$ the closed unit ball of the normed space $X$. Consider a linear subspace $X_p \subset L_p(\Omega, \mu)$ with the induced norm. We define an information operator as a mapping $I\colon BX_p \to Y$ where $Y$ is some set. Let $L$ be a functional on $X_p$ and $S\colon Y \to \mathbb{C}$ is a function on $Y$. The function $S$ defines a recovery algorithm $Lf \approx SIf$.

Consider the following problem of optimal recovery

$$E(L, I, X_p) = \inf_S \sup_{f \in BX_p} |Lf - SIf|. \tag{1}$$

$E(L, I, X_p)$ is called the intrinsic error of recovery. If

$$E(L, I, X_p) = \sup_{f \in BX_p} |Lf - S_* If| = |Lf_* - S_* If| \quad (f_* \in B_p),$$

then $S_*$ is said to be an optimal algorithm and $f_*$ is a worst function.

There are a lot of papers devoted to the recovery problems. The detailed bibliography and the history of the subject can be found in Micchelli and Rivlin [1], [2] and also in Traub and Wozniakowski [3].

In this paper we solve some recovery problems in the Hardy and Bergman spaces in the unit ball of $\mathbb{C}^n$. In particular, some variants of the Schwartz Lemma follow. In the unit disk the similar results were previously obtained for the Hardy spaces in Fisher and Micchelli [4] and for the Bergman spaces in Osipenko and Stessin [5].

In Section 1 we prove the general theorem which connects the worst functions with reproducing kernels in some weighted spaces, where weights are associated with an information operator. In Section 2 this theorem is applied to the recovery problem in the Hardy spaces $H_p(B_n)$ ($B_n$ is the unit ball of $\mathbb{C}^n$) when the information operator is the operator of restriction on some affine subset of $B_n$. Here there arises the astonishing fact that the case $n \le 5$ differs from $n \ge 6$.

The last section is devoted to proving the analogous results for the Bergman spaces.

1. Optimal recovery and reproducing kernels. Let $\Omega$ be a simply connected domain in $\mathbb{C}^n$, $X_p$ be the space of holomorphic in $\Omega$ functions, which belong to $L_p(\overline{\Omega}, \mu)$ (it supposes that for holomorphic function $f \not\equiv 0$ $\|f\|_p \ne 0$). If $\mu(\partial\Omega) \ne 0$, only those holomorphic functions are considered which have boundary values almost everywhere with respect to measure $\mu$. Let $\varphi \in X_p$. Suppose that there exists the reproducing kernel $K_\varphi(z, w)$ with the weight $|\varphi|^{p-2}$, i.e., for every $f \in X_p$ the next equality holds

$$f(z) = \int_\Omega \overline{K_\varphi(z,w)} f(w) |\varphi(w)|^{p-2}\, d\mu(w)$$

and $K_\varphi(z,\cdot) \in X_p$ for every $z \in \Omega$. Note that for every $z \in \Omega$ we have $K_\varphi(z,\cdot) \in X_2(\varphi,p)$, where $X_2(\varphi,p)$ is the Hilbert space of holomorphic in $\Omega$ functions, which satisfy

$$\|f\|_{X_2(\varphi,p)} = \left(\int_\Omega |f(w)|^2 |\varphi(w)|^{p-2}\, d\mu(w)\right)^{1/2} < \infty.$$

Let $A \subset \Omega$. Define the information operator $I$ as the operator of restriction on $A$: $If = f_{|A}$. Put $Lf = f(a)$, where $a \in \Omega \setminus A$. In this case denote $E(a, A, X_p)$ the intrinsic error from (1). Let $M_A$ be the closed linear subspace of $X_2(\varphi,p)$, which contains $K_\varphi(z,\cdot)$ for every $z \in A$:

$$M_A = \begin{cases} \mathrm{cl}(\mathrm{span}\{K_\varphi(z,\cdot),\ z \in A\}), & A \neq \emptyset, \\ \{0\}, & A = \emptyset, \end{cases}$$

and $\mathrm{Pr}_A$ be the orthogonal projection: $\mathrm{Pr}_A \colon X_2(\varphi,p) \to M_A$. It follows from the Riesz theorem that for every $f \in X_2(\varphi,p)$ there exists complex measure $m_f$ such that

$$(\mathrm{Pr}_A f)(\cdot) = \int_A K_\varphi(z,\cdot)\, dm_f(z).$$

Denote $m_a$ the appropriate measure for $f = K_\varphi(a,\cdot)$ and put

$$(2)\quad \phi(w) = K_\varphi(a,w) - \mathrm{Pr}_A K_\varphi(a,w) = K_\varphi(a,w) - \int_A K_\varphi(z,w)\, dm_a(z).$$

We call $\varphi$ to be regular with respect to pair $(a, A)$ if

i) $I\varphi = \varphi_{|A} \equiv 0$;
ii) $\chi = \phi/\varphi \in X_\infty$, $\chi^{-1} = \varphi/\phi \in X_\infty$.

**Theorem 1.** *Let $a \in \Omega$, $A \subset \Omega$, and $\varphi$ is regular with respect to pair $(a, A)$. Put $\alpha = [\chi(a)]^{(2-p)/p}$, $g = \chi^{2/p}\varphi$. Then*

i) *$f(a) \approx S_* If = \alpha \int_A [\chi(z)]^{(p-2)/p} f(z)\, \overline{dm_a(z)}$ is an optimal recovery algorithm,*
ii) *$f^*(w) = |\phi(a)|^{-1/p} g(w)$ is a worst function,*
iii) *$E(a, A, X_p) = |\phi(a)|^{1/p} |\varphi(a)|^{(p-2)/p}$.*

*Proof.* Since $\varphi$ is regular we have $g \in X_p$ and $\chi^{(p-2)/p} f \in X_p$ for every $f \in X_p$. Hence

$$
\begin{aligned}
(3)\quad \alpha \int_{\Omega} \overline{g(w)} |g(w)|^{p-2} f(w)\, d\mu(w) \\
&= \alpha \int_{\Omega} \overline{\phi(w)} [\chi(w)]^{(p-2)/p} f(w) |\varphi(w)|^{p-2}\, d\mu(w) \\
&= \alpha \int_{\Omega} \left[ \overline{K_\varphi(a,w)} - \int_A \overline{K_\varphi(z,w)}\, \overline{dm_a(z)} \right] [\chi(w)]^{(p-2)/p} f(w) \\
&\qquad \times |\varphi(w)|^{p-2}\, d\mu(w) = f(a) - S_* I f.
\end{aligned}
$$

It immediately follows from (3) that

$$
\begin{aligned}
E(a, A, X_p) &\le \sup_{f \in BX_p} |f(a) - S_* I f| \\
&= \sup_{f \in BX_p} \left| \alpha \int_{\Omega} \overline{g(w)} |g(w)|^{p-2} f(w)\, d\mu(w) \right| \le |\alpha| \|g\|_p^{p-1}
\end{aligned}
$$

and $\alpha \|g\|_p^p = g(a)$. Thus $\|g\|_p = \left| \frac{g(a)}{\alpha} \right|^{1/p} = |\phi(a)|^{1/p}$, and $\|f^*\|_p = 1$ (i.e., $f^* \in BX_p$). As $I f^* = f^*_{|A} \equiv 0$ we have for every algorithm S

$$
\begin{aligned}
|f^*(a) - S I f^*| + |-f^*(a) - S I(-f^*)| &= |f^*(a) - S(0)| + |-f^*(a) - S(0)| \\
&\ge 2|f^*(a)|
\end{aligned}
$$

and, therefore $\sup_{f \in BX_p} |f(a) - S I f| \ge |f^*(a)|$,

$$
E(a, A, X_p) \ge |f^*(a)| = \frac{|g(a)|}{\|g\|_p} = |\alpha| \|g\|_p^{p-1} = |\phi(a)|^{1/p} |\varphi(a)|^{(p-2)/p}.
$$

The theorem is proved. □

**Corollary 1.** *Let $K(z,w)$ is a reproducing kernel for functions from $X_p$, $a \in \Omega$, and for every $w \in \Omega$ $K(a,w) \neq 0$. Then*

$$
\sup_{f \in BX_p} |f(a)| = (K(a,a))^{1/p}.
$$

*Proof.* Put $\varphi(w) \equiv 1$. As $K(a,w) \neq 0$, $\varphi$ is regular with respect to $(a, \emptyset)$. It follows from Theorem 1 that

$$
\sup_{f \in BX_p} |f(a)| = \frac{[K(a,a)]^{2/p}}{\|(K(a,\cdot))^{2/p}\|_p}.
$$

Now, note that

$$
\|(K(a,\cdot))^{2/p}\|_p = \left( \int_{\Omega} K(a,w) \overline{K(a,w)}\, d\mu(w) \right)^{1/p} = (K(a,a))^{1/p}.
$$

□

<u>Remark 1</u>. If the conditions of Theorem 1 hold, then

$$E(a, A, X_p) = \sup_{\substack{f \in BX_p \\ f_{|A} \equiv 0}} |f(a)|. \tag{4}$$

Indeed,

$$\sup_{\substack{f \in BX_p \\ f_{|A} \equiv 0}} |f(a)| \geq |f^*(a)| = E(a, A, X_p) = \sup_{f \in BX_p} |f(a) - S_* I f|$$

$$\geq \sup_{\substack{f \in BX_p \\ f_{|A} \equiv 0}} |f(a)|.$$

<u>Remark 2</u>. If $p = 2$ the regularity condition is unnecessary because $\chi^{(p-2)/p} \equiv 1$, and $\chi^{2/p}$ is correctly defined even in the case when $\chi$ has zeros in $\Omega$. More precisely, let $K(z, w)$ be a reproducing kernel with weight 1 for functions from $X_2$, $A$ is a subset of $\Omega$, $a \in \Omega$. By analogy with (2) we define

$$\phi(w) = K(a, w) - \mathrm{Pr}_A K(a, w) = K(a, w) - \int_A K(z, w)\, dm_a(z).$$

The next theorem holds.

**Theorem 1′.**

i) $f(a) \approx \int_A f(z)\, \overline{dm_a(z)}$ *is an optimal algorithm in the recovery problem in* $X_2$*;*

ii) $f^* = |\phi(a)|^{-1/2} \phi$ *is a worst function;*

iii) *the intrinsic error is*

$$E(a, A, X_2) = |\phi(a)|^{1/2}.$$

The meaning of Theorem 1 is that, in some cases, it reduces the problem of finding a worst function and an optimal algorithm to the problem of finding the divisor of a worst function, which connects with the information operator. In fact, the divisor of the worst function is the largest extension of the information set, which does not lead to decreasing the intrinsic error.

2. <u>Recovery in $H_p(B_n)$</u>. Let $B_n$ be the unit ball of $\mathbb{C}^n$

$$B_n = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z|^2 = \sum_{k=1}^{n} |z_k|^2 < 1 \right\}.$$

The Hardy space $H_p(B_n)$ is the set of holomorphic in $B_n$ functions, which satisfy the condition

$$\|f\|_p = \sup_{0<r<1} \left( \int_{|z|=1} |f(rz)|^p \, d\sigma_n(z) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty = \sup_{z \in B_n} |f(z)| < \infty,$$

where $\sigma_n$ is the probability measure on sphere $S^{2n-1}$ which is invariant with respect to the orthogonal group $O(2n)$. It is known (see Rudin [6]) that functions from $H_p(B_n)$ have finite boundary values almost everywhere and so $H_p(B_n)$ can be considered as a subspace of $L_p(\overline{B}_n, \mu)$ where $\mu$ is supported on $\partial B_n = S^{2n-1}$ and coincides with $\sigma_n$ in $S^{2n-1}$.

Let $L$ be an affine plane in $\mathbb{C}^n$, $A = L \cap B_n$. In this section we deal with the recovery of functions from $H_p(B_n)$ at some point $a \in B_n \setminus A$ in the case when $If = f_{|A}$. At the beginning let us consider the case

$$L = \{\, z = (z_1, \dots, z_n) \in \mathbb{C}^n : z_n = 0 \,\}.$$

Let $u \in \mathbb{C}$, $|u| < 1$, $\alpha > 1$, $m$ is a nonnegative integer. Put

$$F_m(\alpha, u) = \sum_{k=0}^{\infty} \frac{\Gamma(m+k+\alpha)}{\Gamma(k+\alpha)} u^k.$$

We can easily prove that

$$F_m(\alpha, u) = \frac{(\alpha-1)\alpha \dots (\alpha+m-1)}{(1-u)^{m+1}} \sum_{k=0}^{m} \frac{(-1)^k}{\alpha+k-1} \binom{m}{k} u^k. \tag{5}$$

**Proposition 1.** *Let $1 \le p < \infty$. Then*
(6)

$$K_{z_n}(z, w) = \frac{1}{(n-1)!(1-\langle w_1', z_1' \rangle)^{n+p/2-1}} F_{n-1}\left(\frac{p}{2}, \frac{\overline{z}_n w_n}{1-\langle w_1', z_1' \rangle}\right),$$

*where $x_1' = (x_1, \dots, x_{n-1}, 0)$ for $x = (x_1, \dots, x_n)$, is the reproducing kernel for functions from $H_p(B_n)$ with the weight $|z_n|^{p-2}$.*

*Proof.* As polynomials are dense in $H_p(B_n)$ it is sufficient to prove that (6) is the reproducing kernel for the polynomials of the special kind $f(z) = f_1(z_1')z_n^k$, where $f_1(z_1')$ is a polynomial, which depends only on $z_1, \dots, z_{n-1}$. Let $\nu_n$ be the normalized Lebesgue measure on $B_n$. We have

$$\begin{aligned}
&\int_{|z|=1} \overline{K_{z_n}(z, w)} f_1(w_1') w_n^k |w_n|^{p-2} \, d\sigma(w) \\
&\quad = \frac{1}{(n-1)!} \int_{B_{n-1}} \frac{(1-|w_1'|^2)^{(p-2)/2} f_1(w_1')}{(1-\langle z_1', w_1' \rangle)^{n+p/2-1}} \, d\nu_{n-1}(w_1') \\
&\quad \times \frac{1}{2\pi i} \int_{|w_n|=(1-|w_1'|^2)^{1/2}} w_n^k \sum_{m=0}^{\infty} \frac{\Gamma(n-1+p/2+m) z_n^m w_n^{-m}}{\Gamma(p/2+m)(1-\langle z_1', w_1' \rangle)^m} \frac{dw_n}{w_n} \\
&\quad = \frac{\Gamma(n+(p-2)/2+k)}{(n-1)!\Gamma((p-2)/2+k+1)} z_n^k \\
&\quad \times \int_{B_{n-1}} \frac{(1-|w_1'|^2)^{k+(p-2)/2} f_1(w_1')}{(1-\langle z_1', w_1' \rangle)^{n+k+(p-2)/2}} \, d\nu_{n-1}(w_1') = f_1(z_1') z_n^k.
\end{aligned}$$

In the last equality we used that

$$\frac{\Gamma(n+s+1)}{\Gamma(n+1)\Gamma(s+1)} \frac{(1-|w|^2)^s}{(1-\langle z,w\rangle)^{n+s+1}}$$

is the reproducing kernel in $H_p(B_n)$ (see Rudin [6]). The proposition is proved. □

In view of $K_{z_n}(a'_1, w) \in M_A$ and

$$\langle K_{z_n}(b,\cdot), K_{z_n}(a,\cdot) - K_{z_n}(a'_1,\cdot)\rangle_{X_2(z_n,p)} = K_{z_n}(b,a) - K_{z_n}(b,a'_1) = 0$$

for every $b = (b_1, \ldots, b_{n-1}, 0)$, we have

$$\mathrm{Pr}_A K_{z_n}(a,w) = K_{z_n}(a'_1, w),$$

$$(7)\quad \begin{aligned} \phi(w) &= K_{z_n}(a,w) - K_{z_n}(a'_1,w) \\ &= \frac{1}{(n-1)!} \frac{\overline{a}_n w_n}{(1-\langle w'_1, a'_1\rangle)^{n+p/2}} F_{n-1}\left(\frac{p}{2}+1, \frac{\overline{a}_n w_n}{1-\langle w'_1, a'_1\rangle}\right). \end{aligned}$$

It follows from the last expression that $\phi(w) = w_n$ is regular with respect to pair $(a, \{w_n = 0\})$ if and only if $F_{n-1}\left(\frac{p}{2}+1, \frac{\overline{a}_n w_n}{(1-\langle w'_1, a'_1\rangle)}\right)$ does not turn into zero in $B_n$. It is not valid in general, nevertheless we shall find the domain (for $a$), where the regularity holds, for every $n$ and $p$.

**Proposition 2.**

i) *If $n \le 5$, then for every $1 \le p < \infty$ and $a \in B_n$ function $F_{n-1}\left(\frac{p}{2}+1, \frac{\overline{a}_n w_n}{(1-\langle w'_1, a'_1\rangle)}\right)$ does not turn into zero in $B_n$.*

ii) *For every $n > 5$ there exist $a \in B_n$ and $p$, $1 \le p < \infty$, such that zero set of $F_{n-1}\left(\frac{p}{2}+1, \frac{\overline{a}_n w_n}{(1-\langle w'_1, a'_1\rangle)}\right)$ is not empty in $B_n$.*

We shall prove this proposition at the end of this section, because the proof is quite technical.

Now we note that

$$\sup_{w\in B_n} \left| \frac{\overline{a}_n w_n}{1-\langle w'_1, a'_1\rangle} \right| = \frac{|a_n|}{\sqrt{1-|a'_1|^2}}$$

and, therefore, in view of (5), the zero-set of $F_{n-1}$ is empty in $B_n$ if and only if polynomial

$$P_{n-1}\left(\frac{p}{2}+1, u\right) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k+p/2} \binom{n-1}{k} u^k$$

has no zeros in the disk $\left\{ |u| < \frac{|a_n|}{\sqrt{1-|a'_1|^2}} \right\}$.

Put

$$\lambda_n(p) = \min\left\{ |u| : P_{n-1}\left(\frac{p}{2}+1, u\right) = 0 \right\},$$
$$\Delta_n^1(p) = \{ a \in B_n : |a_n|^2 < \lambda_n^2(p)(1-|a_1'|^2) \}.$$

It follows from Proposition 2 that $\Delta_n^1(p) = B_n$ if $n \le 5$.

The next lemma immediately follows from theorem 1, proposition 1, and (7).

**Lemma 1.** *Let $1 \le p < \infty$ and $a \in \Delta_n^1(p)$. Then*

i) *the algorithm*

$$f(a) \approx (D_{n1}(p,a))^{\frac{p-2}{p}} f(a_1'),$$

*where*

$$D_{n1}(p,a) = \frac{(1-|a|^2)^n}{\sum_{k=0}^{n-1} \frac{(-1)^k}{2k/p+1}\binom{n-1}{k}|a_n|^{2k}(1-|a_1'|^2)^{n-k}}$$

*is an optimal one;*

ii)

$$f_1^*(p,a,w) = \left( \frac{\Gamma\left(n+\frac{p}{2}\right) D_{n1}(p,a)}{\Gamma(n)\Gamma(p/2+1)} (1-|a_1'|^2)^{n+\frac{p}{2}} \right)^{1/p}$$
$$\times \frac{w_n}{(1-\langle w_1', a_1'\rangle)(1-\langle w,a\rangle)^{2n/p}}$$
$$\times \left( \sum_{k=0}^{n-1} \frac{(-1)^k}{2k/p+1}\binom{n-1}{k}\left(\frac{\bar{a}_n w_n}{1-\langle w_1', a_1'\rangle}\right)^k \right)^{2/p}$$

*is a worst function;*

iii)

$$E(a,\{z_n=0\},H_p(B_n))$$
$$= \left( \frac{\Gamma\left(n+\frac{p}{2}\right)}{\Gamma(n)\Gamma(p/2+1)D_{n1}(p,a)} \right)^{1/p} \frac{|a_n|}{(1-|a_1'|^2)^{n/p+1/2}}.$$

When proving proposition 2 we shall see that $\lambda_n(2) = 2\sin\frac{\pi}{n}$ $(n>1)$. Now we shall provide a lower estimate $\lambda_n(p)$ for every $n$ and $p$. Though being inaccurate this estimation will give us the opportunity to show that for every $n$ and $a \in B_n$ there exists $p_0$ such that $a \in \Delta_n^1(p)$ if $p > p_0$.

**Proposition 3.**

$$\lambda_n(p) \ge \frac{p+2}{p+2n}.$$

*Proof.* In view of (5) we must prove that $F_{n-1}\left(\frac{p}{2}+1,u\right)$ has no zeros in the disk $\left\{|u|<\frac{p+2}{p+2n}\right\}$. Consider the function

$$\phi(v)=\sum_{k=0}^{\infty}\frac{\Gamma(n+k+p/2)}{\Gamma(k+p/2+1)}\left(\frac{p+2}{p+2n}v\right)^k=\sum_{k=0}^{\infty}d_kv^k.$$

We have

$$\frac{d_{k+1}}{d_k}=\frac{(n+k+p/2)(p+2)}{(k+p/2+1)(p+2n)}\leq 1.$$

It is well known (see Khavinson [7]) that $d_k$ being positive and $d_{k+1}\leq d_k$, $\phi(v)$ has no zeros in the unit disk $\{|v|<1\}$ and required statement follows. The proposition is proved. □

**Corollary 2.** *If* $p>p_0=2\frac{n|a_n|-\sqrt{1-|a_1'|^2}}{\sqrt{1-|a_1'|^2}-|a_n|}$ *then* $a\in\Delta_n^1(p)$.

Indeed,

$$\lambda_n(p)\geq\frac{p+2}{p+2n}>\frac{p_0+2}{p_0+2n}=\frac{|a_n|}{\sqrt{1-|a_1'|^2}}.$$

**Lemma 2.** *Let* $a\in B_n$ *and* $p=\infty$. *Then*

i) *the algorithm*

$$f(a)\approx\frac{1-|a|^2}{1-|a_1'|^2}f(a_1')$$

*is an optimal one;*

ii) $f_1^*(\infty,a,w)=\frac{\sqrt{1-|a_1'|^2}w_n}{1-\langle w_1',a_1'\rangle}$ *is a worst function;*

iii) $E(a,\{w_n=0\},H_\infty(B_n))=\frac{|a_n|}{\sqrt{1-|a_1'|^2}}$.

*Proof.* Let $f\in BH_\infty(B_n)$. Consider the function $\varphi(u)=f(a_1,\ldots,a_{n-1},u)$. $\varphi$ is holomorphic in the disk $\{|u|<\sqrt{1-|a_1'|^2}\}$ and $\sup|\varphi(u)|\leq 1$. It follows from Osipenko [8] that

$$\left|\varphi(a_n)-\left(1-\frac{|a_n|^2}{1-|a_1'|^2}\right)\varphi(0)\right|=\left|f(a)-\frac{1-|a|^2}{1-|a_1'|^2}f(a_1')\right|\leq\frac{|a_n|}{\sqrt{1-|a_1'|^2}},$$

and so the error of the algorithm

$$f(a)\approx\frac{1-|a|^2}{1-|a_1'|^2}f(a_1')$$

is not greater than $\dfrac{|a_n|}{\sqrt{1-|a_1'|^2}}$. On the other hand, $f_1^*(\infty, a, w) \in BH_\infty(B_n)$, $If_1^* = f^*_{1\,|w_n=0} = 0$, and hence the intrinsic error can be evaluated as

$$E(a, \{w_n = 0\}, H_\infty(B_n)) \geq |f_1^*(\infty, a, a)| = \frac{|a_n|}{\sqrt{1-|a_1'|^2}}.$$

The proof is complete. □

Now consider the case

$$A = \{\, z = (z_1, \ldots, z_n) \in B_n : z_{n-k+1} = \ldots = z_n = 0 \,\}.$$

For $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ put $w_k' = (w_1, \ldots, w_{n-k}, 0, \ldots, 0)$, $w_k'' = w - w_k' = (0, \ldots, 0, w_{n-k+1}, \ldots, w_n)$. Denote $\Delta_n^k(p) = \{z \in B_n : |z_k''| < \lambda_n(p)\sqrt{1-|z_k'|^2}\}$, $\Delta_n^k(\infty) = B_n$. As in the case $k = 1$, we have $\Delta_n^k(p) = B_n$ for every $p$ if $n \leq 5$.

**Lemma 3.** *Let $a \in \Delta_n^k(p) \setminus A$. Then*

i) *the algorithm*

$$f(a) \approx \begin{cases} (D_{nk}(p,a))^{(p-2)/p} f(a_k'), & 1 \leq p < \infty, \\ \dfrac{1-|a|^2}{1-|a_k'|^2} f(a_k'), & p = \infty, \end{cases}$$

*where*

$$D_{nk}(p,a) = \frac{(1-|a|^2)^n}{\displaystyle\sum_{j=0}^{n-1} \frac{(-1)^j}{2j/p+1} \binom{n-1}{j} |a_k''|^{2j} (1-|a_k'|^2)^{n-j}}$$

*is an optimal one;*

ii)

$$f_k^*(p, a, w) = \left( \frac{\Gamma\left(n + \frac{p}{2}\right) D_{nk}(p,a)}{\Gamma(n)\Gamma(p/2+1)} (1-|a_k'|^2)^{n+\frac{p}{2}} \right)^{1/p}$$
$$\times \frac{\langle w_k'', a_k'' \rangle}{|a_k''|(1-\langle w_k', a_k' \rangle)(1-\langle w, a \rangle)^{2n/p}}$$
$$\times \left( \sum_{j=0}^{n-1} \frac{(-1)^j}{2j/p+1} \binom{n-1}{j} \left( \frac{\langle w_k'', a_k''}{1-\langle w_k', a_k' \rangle} \right)^j \right)^{2/p}, \quad 1 \leq p < \infty,$$

$$f_k^*(\infty, a, w) = \frac{\sqrt{1-|a_k'|^2}}{|a_k''|} \frac{\langle w_k'', a_k'' \rangle}{1-\langle w_k', a_k' \rangle}$$

*is a worst function;*

iii)

$$
\begin{aligned}
&E(a, A, H_p(B_n)) \\
&= \begin{cases} \left( \dfrac{\Gamma(n+p/2)}{\Gamma(n)\Gamma(p/2+1) D_{nk}(p,a)} \right)^{1/p} \dfrac{|a''_k|}{(1-|a'_k|^2)^{n/p+1/2}}, & 1 \le p < \infty, \\ \dfrac{|a''_k|}{\sqrt{1-|a'_k|^2}}, & p = \infty. \end{cases}
\end{aligned}
$$

*Proof.* Let $U$ be the $n \times n$ matrix,

$$
U = \begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ & & & \\ & 0 & & \boxed{C} \end{pmatrix},
$$

where $C$ is the unitary $k \times k$ matrix which transforms $(a_{n-k+1}, \dots, a_n)$ into $(0, \dots, 0, |a''_k|)$. Since the measure $\sigma_n$ being invariant with respect to $U$, the mapping $f(z) \to f(Uz)$ is an isometry of $H_p(B_n)$. Using the unitary transformation with matrix $U$ we reduce the problem of recovering $f(a)$ to the one of recovering $f(Ua)$, using the information, provided by operator $I_U$, $I_U f = f_{|UA} = f_{|A} = If$. ($I_U$ coincides with $I$ because A is a $U$-invariant set). Let $\widetilde{I}$ be the information operator $\widetilde{I} f = f_{|(Uz)_n=0}$. As $\{(Uz)_n = 0\} \supset A$, we evidently have

$$
E(a, A, H_p(B_n)) \ge E(Ua, \{(Uz)_n = 0\}, H_p(B_n)).
$$

Note that $(Ua)'_1 = a'_k \in A$. In accordance with the statement of Lemmas 1 and 2, the optimal algorithm in the recovery problem with the information operator $\widetilde{I}$ is $\mathrm{const}\, f((Ua)'_1) = \mathrm{const}\, f(a'_k)$. This algorithm uses only information $If = f_{|A}$, and so it is an optimal one in the recovery problem with information operator $I$. To finish the proof we note that $\langle (Uw)'_1, (Ua)'_1 \rangle = \langle w'_k, a'_k \rangle$, $(Uw)_n \overline{(Ua)}_n = \langle Uw, Ua \rangle - \langle (Uw)'_1, (Ua)'_1 \rangle = \langle w, a \rangle - \langle w'_k, a'_k \rangle = \langle w''_k, a''_k \rangle$. Now the required statements immediately follow from Lemmas 1 and 2. □

By analogy with Corollary 2 $a \in \Delta_n^k(p)$ if

$$
p > 2 \frac{n|a''_k| - \sqrt{1-|a'_k|^2}}{\sqrt{1-|a'_k|^2} - |a''_k|}.
$$

Now let $A$ be an arbitrary linear affine subset of $B_n$. In the appropriate Hermitian coordinate system $A$ has the following form

(8) $$A = \{\, z \in B_n : z_{n-k+1} = c_{n-k+1}, \dots, z_n = c_n \,\}, \quad 1 \le k \le n,$$

(9) $$\sum_{l=n-k+1}^{n} |c_l|^2 = |c_k''|^2 < 1, \text{ where } c = (0, \dots, 0, c_{n-k+1}, \dots, c_n) \in B_n.$$

Recall (see Rudin [6]) that every automorphism $\varphi$ of $B_n$ determines the isometry $\Phi_\varphi$ of $H_p(B_n)$,

(10) $$(\Phi_\varphi f)(z) = \frac{f(\varphi(z))(1-|b|^2)^{n/p}}{(1-\langle z, b\rangle)^{2n/p}}, \quad b = \varphi^{-1}(0),$$

and every automorphism of $B_n$ is the superposition of some unitary transformation and special mapping $\varphi_b$

$$\varphi_b(z) = \frac{b - P_b z - \sqrt{1-|b|^2}\,Q_b z}{1-\langle z, b\rangle}, \quad b \in B_n,$$

where

$$P_b z = \begin{cases} \dfrac{\langle z, b\rangle}{\langle b, b\rangle} b, & b \ne 0, \\ 0, & b = 0, \end{cases} \qquad Q_b z = z - P_b z.$$

**Proposition 4.** *Let $a \in B_n$, $A \subset B_n$, $\varphi \in \mathrm{Aut}(B_n)$, $D = \varphi(A)$, $d = \varphi(a)$, $I$ and $I_1$ are the information operators: $If = f_{|A}$, $I_1 f = f_{|D}$. If the algorithm $f(a) \approx \int_A f(w)\, d\mu^*(w)$ is an optimal one in the recovery problem*

$$E(a, A, H_p(B_n)) = \inf_S \sup_{f \in BH_p(B_n)} |f(a) - SIf|$$

*and $f^*(w)$ is a worst function in this problem, then*

$$f(d) \approx \int_D f(w)\, d\mu_1^*(w),$$

*where*

$$d\mu_1^*(w) = \begin{cases} \left( \dfrac{1-\langle a, \varphi^{-1}(0)\rangle}{1-\langle \varphi^{-1}(w), \varphi^{-1}(0)\rangle} \right)^{2n/p} d(\mu^* \circ \varphi^{-1})(w), & 1 \le p < \infty, \\ d(\mu^* \circ \varphi^{-1})(w), & p = \infty, \end{cases}$$

*is an optimal algorithm in the problem*

$$E(d, D, H_p(B_n)) = \inf_S \sup_{f \in BH_p(B_n)} |f(d) - SI_1 f|,$$

$f_1^*(w) = \dfrac{f^*(\varphi^{-1}(w))(1-|\varphi(0)|^2)^{n/p}}{(1-\langle w,\varphi(0)\rangle)^{2n/p}}$ *is a worst function for the last problem, and*

$$E(d,D,H_p(B_n)) = \begin{cases} \dfrac{(1-|\varphi(0)|^2)^{n/p}}{|1-\langle\varphi(a),\varphi(0)\rangle|^{2n/p}}, & 1\le p<\infty,\\ E(a,A,H_p(B_n)), & p=\infty. \end{cases}$$

*Proof.* Let $f\in H_p(B_n)$, $1\le p<\infty$. Put $g(z)=(\Phi_\varphi f)(z)$. In view of (10) we have $\|g\|_p=\|f\|_p$ and

$$f(z)=\frac{g(\varphi^{-1}(z))(1-|\varphi(0)|^2)^{n/p}}{(1-\langle z,\varphi(0)\rangle)^{2n/p}},$$

because for every $\varphi\in\mathrm{Aut}(B_n)$

$$1-\langle\varphi(z),\varphi(w)\rangle=\frac{(1-|\varphi^{-1}(0)|^2)(1-\langle z,w\rangle)}{(1-\langle z,\varphi^{-1}(0)\rangle)(1-\langle\varphi^{-1}(0),w\rangle)}$$

(see Rudin [6]). Let $S$ be some algorithm for the information operator $I_1$, then

$$f(d)-SI_1f=\frac{(1-|\varphi(0)|^2)^{n/p}}{(1-\langle\varphi(a),\varphi(0)\rangle)^{2n/p}}(g(a)-\widetilde{S}Ig),$$

where

$$\widetilde{S}Ig=\frac{(1-\langle\varphi(a),\varphi(0)\rangle)^{2n/p}}{(1-|\varphi(0)|^2)^{n/p}}SI_1(\Phi_{\varphi^{-1}}g).$$

Hence

$$E(d,D,H_p(B_n))\ge\frac{(1-|\varphi(0)|^2)^{n/p}}{(1-\langle\varphi(a),\varphi(0)\rangle)^{2n/p}}E(a,A,H_p(B_n)).$$

On the other hand,

$$\begin{aligned} E(d,D,H_p(B_n)) &\le \sup_{f\in BH_p(B_n)}\left|f(d)-\int_D f(w)\,d\mu_1^*(w)\right| \\ &= \sup_{f\in BH_p(B_n)}\left|\frac{g(a)(1-|\varphi(0)|^2)^{n/p}}{(1-\langle\varphi(a),\varphi(0)\rangle)^{2n/p}}\right. \\ &\quad \left.-\int_A\frac{g(w)(1-|\varphi(0)|^2)^{n/p}}{(1-\langle\varphi(w),\varphi(0)\rangle)^{2n/p}}\left(\frac{1-\langle a,\varphi^{-1}(0)\rangle}{1-\langle w,\varphi^{-1}(0)\rangle}\right)^{2n/p}d\mu^*(w)\right| \\ &= \frac{(1-|\varphi(0)|^2)^{n/p}}{|1-\langle\varphi(a),\varphi(0)\rangle|^{2n/p}}\sup_{g\in BH_p(B_n)}\left|g(a)-\int_A g(w)\,d\mu^*(w)\right| \\ &= \frac{(1-|\varphi(0)|^2)^{n/p}}{|1-\langle\varphi(a),\varphi(0)\rangle|^{2n/p}}E(a,A,H_p(B_n)). \end{aligned}$$

The case $p=\infty$ may be worked out in the same way. The proof is over. □

Now we shall apply the last proposition to the recovery problem corresponding to information operator, which is one of the restriction on the set (8). Note, that $\varphi_c$ ($c$ is defined in (9)) transforms the set $D = \{z \in B_n : z_{n-k+1} = \ldots = z_n = 0\}$ into the set (8).

Put

$$\Delta_n^k(p,c) = \varphi_c(\Delta_n^k(p)) = \left\{ z \in B_n : \left| \left( \sqrt{1-|c|^2}z - c + \frac{\langle z,c\rangle}{1+\sqrt{1-|c|^2}}c \right)''_k \right| < \lambda_n(p)\sqrt{|1-\langle z,c\rangle|^2 + (1-|c|^2)|z'_k|^2} \right\},$$

$$a_c = \varphi_c(a), \quad \widetilde{a}_c = \varphi_c((a_c)'_k) = \varphi_c\left( -\frac{\sqrt{1-|c|^2}}{1-\langle a,c\rangle}a'_k \right) = \frac{1-|c|^2}{1-\langle a,c\rangle}a'_k + c.$$

It follows from Proposition 2 that $\Delta_n^k(p,c) = B_n$ if $n < 5$.

**Theorem 2.** *Let $a \in \Delta_n^k(p,c) \setminus A$. Then*

i) *the algorithm*

$$f(a) \approx \begin{cases} \left( \dfrac{1-|c|^2}{1-\langle a,c\rangle} \right)^{2n/p} (D_{nk}(p,a_c))^{(p-2)/p} f(\widetilde{a}_c), & 1 \le p < \infty, \\ \dfrac{1-|a_c|^2}{1-|(a_c)'_k|^2} f(\widetilde{a}_c), & p = \infty, \end{cases}$$

*is an optimal one;*

ii)

$$f_k^*(c;p,a,w) = \begin{cases} \dfrac{(1-|c|^2)^{n/p}}{(1-\langle w,c\rangle)^{2n/p}} f_k^*(p,a_c,\varphi_c(w)), & 1 \le p < \infty, \\ f_k^*(\infty,a_c,\varphi_c(w)), & p = \infty, \end{cases}$$

*is a worst function;*

iii)

$$E(a,A,H_p(B_n)) = \begin{cases} \left( \dfrac{(1-|c|^2)^n \Gamma\left(n+\dfrac{p}{2}\right)}{|1-\langle a,c\rangle|^{2n}\Gamma(n)\Gamma(p/2+1)D_{nk}(p,a_c)} \right)^{1/p} \times \dfrac{|(a_c)''_k|}{(1-|(a_c)'_k|^2)^{\frac{n}{p}+\frac{1}{2}}}, & 1 \le p < \infty, \\ \dfrac{|(a_c)''_k|}{\sqrt{1-|(a_c)'_k|^2}}, & p = \infty. \end{cases}$$

*Proof.* As $\varphi_c \circ \varphi_c = \mathrm{id}$ (see Rudin [6]), we have $\varphi_c(\varphi_c(a)) = a$, $\varphi_c^{-1}(0) = \varphi_c(0) = c$. By Proposition 4 we can reduce the problem of recovery at the point $d = \varphi_c(a)$ and information operator to be $If = f_{|D}$ to recovery at the point $a$ and information set to be $A$. We obtain from Proposition 4 that an optimal algorithm is

$$F(a) \approx \gamma^{2n/p}(D_{nk}(p, a_c))^{(p-2)/p} f(\widetilde{a}_c),$$

where

$$\gamma = \frac{1 - \langle \varphi_c(a), \varphi_c^{-1}(0)\rangle}{1 - \langle \varphi_c^{-1}(\widetilde{a}_c), \varphi_c^{-1}(0)\rangle}$$

and

$$E(a, A, H_p(B_n)) = \frac{(1 - |\varphi_c(0)|^2)^{n/p}}{|1 - \langle \varphi_c(a_c), \varphi_c(0)\rangle|^{2n/p}} E(a_c, D, H_p(B_n)).$$

Note that

$$\gamma = \frac{1 - \langle \varphi_c(a), \varphi_c(0)\rangle}{1 - \langle (a_c)'_k, \varphi_c(0)\rangle} = 1 - \langle \varphi_c(a), \varphi_c(0)\rangle = \frac{1 - |c|^2}{1 - \langle a, c\rangle},$$

$1 - |\varphi_c(0)|^2 = 1 - |c|^2$, $1 - \langle \varphi_c(a), \varphi_c(0)\rangle = 1 - \langle a, c\rangle$ and now the required statement follows for $1 \le p < \infty$. The case $p = \infty$ is quite analogous. The theorem is proved. $\square$

The following theorem is a generalization of the Schwartz Lemma for functions from $H_p(B_n)$.

**Theorem 3.** *Let $f \in BH_p(B_n)$, $f(0) = 0$, $1 \le p < \infty$, $a \in \Delta_n^n(p) = \{z \in B_n : |z| < \lambda_n(p)\}$. Then*

$$|f(a)| \le |a| \left( \frac{\Gamma\left(n + \dfrac{p}{2}\right)}{\Gamma(n)\Gamma(p/2+1)(1-|a|^2)^n} \sum_{j=0}^{n-1} \frac{(-1)^j}{2j/p+1} \binom{n-1}{j} |a|^{2j} \right)^{1/p}.$$

<u>Remark</u>. The case $p = \infty$ is obviously trivial: $|f(a)| \le |a|$ is the classical Schwartz Lemma.

*Proof.* From (4) we have

$$|f(a)| \le E(a, \{0\}, H_p(B_n)) = \left( \frac{\Gamma\left(n + \dfrac{p}{2}\right)}{\Gamma(n)\Gamma(p/2+1) D_{nn}(p, a)} \right)^{1/p} |a|.$$

Now the required statement follows from the definition of $D_{nn}(p, a)$. $\square$

Using Theorem $1'$ instead of Theorem 1 in the case $p = 2$ we can eliminate the condition $a \in \Delta_n^k(2, c)$. The following two theorems may be proved using the same arguments as in Theorems 2 and 3, respectively.

**Theorem $2'$.** *Let $a \in B_n$ and $p = 2$. Then*

i) *the algorithm*

$$f(a) \approx \left(\frac{1-|c|^2}{1-\langle a, c\rangle}\right)^n f(\widetilde{a}_c)$$

*is an optimal one;*

ii) *a worst function is*

$$f_k^*(c; 2, a, w) = \frac{(1-|c|^2)^{n/2}}{(1-\langle w, c\rangle)^n} f_k^*(2, a_c, \varphi_c(w)),$$

*where*

$$f_k^*(2, a, z) = \frac{(1-\langle z, a\rangle)^{-n} - (1-\langle z_k', a_k'\rangle)^{-n}}{\sqrt{(1-|a|^2)^{-n} - (1-|a_k'|^2)^{-n}}};$$

iii)

$$E(a, A, H_2(B_n)) = \sqrt{(1-|a|^2)^{-n} - \left(\frac{|1-\langle a, c\rangle|^2}{1-|c|^2} - |a_k'|^2\right)^{-n}}.$$

**Theorem 3′.** *Let $a \in B_n$, $f \in BH_2(B_n)$, $f(0) = 0$. Then*

$$|f(a)| \le \sqrt{(1-|a|^2)^{-n} - 1}.$$

Now we can easily demonstrate the difference between the cases $n \ge 6$ and $n < 6$. Let us consider recovery in $H_2(B_n)$, when the information operator is, for example, $If = f_{|\{z_n=0\}}$. If $n < 6$ the set $\{z_n = 0\}$ is the zero-set of the worst function, but if $n > 6$ there are extra sets: the divisor of the worst function $f_1^*(2, a, w)$ in $B_n$ is (see Lemma 1)

$$D_n = \bigcup_{\substack{k\in\mathbb{Z} \\ |k|<n/6}} \{\, w \in B_n : \langle w, a\rangle - e^{2\pi ki/n}\langle w_1', a_1'\rangle = 1 - e^{2\pi ki/n} \,\}.$$

It means that in the case $n \le 6$ any additional information leads to decreasing of the intrinsic error. On the contrary if $n > 6$ there exists extension of information operator, which does not decrease the intrinsic error of recovery.

Now we shall prove Proposition 2.

*Proof of Proposition* 2. To prove *i)* we must prove that $P_k(\alpha, u)$, $k = 1, 2, 3, 4$, has no zeros in the unit disk, if $\alpha > \dfrac{3}{2}$. We have

$$P_1(\alpha, u) = \frac{1}{\alpha-1} - \frac{u}{2}, \quad P_2(\alpha, u) = \frac{1}{\alpha-1} - \frac{2u}{\alpha} + \frac{u^2}{\alpha+1},$$

$$P_3(\alpha, u) = \frac{1}{\alpha-1} - \frac{3u}{\alpha} + \frac{3u^2}{\alpha+1} - \frac{u^3}{\alpha+2},$$

$$P_4(\alpha, u) = \frac{1}{\alpha-1} - \frac{4u}{\alpha} + \frac{6u^2}{\alpha+1} - \frac{4u^3}{\alpha+2} + \frac{u^4}{\alpha+3}.$$

It is easy to see that $P_1(\alpha, u)$ and $P_2(\alpha, u)$ have no zeros in the unit disk. Next we observe that $P_3(\alpha, u)$ has only one real zero which belongs to the interval $(1, (\alpha+2)/(\alpha+1))$, and so the modulus of each other zero of $P_3(\alpha, u)$ is greater than 1.

Now let us consider $P_4(\alpha, u)$. We can easily check that

$$P_4(\alpha, u) = \int_0^1 t^{\alpha-2}(1-ut)^4\, dt$$

which yields that $P_4(\alpha, u)$ has no real zeros. Let $x_1$, $\widetilde{x}_1$, $x_2$, $\widetilde{x}_2$ be the roots of $P_4(\alpha, u)$. In view of the Viete theorem,

$$\begin{cases} \operatorname{Re} x_1 + \operatorname{Re} x_2 = 2\dfrac{\alpha+3}{\alpha+2}, \\ |x_1|^2 + |x_2|^2 + 4\operatorname{Re} x_1 \operatorname{Re} x_2 = 6\dfrac{\alpha+3}{\alpha+1}, \\ |x_1|^2 \operatorname{Re} x_2 + |x_2|^2 \operatorname{Re} x_1 = 2\dfrac{\alpha+3}{\alpha}, \\ |x_1|^2|x_2|^2 = \dfrac{\alpha+3}{\alpha+1}. \end{cases}$$

Suppose $|x_1| = 1$. From the first, third, and fourth equations of this system we find that $\operatorname{Re} x_1 = (\alpha-1)(\alpha+3)\alpha^{-1}(\alpha+2)^{-1}$, $\operatorname{Re} x_2 = (\alpha+1)(\alpha+3)\alpha^{-1}(\alpha+2)^{-1}$ and then from the second equation we obtain $\alpha^2 + 2\alpha + 3 = 0$. This contradiction shows that $P_4(\alpha, u)$ has no roots, which modulus is equal to 1. Let $\alpha = \dfrac{3}{2}$. In this case,

$$\begin{cases} \operatorname{Re} x_1 + \operatorname{Re} x_2 = \dfrac{18}{7}, \\ |x_1|^2 + |x_2|^2 + 4\operatorname{Re} x_1 \operatorname{Re} x_2 = \dfrac{54}{5}, \\ |x_1|^2 \operatorname{Re} x_2 + |x_2|^2 \operatorname{Re} x_1 = 6, \\ |x_1|^2|x_2|^2 = 9. \end{cases}$$

It follows from the first, third, and fourth equations of last system that $\operatorname{Re} x_1 = 6\left(|x_2|^2 - 27/7\right)/\left(|x_2|^4 - 9\right)$. If $|x_1| < 1$, then $|x_2|^2 > 9$ and therefore, $\operatorname{Re} x_l > 0$, $\operatorname{Re} x_2 > \dfrac{18}{7} - |x_1| > 0$. Now $|x_1| < 1$ yields $|x_1|^2 + |x_2|^2 = |x_1|^2 + \dfrac{9}{|x_1|^2} > 10$, $0 < \operatorname{Re} x_1 \operatorname{Re} x_2 < \dfrac{1}{5}$. As $\operatorname{Re} x_1 \operatorname{Re} x_2 > 0$ we have $|x_1|^2 + \dfrac{9}{|x_1|^2} < \dfrac{54}{5}$ and hence $|x_1|^2 > \dfrac{27 - 6\sqrt{14}}{5} > 0.9$, $|x_2|^2 < 10$, $\operatorname{Re} x_2 + 10\operatorname{Re} x_1 > |x_1|^2 \operatorname{Re} x_2 + |x_2|^2 \operatorname{Re} x_1 = 6$. Combining this equality with the equation $\operatorname{Re} x_2 + \operatorname{Re} x_1 = 18/7$, we find

$$\operatorname{Re} x_1 > \frac{8}{21} > \frac{1}{5}, \quad \operatorname{Re} x_2 = \frac{18}{7} - \operatorname{Re} x_1 > 1$$

which contradicts $\operatorname{Re} x_1 \operatorname{Re} x_2 < \dfrac{1}{5}$. Thus we have proved that $P_4\left(3/2, u\right)$ has no roots in the unit disk.

Finally, suppose the set of those $\alpha > 3/2$ that $P_4(\alpha, u)$ has zeros inside the unit disk is not empty. Let $\alpha_0$ be the infimum of those $\alpha$. One can easily check that $P_4(\alpha, u)$ has no multiple zeros and hence zeros of $P_4(\alpha, u)$ depend on $\alpha$ differentially. Therefore $P_4(\alpha_0, u)$ has the root, which modulus is equal to 1, but we proved before it is impossible and i) is proved.

To prove (ii) note that

$$P_{n-1}(2, u) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} \binom{n-1}{k} u^k = \frac{1-(1-u)^n}{nu}.$$

Thus $u_{n-1} = 1 - e^{2\pi i/n}$ is the root of $P_{n-1}(2, u)$. As $|1 - e^{2\pi i/n}| = 2\sin(\pi/n)$, $|u_{n-1}| < 1$ if $n \geq 7$, and we must only prove that $P_5(\alpha, u)$ has zeros inside the unit disk for some $\alpha \geq 3/2$. Let $u(\alpha)$ be the root of $P_5(\alpha, u)$ so $P_5(\alpha, u(\alpha)) \equiv 0$ and hence

$$\frac{du(\alpha)}{d\alpha} = \frac{1}{5} \frac{\dfrac{u^5(\alpha)}{(\alpha+4)^2} - 5\dfrac{u^4(\alpha)}{(\alpha+3)^2} + 10\dfrac{u^3(\alpha)}{(\alpha+2)^2} - 10\dfrac{u^2(\alpha)}{(\alpha+1)^2}}{\dfrac{u^4(\alpha)}{\alpha+4} - 4\dfrac{u^3(\alpha)}{\alpha+3} + 6\dfrac{u^2(\alpha)}{\alpha+2} - 4\dfrac{u(\alpha)}{\alpha+1} + \dfrac{1}{\alpha}}$$

$$+ \frac{1}{5} \frac{5\dfrac{u(\alpha)}{\alpha^2} - \dfrac{1}{(\alpha-1)^2}}{\dfrac{u^4(\alpha)}{\alpha+4} - 4\dfrac{u^3(\alpha)}{\alpha+3} + 6\dfrac{u^2(\alpha)}{\alpha+2} - 4\dfrac{u(\alpha)}{\alpha+1} + \dfrac{1}{\alpha}}.$$

$u_5 = e^{-\pi i/3}$ is a root of $P_5(2, u)$. Substituting $u(\alpha) = e^{-\pi i/3}$, $\alpha = 2$ in the last expression we obtain

$$\left.\frac{du(\alpha)}{d\alpha}\right|_{\alpha=2} = \frac{119 + 56i\sqrt{3}}{360}$$

and, therefore, if $\alpha > 2$, $\alpha$ is sufficiently close to 2 then $|\alpha| < 1$. The proposition is proved. $\square$

3. Recovery in the Bergman space. The Bergman space $A_p(B_n)$ is the set of holomorphic in $B_n$ functions which satisfy

$$\|f\|_{A_p} = \left( \int_{B_n} |f(z)|^p \, d\nu_n(z) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

where $\nu_n$ is the normalized Lebesgue measure in $B_n$. $A_\infty(B_n)$ is the same as $H_\infty(B_n)$. In this section we shall extend the results of the previous section to the Bergman space $A_p(B_n)$. Let the linear affine plane $M$ has equations $z_{n-k+1} = c_{n-k+1}, \ldots, z_n = c_n$, $1 \leq k \leq n$, where $c = (0, \ldots, 0, c_{n-k+1}, \ldots, c_n) \in B_n$.

Let us consider the recovery problem (1), where $X_p = A_p(B_n)$, $Lf = f(a)$, $a \in B_n$, and $If = f_{|A}$, where $A = M \cap B_n$. Let $e$ be the standard embedding $e\colon \mathbb{C}^n \to \mathbb{C}^{n+1}$, $ez = (0, z)$. Denote $\widetilde{\Delta}_n^k(p, c) = \{z \in B_n : ez \in \Delta_{n+1}^k(p, ec)\}$. Note that $\widetilde{\Delta}_n^k(p, c) = B_n$ if $n \le 4$.

**Theorem 4.** *Let $a \in \widetilde{\Delta}_n^k(p, c)$ then*

i) *an optimal algorithm is*

$$f(a) \approx s_0 f(\widetilde{a}_c),$$

*where*

$$s_0 = \left(\frac{1 - |c|^2}{1 - \langle a, c\rangle}\right)^{\frac{2(n+1)}{p}} (D_{n+1,k}(p, \varphi_{ec}(ea)))^{\frac{p-2}{p}};$$

ii) *a worst function is*

$$g^*(w) = \frac{(1 - |c|^2)^{(n+1)/p}}{(1 - \langle w, c\rangle)^{2(n+1)/p}} f_k^*(p, \varphi_{ec}(ea), \varphi_{ec}(ew));$$

iii) *the intrinsic error is*

$$E(a, A, A_p(B_n)) = \left(\frac{(1 - |c|^2)^{n+1}\Gamma\left(n + \frac{p}{2} + 1\right)}{|1 - \langle a, c\rangle|^{2(n+1)}\Gamma(n+1)\Gamma(p/2+1)D_{n+1,k}(p, \varphi_{ec}(ea))}\right)^{1/p} \times \frac{|(a_c)_k''|}{(1 - |(a_c)_k'|^2)^{\frac{n+1}{p} + \frac{1}{2}}}.$$

*Proof.* Recall (Rudin [6]) that there exists the linear isometric embedding

$$E\colon A_p(B_n) \to H_p(B_{n+1}),$$
$$(Eg)(z_0, z) = g(z), \quad z_0 \in \mathbb{C}, \quad z \in B_n, \quad |z_0|^2 + |z|^2 < 1.$$

If $g \in BA_p(B_n)$ then $Eg \in BH_p(B_{n+1})$. As $\widetilde{ea}_{ec} = \varphi_{ec}((ea_{ec})_k') = \dfrac{1 - |c|^2}{1 - \langle a, c\rangle}$ $(ea_k' + ec = (0, \widetilde{a}_c))$ we have (using Theorem 2)

$$(11) \quad E(a, A, A_p(B_n)) \le |g(a) - s_0 g(\widetilde{a}_c)| = |Eg(ea) - s_0 Eg(\widetilde{ea}_{ec})| \le E(ea, eA, H_p(B_{n+1})).$$

On the other hand, $Eg^*$ turns (11) in the equality and in view of $Eg^*(\widetilde{ea}_{ec}) = g^*(\widetilde{a}_c) = 0$ we have

$$E(a, A, A_p(B_n)) \ge |g^*(a)| = |(Eg^*)(ea)| = E(ea, eA, H_p(B_{n+1})).$$

Now all required statements follow from Theorem 2. The theorem is proved. □

The following theorem is the variant of the Schwartz Lemma for the Bergman spaces. Its proof is analogous to the proof of Theorem 3.

**Theorem 5.** *Let* $g \in BA_p(B_n)$, $1 \le p < \infty$, $g(0) = 0$, $a \in B_n$, $|a| < \lambda_{n+1}(p)$. *Then*

$$|g(a)| \le |a| \left( \frac{\Gamma\left(n + \frac{p}{2} + 1\right)}{\Gamma(n+1)\Gamma(p/2+1)(1-|a|^2)^{n+1}} \sum_{j=0}^{n} \frac{(-1)^j}{2j/p+1} \binom{n}{j} |a|^{2j} \right)^{1/p}.$$

*In particular, if* $n \le 4$ *this inequality holds for every* $a \in B_n$.

Note that using Theorem $1'$ instead of Theorem 1, we can obtain the results analogous to Theorems $2'$ and $3'$ in the case of $A_2(B_n)$.

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