EXACT n-WIDTHS OF HARDY–SOBOLEV CLASSES

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ABSTRACT. Let $\tilde{h}^r_{\infty,\beta}$ and $\widetilde{H}^r_{\infty,\beta}$ denote those 2π-periodic, real-valued functions on $\mathbb{R}$ that are analytic in the strip $|\operatorname{Im} z| < \beta$ and satisfy the restrictions $|\operatorname{Re} f^{(r)}(z)| \le 1$ and $|f^{(r)}(z)| \le 1$, respectively. We determine the Kolmogorov, linear, and Gel'fand widths of $\tilde{h}^r_{\infty,\beta}$ in $L_q[0,2\pi]$, $1 \le q \le \infty$, and $\widetilde{H}^r_{\infty,\beta}$ in $L_\infty[0,2\pi]$.

INTRODUCTION

The Kolmogorov n-width of a subset A of a normed linear space X is defined by

$$d_n(A,X) := \inf_{X_n} \sup_{x\in A} \inf_{y\in X_n} \|x-y\|,$$

where X_n runs over all n-dimensional subspaces of X. The linear n-width of A in X is defined by

$$\lambda_n(A,X) := \inf_{P_n} \sup_{x\in A} \|x-P_nx\|,$$

where P_n varies over all bounded linear operators mapping X into itself whose range has dimension n or less. The Gel'fand n-width is given by

$$d^n(A,X) := \inf_{X^n} \sup_{x\in A\cap X^n} \|x\|,$$

where the infimum is taken over all subspaces X^n of X of codimension n (here we assume that $0 \in A$).

Set $S_\beta := \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$. For integer $r \ge 0$ denote by $\tilde{h}^r_{\infty,\beta}$ ($\widetilde{H}^r_{\infty,\beta}$) the set of 2π-periodic functions, real-valued on $\mathbb{R}$ and analytic in the strip S_β which satisfy the conditions $|\operatorname{Re} f^{(r)}(z)| \le 1$ ($|f^{(r)}(z)| \le 1$). For $r = 0$ we shall omit the upper index in the notation of these classes. In this paper we determine the exact values of the Kolmogorov, linear, and Gel'fand widths of $\tilde{h}^r_{\infty,\beta}$ in $L_q := L_q[0,2\pi]$, $1 \le q \le \infty$, and $\widetilde{H}^r_{\infty,\beta}$ in L_∞. We also show that the n-widths of the subset of analytic functions on $[0,2\pi)$ from the Sobolev class $\widetilde{W}^r_\infty$ are the same as for $\widetilde{W}^r_\infty$.

For $q = \infty$ the n-widths of the class $\tilde{h}_{\infty,\beta}$ were obtained by V. M. Tikhomirov [11]. For $1 \le q \le \infty$ the exact values of the even n-widths of the classes $\tilde{h}_{\infty,\beta}$ and $\widetilde{H}_{\infty,\beta}$ were determined in [9] and [7].

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In the nonperiodic case for the functions which are analytic on the open unit disk, real-valued on $(-1,1)$, and which satisfy the restriction $|f^{(r)}(z)| \le 1$, the n-widths were obtained in [4]. The results concerning the n-widths of the classes of analytic functions of several variables whose rth radial derivative is bounded may be found in [3] and [8].

1. Exact n-Widths of $\widetilde{h}^r_{\infty,\beta}$

If $f \in \tilde{h}_{\infty,\beta}$, then (see [1, p. 269])

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} K_\beta(z-t)\operatorname{Re} f(t+i\beta)\,dt,$$

where

$$K_\beta(z) = 1 + 2\sum_{k=1}^{\infty}\frac{\cos kz}{\cosh k\beta}.$$

Set

$$(f*g)(z) := \frac{1}{2\pi}\int_0^{2\pi} f(z-t)g(t)\,dt,$$

$$D_r(t) := 2\sum_{k=1}^{\infty}\frac{\cos(kt-\pi r/2)}{k^r}, \quad r = 1,2,\dots\ .$$

Using the representation

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt + (D_r * f^{(r)})(z),$$

for $r \ge 1$ we have

$$\tilde{h}^r_{\infty,\beta} = \{\, a + G_{r,\beta} * h : \|h\|_\infty \le 1,\ h \perp 1,\ a \in \mathbb{R} \,\},$$

where $G_{r,\beta} := D_r * K_\beta$.

We say that a real, 2π-periodic, continuous function G satisfies Property B (cf. [10, p. 129]) if for every choice of $0 \le t_1 < \ldots < t_m < 2\pi$ and each m, the subspace

$$X_m := \left\{ b + \sum_{j=1}^{m} b_j G(\cdot - t_j) : \sum_{j=1}^{m} b_j = 0 \right\}$$

is of dimension m, and for every $f \in X_{2m+1}$, $f \not\equiv 0$, the number of cyclic sign changes $S_c(f) \le 2m$.

Set

$$\Lambda_{2n} := \{\, \xi : \xi = (\xi_1,\dots,\xi_{2n}),\ 0 \le \xi_1 < \ldots < \xi_{2n} < 2\pi \,\}.$$

For each $\xi \in \Lambda_{2n}$ we define

$$h_\xi(t) := (-1)^j, \quad t \in [\xi_{j-1},\xi_j), \quad j = 1,\dots,2n+1,$$

where $\xi_0 := 0$, $\xi_{2n+1} := 2\pi$. Denote by $h_n(t)$ the function h_ξ for $\xi_j = (j-1)\pi/n$, $j = 1,\dots,2n$.

To calculate the exact n-widths of $\tilde{h}^r_{\infty,\beta}$ we need the following theorem of A. Pinkus [10, p. 180, 182].

Theorem 1. *Let G satisfy Property B. Set*

$$\widetilde{\mathcal{B}}_\infty := \{\, a + G * h : \|h\|_\infty \le 1,\ h \perp 1,\ a \in \mathbb{R} \,\}.$$

Then:

(*i*)

$$d_{2n-1}(\widetilde{\mathcal{B}}_\infty, L_\infty) = \lambda_{2n-1}(\widetilde{\mathcal{B}}_\infty, L_\infty) = d^{2n-1}(\widetilde{\mathcal{B}}_\infty, L_\infty) = \|G * h_n\|_\infty;$$

(*ii*) *for each $1 \le q \le \infty$*

$$d_{2n}(\widetilde{\mathcal{B}}_\infty, L_q) = \lambda_{2n}(\widetilde{\mathcal{B}}_\infty, L_q) = d^{2n}(\widetilde{\mathcal{B}}_\infty, L_q) = \|G * h_n\|_q.$$

We say that a real, 2π-periodic, continuous function $\mathcal{K} \in NCVD$ (nondegenerate cyclic variation diminishing) if $S_c(\mathcal{K} * f) \le S_c(f)$ for all real, 2π-periodic, piecewise continuous functions f, and

$$\dim \operatorname{span} \{\, \mathcal{K}(t_1 - \cdot), \dots, \mathcal{K}(t_n - \cdot) \,\} = n$$

for every choice of $0 \le t_1 < \dots < t_n < 2\pi$ and all n. It is known (see [10, p. 128, 133]) that $K_\beta \in NCVD$ and for each $r \ge 2$, D_r satisfies Property B ($D_1(x)$ also satisfies all conditions of Property B except that it is not continuous at $x = 0$). Therefore, $G_{r,\beta}$ satisfies Property B for every $r \ge 1$.

The Euler perfect splines are defined by

$$\varphi_{n,r}(t) := \frac{4}{\pi n^r} \sum_{k=0}^{\infty} \frac{\sin((2k+1)nt - \pi r/2)}{(2k+1)^{r+1}}, \quad r = 0, 1, \dots\ .$$

Several properties of this splines may be found, for example, in [6, p. 104]. In particular,

$$\varphi_{n,0} = h_n, \quad \varphi_{n,r} = D_r * h_n, \quad r = 1, 2, \dots\ .$$

Put

$$\varphi^\beta_{n,r}(t) := (K_\beta * \varphi_{n,r})(t) = \frac{4}{\pi n^r} \sum_{k=0}^{\infty} \frac{\sin((2k+1)nt - \pi r/2)}{(2k+1)^{r+1} \cosh((2k+1)n\beta)}.$$

It was proved in [9] that

$$\varphi^\beta_{n,0}(t) = (K_\beta * h_n)(t) = \frac{4}{\pi} \arctan\left(\sqrt{\lambda}\, \operatorname{sn}\left(\frac{2n\Lambda}{\pi} t, \lambda\right)\right), \tag{1}$$

where Λ is the complete elliptic integral of the first kind with modulus

$$\lambda = 4e^{-2\beta n} \left(\frac{\sum_{k=0}^{\infty} e^{-4\beta n k(k+1)}}{1 + 2\sum_{k=1}^{\infty} e^{-4\beta n k^2}} \right)^2. \tag{2}$$

By analogy to the Euler perfect splines it can be shown that

$$K^\beta_{n,r} := \|\varphi^\beta_{n,r}\|_\infty = \frac{4}{\pi n^r} \sum_{k=0}^{\infty} \frac{(-1)^{k(r+1)}}{(2k+1)^{r+1} \cosh((2k+1)n\beta)}.$$

For $r \geq 1$, $\varphi^\beta_{n,r} = G_{r,\beta} * h_n$. Thus from Theorem 1 ($r \geq 1$) and Theorems 4.8 and 4.9 of [10, p. 179, 180] ($r = 0$) we obtain the following result.

Theorem 2. *Let* $r \geq 0$. *Then:*

(*i*)

$$d_{2n-1}(\tilde{h}^r_{\infty,\beta}, L_\infty) = \lambda_{2n-1}(\tilde{h}^r_{\infty,\beta}, L_\infty) = d^{2n-1}(\tilde{h}^r_{\infty,\beta}, L_\infty) = K^\beta_{n,r};$$

(*ii*) *for each* $1 \leq q \leq \infty$

$$d_{2n}(\tilde{h}^r_{\infty,\beta}, L_q) = \lambda_{2n}(\tilde{h}^r_{\infty,\beta}, L_q) = d^{2n}(\tilde{h}^r_{\infty,\beta}, L_q) = \|\varphi^\beta_{n,r}\|_q.$$

For integer $r \geq 1$ denote by $\widetilde{W}^r_\infty$ the class of real, 2π-periodic functions whose $(r-1)$st derivative is absolutely continuous and whose rth derivative satisfies the condition $|f^{(r)}(t)| \leq 1$. Let $\widetilde{A}^r_\infty$ be the set of functions from $\widetilde{W}^r_\infty$ which are analytic on $[0, 2\pi)$.

Theorem 3. *Let* $r \geq 1$. *Then:*

(*i*)

$$d_{2n-1}(\widetilde{A}^r_\infty, L_\infty) = \lambda_{2n-1}(\widetilde{A}^r_\infty, L_\infty) = d^{2n-1}(\widetilde{A}^r_\infty, L_\infty) = \frac{K_r}{n^r}, \tag{3}$$

where

$$K_r := \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k(r+1)}}{(2k+1)^{r+1}};$$

(*ii*) *for each* $1 \leq q \leq \infty$

$$d_{2n}(\widetilde{A}^r_\infty, L_q) = \lambda_{2n}(\widetilde{A}^r_\infty, L_q) = d^{2n}(\widetilde{A}^r_\infty, L_q) = \|\varphi_{n,r}\|_q. \tag{4}$$

Proof. For the class $\widetilde{W}^r_\infty$ the equations analogous to (3) and (4) are a well-known fact (the details and references may be found in [10] and [6]). Since $\widetilde{A}^r_\infty \subset \widetilde{W}^r_\infty$ it is sufficient to prove the lower bound. For all $\beta > 0$, $\tilde{h}^r_{\infty,\beta} \subset \widetilde{A}^r_\infty$. Therefore, the lower bound follows from Theorem 2 and the obvious equations

$$\lim_{\beta \to 0} K^\beta_{n,r} = \frac{K_r}{n^r}, \quad \lim_{\beta \to 0} \|\varphi^\beta_{n,r}\|_q = \|\varphi_{n,r}\|_q.$$

The theorem is proved. □

2. EXACT n-WIDTHS OF $\widetilde{H}^r_{\infty,\beta}$

To calculate the exact n-widths of $\widetilde{H}^r_{\infty,\beta}$ we shall need some preliminary results.

Proposition 4. *Let φ be a continuous, odd, and strictly increasing function defined on* $[-1, 1]$. *Put*

$$\Lambda^\varphi_{2n} := \{\, \xi \in \Lambda_{2n} : \varphi(K_\beta * h_\xi) \perp 1 \,\}.$$

Then

$$\inf\{\, \|a + D_r * \varphi(K_\beta * h_\xi)\|_\infty : a \in \mathbb{R},\ \xi \in \Lambda^\varphi_{2n} \,\} = \|D_r * \varphi(K_\beta * h_n)\|_\infty.$$

Proof. Let $\xi \in \Lambda^\varphi_{2n}$. Set

$$f_\xi := D_r * \varphi(K_\beta * h_\xi), \quad f_n := D_r * \varphi(K_\beta * h_n).$$

Suppose that there exists an $a \in \mathbb{R}$ and a $\xi \in \Lambda^\varphi_{2n}$ for which $\|a + f_\xi\|_\infty < \|f_n\|_\infty$. As $f_n(t + \pi/n) = -f_n(t)$ there exist at least $2n$ points $0 \le t_1 < \ldots < t_{2n} < 2\pi$ such that

$$f_n(t_j) = \varepsilon(-1)^j \|f_n\|_\infty, \quad j = 1, \ldots, 2n,$$

where $\varepsilon = 1$ or -1. Denote by $Z(f)$ the number of distinct zeros of a function f on $[0, 2\pi)$. We have

$$Z\big(f_n(\cdot + \alpha) \pm a \pm f_\xi(\cdot)\big) \ge 2n$$

for every $\alpha \in \mathbb{R}$. By Rolle's Theorem

$$S_c\big(f_n^{(r)}(\cdot + \alpha) \pm f_\xi^{(r)}(\cdot)\big) = S_c\big(\varphi\left((K_\beta * h_n)(\cdot + \alpha)\right) \pm \varphi\left((K_\beta * h_\xi)(\cdot)\right)\big) \\ \ge 2n.$$

Since φ is an odd and strictly increasing function,

$$S_c\big((K_\beta * h_n)(\cdot + \alpha) \pm (K_\beta * h_\xi)(\cdot)\big) \ge 2n.$$

From the fact that $K_\beta \in NCVD$ it follows that

$$S_c\big(h_n(\cdot + \alpha) \pm h_\xi(\cdot)\big) \ge 2n.$$

Using Lemma 4.1 of [10, p. 170], we obtain the existence of an $\alpha \in \mathbb{R}$ and $\varepsilon = 1$ or -1 for which

$$S_c\big(h_n(\cdot + \alpha) - \varepsilon h_\xi(\cdot)\big) \le 2(n - 1).$$

This contradiction proves the proposition. ☐

Set $\varphi_0(z) := \tan \frac{\pi}{4} z$,

$$\Phi^\beta_{n,0} := \varphi_0(K_\beta * h_n), \quad \Phi^\beta_{n,r} := D_r * \varphi_0(K_\beta * h_n), \quad r = 1, 2, \ldots\ .$$

In view of (1) and the representation (see [2, p. 266])

$$\operatorname{sn}\left(\frac{2n\Lambda}{\pi} t, \lambda\right) = \frac{\pi}{\lambda\Lambda} \sum_{k=0}^{\infty} \frac{\sin((2k+1)nt)}{\sinh((2k+1)2n\beta)}$$

we have

$$\Phi^\beta_{n,r}(t) = \frac{\pi}{\sqrt{\lambda}\Lambda n^r} \sum_{k=0}^{\infty} \frac{\sin((2k+1)nt - \pi r/2)}{(2k+1)^r \sinh((2k+1)2n\beta)}, \quad r = 0, 1, \ldots\ .$$

It also follows from (1) that

$$\Phi^\beta_{n,0}(t) = \sqrt{\lambda}\, \operatorname{sn}\left(\frac{2n\Lambda}{\pi} t, \lambda\right).$$

Using the same arguments as for $\varphi^\beta_{n,r}$, we obtain

$$\|\Phi^\beta_{n,r}\|_\infty = \frac{\pi}{\sqrt{\lambda}\Lambda n^r}\sum_{k=0}^{\infty}\frac{(-1)^{k(r+1)}}{(2k+1)^r\sinh((2k+1)2n\beta)}.$$

Put

$$t^{(j)}_{n,r} := \begin{cases} \dfrac{\pi(j-1)}{n}, & r=2m, \\ \dfrac{\pi(j-1)}{n}+\dfrac{\pi}{2n}, & r=2m+1, \end{cases} \qquad j=1,\dots,2n.$$

It is easily seen that $\Phi^\beta_{n,r}(t^{(j)}_{n,r})=0$, $j=1,\dots,2n$.

Proposition 5. *For all $t\in[0,2\pi)$ and $r\ge 0$*

$$\sup\left\{\,|f(t)| : f\in\widetilde{H}^r_{\infty,\beta},\ f(t^{(j)}_{n,r})=0,\ j=1,\dots,2n\right\} = |\Phi^\beta_{n,r}(t)|.$$

Proof. Suppose there exists a $t^*\in[0,2\pi)$ and a function $f_0\in\widetilde{H}^r_{\infty,\beta}$ for which $f_0(t^{(j)}_{n,r})=0$, $j=1,\dots,2n$, and $|f_0(t^*)|>|\Phi^\beta_{n,r}(t^*)|$. Set

$$\rho := \Phi^\beta_{n,r}(t^*)/f_0(t^*), \qquad F := \Phi^\beta_{n,r} - \rho f_0.$$

The function F has at least $2n+1$ distinct zeros at the points $t^{(j)}_{n,r}$, $j=1,\dots,2n$, and t^*. By Rolle's Theorem

$$F^{(r)}(t) = \sqrt{\lambda}\,\mathrm{sn}\left(\frac{2n\Lambda}{\pi}t,\lambda\right) - \rho f_0^{(r)}(t)$$

has at least $2n+1$ zeros on $[0,2\pi)$.

Denote by $H_\infty(\Delta_\beta)$ the set of functions which are analytic on the annulus

$$\Delta_\beta := \left\{\, z\in\mathbb{C} : e^{-\beta}<|z|<e^{\beta}\,\right\}$$

and which satisfy the condition $|f(z)|\le 1$, $z\in\Delta_\beta$. If $f(z)\in\widetilde{H}_{\infty,\beta}$, then $f\left(\dfrac{1}{i}\log z\right)\in H_\infty(\Delta_\beta)$. It is easy to check that the function

$$G(z) := \sqrt{\lambda}\,\mathrm{sn}\left(\frac{2n\Lambda}{\pi i}\log z,\lambda\right)$$

has exactly $2n$ zeros on Δ_β. Since on $\partial\Delta_\beta$

$$\left|G(z) - F^{(r)}\left(\frac{1}{i}\log z\right)\right| = \left|\rho f_0^{(r)}\left(\frac{1}{i}\log z\right)\right| \le |\rho| < 1 \equiv |G(z)|,$$

Rouche's Theorem implies that $F^{(r)}(t)$ has $2n$ or fewer zeros on $[0,2\pi)$. We thus reach a contradiction, which proves the proposition. ☐

Theorem 6. *For all integers* $r \ge 0$

$$d_{2n}(\widetilde{H}^r_{\infty,\beta}, L_\infty) = \lambda_{2n}(\widetilde{H}^r_{\infty,\beta}, L_\infty) = d^{2n}(\widetilde{H}^r_{\infty,\beta}, L_\infty)$$
$$= \frac{\pi}{\sqrt{\lambda}\Lambda n^r} \sum_{k=0}^{\infty} \frac{(-1)^{k(r+1)}}{(2k+1)^r \sinh((2k+1)2n\beta)},$$

where Λ *is the complete elliptic integral of the first kind with modulus* λ *defined by* (2).

Proof. The case $r = 0$ follows from [7] where the equalities

$$d_{2n}(\widetilde{H}_{\infty,\beta}, L_q) = \lambda_{2n}(\widetilde{H}_{\infty,\beta}, L_q) = d^{2n}(\widetilde{H}_{\infty,\beta}, L_q) = \|\Phi^{\beta}_{n,0}\|_q, \quad 1 \le q \le \infty,$$

were proved. So we shall assume that $r \ge 1$.

We shall first prove the lower bound for the Kolmogorov widths. Set

$$S^{2n} := \left\{ x = (x_1, \dots, x_{2n+1}) \in \mathbb{R}^{2n+1} : \sum_{k=1}^{2n+1} |x_k| = 2\pi \right\},$$

$$\tau_0(x) := 0, \quad \tau_j(x) := \sum_{k=1}^{j} |x_k|, \quad j = 1, \dots, 2n+1.$$

For each $x \in S^{2n}$ put

$$g_x(t) := \operatorname{sign} x_j, \quad \tau_{j-1}(x) \le t < \tau_j(x), \quad j = 1, \dots, 2n+1,$$
$$f_x := D_r * \varphi_0(K_\beta * g_x).$$

Let X_{2n} be any $2n$-dimensional subspace of L_q, $1 < q < \infty$, such that $1 \in X_{2n}$. Suppose that $X_{2n} = \operatorname{span}\{f_1, \dots, f_{2n}\}$ and $f_1(t) \equiv 1$. Let $a_1(x), \dots, a_{2n}(x)$ be the coefficients of $f_1, \dots, f_{2n}$, respectively, in the best approximation to f_x from X_{2n}. The mapping

$$A(x) := (b(x), a_2(x), \dots, a_{2n}(x)),$$

where

$$b(x) := \int_0^{2\pi} \varphi_0\big((K_\beta * g_x)(t)\big)\, dt,$$

is a continuous map of S^{2n} into $\mathbb{R}^{2n}$. By Borsuk's Theorem there exists an $x^* \in S^{2n}$ for which $A(x^*) = 0$. As the function $\varphi_0(z) = \tan\frac{\pi}{4}z$ maps the strip $|\operatorname{Re} z| < 1$ conformally onto the open unit disk, for all $x \in S^{2n}$, $f_x \in \widetilde{H}^r_{\infty,\beta}$. Thus,

$$\sup_{f\in\widetilde{H}^r_{\infty,\beta}} \inf_{g\in X_{2n}} \|f-g\|_q \ge \sup_{x\in S^{2n}} \inf_{g\in X_{2n}} \|f_x - g\|_q \ge \|f_{x^*} - a_1(x^*)\|_q$$
$$\ge \inf\left\{ \|a + D_r * \varphi_0(K_\beta * h_\xi)\|_q : a \in \mathbb{R},\ \xi \in \Lambda^{\varphi_0}_{2n} \right\}. \tag{5}$$

If $1 \notin X_{2n}$, then the left-hand side of (5) is equal to $+\infty$. Hence,

$$d_{2n}(\widetilde{H}^r_{\infty,\beta}, L_q) \ge \inf\left\{ \|a + D_r * \varphi_0(K_\beta * h_\xi)\|_q : a \in \mathbb{R},\ \xi \in \Lambda^{\varphi_0}_{2n} \right\}.$$

By passing to the limit $q \to \infty$ we obtain

$$d_{2n}(\widetilde{H}^r_{\infty,\beta}, L_\infty) \geq \|\Phi^\beta_{n,r}\|_\infty.$$

Let us now prove the lower bound for the Gel'fand widths. Suppose that

$$X^{2n} := \{\, f \in L_\infty : \langle l_j, f\rangle = 0,\ j = 1, \dots, 2n,\ l_j \in L'_\infty \,\}.$$

If $\langle l_j, 1\rangle = 0$, $j = 1, \dots, 2n$, then

$$\sup_{f \in \widetilde{H}^r_{\infty,\beta} \cap X^{2n}} \|f\|_\infty = \infty.$$

Assume that $\langle l_1, 1\rangle \neq 0$. Set

$$L_j := l_j - \frac{\langle l_j, 1\rangle}{\langle l_1, 1\rangle} l_1, \quad j = 2, \dots, 2n.$$

For each $x \in S^{2n}$ denote by A_1 the mapping

$$A_1(x) := (b(x), \langle L_2, f_x\rangle, \dots, \langle L_{2n}, f_x\rangle).$$

Since $A_1\colon S^{2n} \to \mathbb{R}^{2n}$ is an odd and continuous map, by Borsuk's Theorem there exists an x^* for which $A_1(x^*) = 0$. Then

$$f^* := f_{x^*} - \frac{\langle l_1, f_{x^*}\rangle}{\langle l_1, 1\rangle} \in X^{2n}.$$

Consequently,

$$\begin{aligned} &\sup_{f \in \widetilde{H}^r_{\infty,\beta} \cap X^{2n}} \|f\|_\infty \geq \|f^*\|_\infty \\ &\quad \geq \inf\{\, \|a + D_r * \varphi_0(K_\beta * h_\xi)\|_\infty : a \in \mathbb{R},\ \xi \in \Lambda^{\varphi_0}_{2n} \,\} \geq \|\Phi^\beta_{n,r}\|_\infty. \end{aligned}$$

Thus,

$$d^{2n}(\widetilde{H}^r_{\infty,\beta}, L_\infty) \geq \|\Phi^\beta_{n,r}\|_\infty.$$

Now let us prove the upper bound for the linear widths. We shall use a modification of the proof for the nonperiodic case from [4]. Denote by E_p the set of functions 2π-periodic and analytic on S_β which satisfy

$$\begin{aligned} \|f\|_{E_p} &:= \sup_{0<\rho<1} \left(\frac{1}{4\pi} \int_\Gamma |f(\rho z)|^p |dz|\right)^{1/p} < \infty, \quad 1 \leq p < \infty, \\ \|f\|_{E_\infty} &:= \sup_{z \in S_\beta} |f(z)| < \infty, \quad p = \infty, \end{aligned}$$

where $\Gamma := [2\pi + i\beta, i\beta] \cup [-i\beta, 2\pi - i\beta]$. If the 2π-periodic functions $\omega(z), \omega_1(z) \in C(\Gamma)$ then, using the mapping $z = \dfrac{1}{i}\log w$, it can be proved (see [5]) that for all $1 < p \le \infty$

$$(6)\quad \sup\left\{ \left|\frac{1}{4\pi}\int_\Gamma f(z)\omega(z)\,dz\right| : \|f\|_{E_p} \le 1,\ \int_\Gamma f(z)\omega_1(z)\,dz = 0 \right\}$$
$$= \inf\left\{ \left(\frac{1}{4\pi}\int_\Gamma |\omega(z) - c\omega_1(z) - \chi(z)|^q |dz|\right)^{1/q} : c \in \mathbb{C},\ \chi \in E_q \right\}$$
$$= \|\omega\|_{L_q(\Gamma)/E_{q,1}},$$

where $1/p + 1/q = 1$, $E_{q,1} := E_{q|_\Gamma} + \operatorname{span}\{\omega_1\}$, and $E_{q|_\Gamma}$ is the space of boundary values on Γ of functions from E_q.

By the same mapping $z = \frac{1}{i}\log w$ and the Residue Theorem we obtain

$$(7)\qquad f(t) = \frac{1}{2\pi}\int_\Gamma \frac{f(z)e^{iz}}{e^{iz} - e^{it}}\,dz, \quad t \in S_\beta,$$

for all $f \in E_\infty$. For $x, t_1 \in [0, 2\pi)$ we define

$$K(z,x) := \frac{1}{\pi}\int_0^{2\pi} \frac{D_r(x-t)e^{iz}}{e^{iz} - e^{it}}\,dt, \quad V(z,x) := K(z,x) - K(z,t_1).$$

From (7) it follows that all functions 2π-periodic and analytic on S_β whose rth derivative lies in E_∞ satisfy the following equation

$$(8)\qquad f(x) - f(t_1) = \frac{1}{4\pi}\int_\Gamma V(z,x)f^{(r)}(z)\,dz.$$

Set

$$\omega_1(z) := \begin{cases} 1, & z \in [2\pi + i\beta, i\beta], \\ 0, & z \in [-i\beta, 2\pi - i\beta]. \end{cases}$$

Then the condition $f \perp 1$ can be written in the form

$$\int_\Gamma f(z)\omega_1(z)\,dz = 0.$$

For distinct points $t_1, t_2, \dots, t_{2n} \in [0, 2\pi)$ put

$$D_q(x) := \inf_{c_2,\dots,c_{2n}} \left\| V(\cdot, x) - \sum_{j=2}^{2n} c_j V(\cdot, t_j) \right\|_{L_q(\Gamma)/E_{q,1}}.$$

Since $L_q(\Gamma)/E_{q,1}$ is uniformly convex for $1 < q < \infty$ there are continuous functions on $[0, 2\pi]$, $c_2(x), \dots, c_{2n}(x)$, such that

$$D_q(x) = \left\| V(\cdot, x) - \sum_{j=2}^{2n} c_j(x) V(\cdot, t_j) \right\|_{L_q(\Gamma)/E_{q,1}}.$$

It follows from (6) and (8) that

$$\begin{aligned}
&\sup_{x\in[0,2\pi]} D_q(x)\\
&\quad = \sup_{\|f\|_{E_p}\le 1,\ f\perp 1}\left\|\frac{1}{4\pi}\int_\Gamma\Big(V(z,\cdot)-\sum_{j=2}^{2n}c_j(\cdot)V(z,t_j)\Big)f(z)\,dz\right\|_\infty\\
&\quad \ge \sup_{\|f^{(r)}\|_{E_\infty}\le 1}\left\|\frac{1}{4\pi}\int_\Gamma\Big(V(z,\cdot)-\sum_{j=2}^{2n}c_j(\cdot)V(z,t_j)\Big)f^{(r)}(z)\,dz\right\|_\infty\\
&\ge \sup_{f\in\widetilde{H}^r_{\infty,\beta}}\left\|f(\cdot)-f(t_1)-\sum_{j=2}^{2n}c_j(\cdot)(f(t_j)-f(t_1))\right\|_\infty \ge \lambda_{2n}(\widetilde{H}^r_{\infty,\beta},L_\infty).
\end{aligned}$$

The function D_q is continuous on $[0,2\pi]$, $1\le q<\infty$, and $D_q \searrow D_1$ uniformly as $q\searrow 1$. Letting q decrease to 1, we obtain

$$\begin{aligned}
&\lambda_{2n}(\widetilde{H}^r_{\infty,\beta},L_\infty)\le \sup_{x\in[0,2\pi]} D_1(x)\\
&\quad = \left\|\inf_{c_2,\dots,c_{2n}}\sup_{\|f^{(r)}\|_{E_\infty}\le 1}\left|\frac{1}{4\pi}\int_\Gamma\Big(V(z,\cdot)-\sum_{j=2}^{2n}c_jV(z,t_j)\Big)f^{(r)}(z)\,dz\right|\right\|_\infty\\
&\qquad\qquad = \sup_{x\in[0,2\pi]}\sigma(x),
\end{aligned}$$

where

$$\begin{aligned}
(9)\quad \sigma(x) := \sup\Bigg\{\ &\left|\frac{1}{4\pi}\int_\Gamma V(z,x)f^{(r)}(z)\,dz\right| : \|f^{(r)}\|_{E_\infty}\le 1,\\
&\int_\Gamma V(z,t_j)f^{(r)}(z)\,dz=0,\ j=2,\dots,2n\Bigg\}.
\end{aligned}$$

Using the same methods as in [5], it can be shown that the solution of (9) is unique up to a factor $e^{i\alpha}$, $\alpha\in\mathbb{R}$. By (8) we have

$$\begin{aligned}
\sigma(x)=\sup\big\{\,|f(x)-f(t_1)| : \|f^{(r)}\|_{E_\infty}\le 1,\ &f(t_j)-f(t_1)=0,\\
&j=2,\dots,2n\,\big\}.
\end{aligned}$$

Therefore, if $f^*(z)$ is a solution of (9), then $\overline{f^*(\bar z)}$ is also a solution of (9). Consequently, there exists an extremal function which is real on $\mathbb{R}$. Thus,

$$\sigma(x)=\inf_{c_2,\dots,c_{2n}}\sup_{f\in\widetilde{H}^r_{\infty,\beta}}\left|f(x)-f(t_1)-\sum_{j=2}^{2n}c_j(f(t_j)-f(t_1))\right|.$$

Put

$$\sigma_1(x):=\inf_{c_1,\dots,c_{2n}}\sup_{f\in\widetilde{H}^r_{\infty,\beta}}\left|f(x)-\sum_{j=1}^{2n}c_jf(t_j)\right|.$$

Obviously, $\sigma_1(x) \le \sigma(x)$. On the other hand, if $\sum_{j=1}^{2n} c_j \ne 1$, then

$$\sup_{f\in \widetilde{H}^r_{\infty,\beta}} \Big| f(x) - \sum_{j=1}^{2n} c_j f(t_j) \Big| \ge \sup_{c\in\mathbb{R}} \Big| c\Big(1 - \sum_{j=1}^{2n} c_j\Big)\Big| = \infty.$$

Hence, $\sigma_1(x) = \sigma(x)$. Now we have

$$\begin{aligned} \lambda_{2n}(\widetilde{H}^r_{\infty,\beta}, L_\infty) &\le \sup_{x\in[0,2\pi]} \sigma_1(x) \\ &= \sup_{x\in[0,2\pi]} \sup \Big\{ |f(x)| : f \in \widetilde{H}^r_{\infty,\beta},\ f(t_j) = 0,\ j = 1,\dots,2n \Big\}. \end{aligned}$$

For $t_j = t^{(j)}_{n,r}$, $j = 1,\dots,2n$, it follows from Proposition 5 that

$$\lambda_{2n}(\widetilde{H}^r_{\infty,\beta}, L_\infty) \le \|\Phi^\beta_{n,r}\|_\infty.$$

Since $\lambda_{2n} \ge d_{2n}$ and $\lambda_{2n} \ge d^{2n}$ we obtain

$$d_{2n}(\widetilde{H}^r_{\infty,\beta}, L_\infty) = \lambda_{2n}(\widetilde{H}^r_{\infty,\beta}, L_\infty) = d^{2n}(\widetilde{H}^r_{\infty,\beta}, L_\infty) = \|\Phi^\beta_{n,r}\|_\infty.$$

The theorem is proved. □

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References

[1] Akhiezer, N. I.: Lectures on the Theory of Approximation. Nauka, Moscow 1965

[2] Akhiezer, N. I.: Elements of the Theory of Elliptic Functions. Nauka, Moscow 1970

[3] Farkov, Yu. A.: The N-widths of Hardy–Sobolev spaces of several complex variables. J. Approx. Theory **75** (1993) 183–197

[4] Fisher, S. D.: Envelopes, widths, and Landau problems for analytic functions. Constr. Approx. **5** (1989) 171-187

[5] Khavinson, S. Ya.: Two Papers on Extremal Problems in Complex Analysis. Amer. Math. Soc. Translations, Ser. 2, No. 129 1985

[6] Korneichuk, N. P.: Exact Constants in the Theory of Approximation. Nauka, Moscow 1987

[7] Osipenko, K. Yu.: On n-widths, optimal quadrature formulas, and optimal recovery of functions analytic in a strip. Izv. Russian Akad. Nauk Ser. Mat. **58** (1994) 55–79

[8] Osipenko, K. Yu.: On N-widths of holomorphic functions of several variables. J. Approx. Theory (to appear)

[9] Osipenko, K. Yu.: Exact values of n-widths and optimal quadratures on classes of bounded analytic and harmonic functions. J. Approx. Theory (to appear)

[10] Pinkus, A.: n-Widths in Approximation Theory. Berlin, Springer-Verlag 1985

[11] Tikhomirov, V. M.: Diameters of sets in function spaces and the theory of best approximations. Uspekhi Mat. Nauk **15** (1960) 81–120; English transl. in Russian Math. Surveys **15** (1960) 75–111