

ISMAGILOV TYPE THEOREMS FOR LINEAR, GEL'FAND AND BERNSTEIN n -WIDTHS

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ABSTRACT. Using a variational principle for s -numbers, we obtain estimates for the linear, Gel'fand and Bernstein n -widths. A simple proof of some results concerned with the exact values of n -widths of diagonal operators is given. We also calculate the exact values of the Bernstein n -widths for the Hardy–Sobolev classes.

1. INTRODUCTION

Let X, Y be normed linear spaces and $T: X \rightarrow Y$ be a bounded linear operator. The linear λ_n , Gel'fand d^n and Bernstein b_n n -widths of the operator T are defined by

$$\lambda_n(T) := \inf_{P_n} \sup_{x \in BX} \|Tx - P_n x\|_Y, \quad d^n(T) := \inf_{X^n} \sup_{x \in BX^n} \|Tx\|_Y,$$
$$b_n(T) := \sup_{X_{n+1}} \inf_{\substack{x \in X_{n+1} \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X},$$

where P_n is any linear operator mapping X into Y of rank at most n , BX is the closed unit ball of X , X^n runs over all n -codimensional subspaces of X and X_{n+1} runs over all $(n+1)$ -dimensional subspaces of X .

In Osipenko and Stessin [1] the exact values of the linear and Gel'fand n -widths of the Hardy classes H_2 were obtained. A method of the proof was very close to the one from Ismagilov's Theorem [2] (see also [3, p. 93]). After the paper [1] several results were obtained for $\lambda_n(T)$ and $d^n(T)$ where T is a map from a Hilbert space H into $C(E)$ (see [4]–[6]). Parfenov [7] solved an analogous problem for the Bernstein n -widths $b_n(T)$ where $T: L_\infty(E, \nu) \rightarrow H$ and ν is a probability measure on E .

In this paper we show that many of these results can be obtained, using a general principle concerned with extremal properties of s -numbers. Section 2 is devoted to this principle. In Section 3 we prove the estimates of the linear, Gel'fand and Bernstein n -widths. In Section 4 we give a simple proof of two results about the exact values of n -widths for diagonal operators in the discrete case. Finally, in

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Section 5 we calculate the Bernstein n -widths of the Hardy–Sobolev classes.

2. VARIATIONAL PRINCIPLE FOR s -NUMBERS

Let H and H_1 be Hilbert spaces and $T: H \rightarrow H_1$ a bounded linear operator. Suppose that

$$T'T\varphi_k = \lambda_k\varphi_k, \quad k = 1, 2, \dots,$$

where $\lambda_1 \geq \lambda_2 \geq \dots > 0$ and $\{\varphi_k\}$ form a complete orthonormal basis for the range of $T'T$ (a sufficient condition is that T be a compact operator). The values $s_k(T) = \sqrt{\lambda_k}$ are called the s -numbers of T .

Set $\psi_k := s_k^{-1}(T)T\varphi_k$. Note that $\{\psi_k\}$ is an orthonormal system in H_1 . Then there exists the Schmidt decomposition of T (see, for example, [8]) which is given by

$$T = \sum_{k=1}^{\infty} s_k(T)(\cdot, \varphi_k)\psi_k.$$

Theorem 1. *Let T be as above. Then*

$$(1) \quad \sum_{k=1}^n s_k^2(T) = \max_{\{e_k\}_1^n} \sum_{k=1}^n \|Te_k\|_{H_1}^2,$$

where the maximum is taken over all orthonormal systems $\{e_k\}_1^n$ in H . Furthermore,

$$(2) \quad \sum_{k=n+1}^{\infty} s_k^2(T) = \min_{\{e_k\}_1^{\infty}} \sum_{k=1}^{\infty} \|Te_k\|_{H_1}^2,$$

where the minimum is taken over all orthonormal systems $\{e_k\}_1^{\infty}$ in H such that $\text{codim span}\{e_k\}_1^{\infty} \leq n$.

For a compact operator T this theorem was proved in Parfenov [6]. In our case the proof is almost the same because it does not so much depend on the compactness of T as on the fact that the eigenvectors $\{\varphi_k\}$ of $T'T$ form an orthonormal basis for the range of $T'T$.

We remark that both parts in (2) are finite iff

$$\sum_{k=1}^{\infty} s_k^2(T) < \infty.$$

In Parfenov [6] Theorem 1 was the basic tool in calculating the Gel'fand n -widths of operators $T: H \rightarrow L_{\infty}(E, \nu)$. Similar results were obtained in Osipenko [5], using Ismagilov's Theorem and the duality between the Kolmogorov and Gel'fand n -widths.

In order to estimate the Bernstein n -widths we need the following properties of s -numbers.

Theorem 2. *Let T be as in Theorem 1 and $\text{Ker } T = 0$. Then*

$$(3) \quad \sum_{k=1}^n s_k^{-2}(T) = \min_{\{f_k\}_1^n} \sum_{k=1}^n \|T^{-1}f_k\|_H^2,$$

where the minimum is taken over all orthonormal systems $\{f_k\}_1^n$ in $T(H)$. Furthermore, if $\dim H = N < \infty$, then

$$(4) \quad \sum_{k=n+1}^N s_k^{-2}(T) = \max_{\{f_k\}_1^{N-n}} \sum_{k=1}^{N-n} \|T^{-1}f_k\|_H^2,$$

where the maximum is taken over all orthonormal systems $\{f_k\}_1^{N-n}$ in $T(H)$.

Proof. Let $\{f_k\}_1^n$ be any orthonormal system in $T(H)$. Set $L_n := \text{span}\{T^{-1}f_k\}_1^n$, $\tilde{L}_n := \text{span}\{f_k\}_1^n$, and $T_n := T|_{L_n}$. Suppose that the Schmidt decomposition of T_n has the form

$$T_n = \sum_{k=1}^n s_k(T_n)(\cdot, \varphi'_k)\psi'_k.$$

The value

$$\left(\sum_{k=1}^n \|T_n^{-1}f_k\|_H^2 \right)^{1/2}$$

is the Hilbert–Schmidt norm of T_n^{-1} and does not depend on the choice of the orthonormal basis in \tilde{L}_n . Therefore,

$$\sum_{k=1}^n \|T^{-1}f_k\|_H^2 = \sum_{k=1}^n \|T_n^{-1}f_k\|_H^2 = \sum_{k=1}^n \|T_n^{-1}\psi'_k\|_H^2 = \sum_{k=1}^n s_k^{-2}(T_n).$$

Let P_n be an orthoprojector in H onto L_n . Using the properties of s -numbers of bounded linear operators (see [8, p. 82]) we have

$$s_k(T_n) = s_k(T \circ P_n) \leq \|P_n\| s_k(T) = s_k(T).$$

Thus

$$\sum_{k=1}^n \|T^{-1}f_k\|_H^2 \geq \sum_{k=1}^n s_k^{-2}(T).$$

If $f_k = \psi_k$, $k = 1, \dots, n$, then

$$\sum_{k=1}^n \|T^{-1}\psi_k\|_H^2 = \sum_{k=1}^n s_k^{-2}(T).$$

The equality (3) is proved.

To prove (4) note that

$$s_k(T^{-1}) = s_{N-k+1}^{-1}(T), \quad k = 1, \dots, N.$$

Now (4) follows from (1). The theorem is proved. \square

3. ESTIMATES OF LINEAR, GEL'FAND AND BERNSTEIN n -WIDTHS

To obtain estimates of n -widths we need the following simple result.

Lemma 1. *Let H be a Hilbert space, $\omega := \dim H$ and $T: H \rightarrow C(E)$ a bounded linear operator. Then*

$$\|T\|_{H \rightarrow C(E)} = \sup_{z \in E} \left(\sum_{k=1}^{\omega} |(Te_k)(z)|^2 \right)^{1/2}$$

for any orthonormal basis $\{e_k\}_1^{\omega}$ in H .

Proof. We have

$$\begin{aligned} \|T\|_{H \rightarrow C(E)} &= \sup_{h \in BH} \sup_{z \in E} |(Th)(z)| = \sup_{z \in E} \sup_{h \in BH} |(Th)(z)| \\ &= \sup_{z \in E} \sup_{\{c_k\}_1^{\omega} \in Bl_2} \left| \sum_{k=1}^{\omega} c_k (Te_k)(z) \right| = \sup_{z \in E} \left(\sum_{k=1}^{\omega} |(Te_k)(z)|^2 \right)^{1/2}. \end{aligned}$$

The lemma is proved. \square

Let H be a Hilbert space of functions defined on some set Ω . A function $K(z, w)$ defined on $\Omega \times \Omega$ is called a reproducing kernel of H if for each $w \in \Omega$, $K(z, w) \in H$, and for all $f \in H$

$$f(w) = (f(\cdot), K(\cdot, w))_H.$$

It is a well-known fact that if the $\{\varphi_k\}_1^{\omega}$ form an orthonormal basis in H , then

$$K(z, w) = \sum_{k=1}^{\omega} \varphi_k(z) \overline{\varphi_k(w)}.$$

Suppose that Ω is a topological space, $E \subset \Omega$ and $Tf := f|_E$ is a bounded linear operator from H into $C(E)$. Then from Lemma 1 we obtain

$$\|T\|_{H \rightarrow C(E)} = \sup_{z \in E} (K(z, z))^{1/2}.$$

Theorem 3 ([5], [6]). *Let H be a Hilbert space, E a topological space with probability measure ν such that $\text{supp } \nu = E$, and $T: H \rightarrow C(E)$ a bounded linear operator. Define $T_0: H \rightarrow L_2(E, \nu)$ by the equation $T_0 h := Th$. Assume that*

$$(5) \quad T_0' T_0 \varphi_k = s_k^2 \varphi_k, \quad k = 1, 2, \dots,$$

where $s_1 \geq s_2 \geq \dots > 0$ and $\{\varphi_k\}$ is an orthonormal basis for the range of $T_0' T_0$. Then

$$\left(\sum_{k=n+1}^{\infty} s_k^2 \right)^{1/2} \leq \lambda_n(T) = d^n(T) \leq \sup_{z \in E} \left(\sum_{k=n+1}^{\infty} |(T\varphi_k)(z)|^2 \right)^{1/2}.$$

Proof. Since $\text{supp } \nu = E$, $\text{Ker } T_0' T_0 = \text{Ker } T_0 = \text{Ker } T$ and we can assume, without loss of generality, that $\{\varphi_k\}$ is an orthonormal basis in H . From the definition of the Gel'fand n -width it follows that

$$d^n(T) = \inf_{H^n} \|T\|_{H^n \rightarrow C(E)},$$

where H^n runs over all n -codimensional subspaces of H . Consider $H^n = \{\varphi_k\}_{n+1}^\infty$. We obtain from Lemma 1

$$d^n(T) \leq \sup_{z \in E} \left(\sum_{k=n+1}^{\infty} |(T\varphi_k)(z)|^2 \right)^{1/2}.$$

Let H^n be any n -codimensional subspace of H . Suppose that $\{\varphi'_k\}$ is an orthonormal basis in H^n . Using Lemma 1 and (2), we have

$$\begin{aligned} \|T\|_{H^n \rightarrow C(E)}^2 &= \sup_{z \in E} \sum_{k=1}^{\infty} |(T\varphi'_k)(z)|^2 \geq \int_E \sum_{k=1}^{\infty} |(T\varphi'_k)(z)|^2 d\nu(z) \\ &= \sum_{k=1}^{\infty} \|T_0\varphi'_k\|_{L_2(E,\nu)}^2 \geq \sum_{k=n+1}^{\infty} s_k^2. \end{aligned}$$

Thus,

$$d^n(T) \geq \left(\sum_{k=n+1}^{\infty} s_k^2 \right)^{1/2}.$$

The equality $\lambda_n(T) = d^n(T)$ is the well-known fact for operators defined on Hilbert spaces (see, [3, p. 33]). The theorem is proved. \square

Now we will obtain the similar estimates for the Bernstein n -widths.

Theorem 4. *Let H , E , and ν be as above, H_1 be a subspace of $L_2(E,\nu)$, and $X_E \subset H_1$ be a subspace of $C(E)$. Assume that a bounded linear operator $T_0: H_1 \rightarrow H$ satisfies the conditions (5) where $s_1 \geq s_2 \geq \dots > 0$, $\{\varphi_k\}$ is an orthonormal basis for the range of $T_0' T_0$, and $\varphi_k \in X_E$, $k = 1, 2, \dots$. Define $T: X_E \rightarrow H$ by the equation $Tf := T_0 f$. Then*

$$(6) \quad \left(\sup_{z \in E} \sum_{k=1}^{n+1} s_k^{-2} |\varphi_k(z)|^2 \right)^{-1/2} \leq b_n(T) \leq \left(\sum_{k=1}^{n+1} s_k^{-2} \right)^{-1/2}.$$

Proof. Let $L_{n+1} \subset X_E$ and $\dim L_{n+1} = n+1$. Consider the operator $T_{n+1} := T|_{L_{n+1}}$. If $\text{Ker } T_{n+1} \neq 0$, then

$$\inf_{\substack{f \in L_{n+1} \\ f \neq 0}} \frac{\|Tf\|_H}{\|f\|_{C(E)}} = 0.$$

Suppose that $\text{Ker } T_{n+1} = 0$. Then we can define the operator $T_{n+1}^{-1}: T(L_{n+1}) \rightarrow L_{n+1}$. Using Lemma 1 and (4), for any orthonormal system $\{e_k\}_1^{n+1}$ in $T(L_{n+1})$ we have

$$\begin{aligned} \inf_{\substack{f \in L_{n+1} \\ f \neq 0}} \frac{\|Tf\|_H}{\|f\|_{C(E)}} &= \inf_{\substack{g \in T(L_{n+1}) \\ g \neq 0}} \frac{\|g\|_H}{\|T_{n+1}^{-1}g\|_{C(E)}} = \|T_{n+1}^{-1}\|_{T(L_{n+1}) \rightarrow C(E)}^{-1} \\ &= \left(\sup_{z \in E} \sum_{k=1}^{n+1} |(T_{n+1}^{-1}e_k)(z)|^2 \right)^{-1/2} \leq \left(\sum_{k=1}^{n+1} \int_E |(T_{n+1}^{-1}e_k)(z)|^2 d\nu(z) \right)^{-1/2} \\ &= \left(\sum_{k=1}^{n+1} \|T_{n+1}^{-1}e_k\|_{H_1}^2 \right)^{-1/2} \leq \left(\sum_{k=1}^{n+1} s_k^{-2}(T_{n+1}) \right)^{-1/2}. \end{aligned}$$

Since $s_k(T_{n+1}) \leq s_k(T_0) = s_k$, $k = 1, \dots, n+1$, we obtain

$$b_n(T) \leq \left(\sum_{k=1}^{n+1} s_k^{-2} \right)^{-1/2}.$$

Let $L_{n+1} = \text{span}\{\varphi_k\}_1^{n+1}$. Then $\psi_k := s_k^{-1}T\varphi_k$, $k = 1, \dots, n+1$, form an orthonormal system in $T(L_{n+1})$. Thus,

$$\begin{aligned} b_n(T) &\geq \inf_{\substack{f \in L_{n+1} \\ f \neq 0}} \frac{\|Tf\|_H}{\|f\|_{C(E)}} = \left(\sup_{z \in E} \sum_{k=1}^{n+1} |(T_{n+1}^{-1}\psi_k)(z)|^2 \right)^{-1/2} \\ &= \left(\sup_{z \in E} \sum_{k=1}^{n+1} s_k^{-2} |\varphi_k(z)|^2 \right)^{-1/2}. \end{aligned}$$

The theorem is proved. \square

4. n -WIDTHS OF DIAGONAL OPERATORS

Let $T: l_2 \rightarrow l_\infty$ be the diagonal operator

$$(7) \quad T(\{x_k\}_1^\infty) := \{\lambda_k x_k\}_1^\infty,$$

where $\lambda_1 \geq \lambda_2 \geq \dots > 0$. Smolyak [9] (in the finite-dimensional case) proved that

$$(8) \quad d^n(T) = \sup_{m > n} \left(\frac{m-n}{\sum_{k=1}^m \lambda_k^{-2}} \right)^{1/2}.$$

In dual terms this result was obtained by Sofman [10] (see also [11]). We will show that the lower bound in (8) easily follows from Theorem 3.

Denote by $\{e_k\}_1^\infty$ the standard basis of l_2 . Fix any $m > n$. Let $T_m: l_2^m \rightarrow l_\infty^m$ be the operator defined by

$$T_m(\{x_k\}_1^m) := \{\lambda_k x_k\}_1^m.$$

It is easy to see that $d^n(T) \geq d^n(T_m)$. Define the probability measure ν_m on the set $\{1, 2, \dots, m\}$ as

$$\nu_m(\{j\}) := \lambda_j^{-2} \left(\sum_{k=1}^m \lambda_k^{-2} \right)^{-1}.$$

Denote by T_{m0} the operator T_m regarded as an operator from l_2^m into $l_2^m(\nu_m)$. Then

$$T_{m0}' T_{m0} e_j = \left(\sum_{k=1}^m \lambda_k^{-2} \right)^{-1} e_j, \quad j = 1, \dots, m.$$

From Theorem 3

$$d^n(T_m) \geq \left(\frac{m-n}{\sum_{k=1}^m \lambda_k^{-2}} \right)^{1/2}.$$

Thus

$$\lambda_n(T) = d^n(T) \geq \sup_{m>n} d^n(T_m) \geq \left(\frac{m-n}{\sum_{k=1}^m \lambda_k^{-2}} \right)^{1/2}.$$

Note that the values (8) are also related to linear stochastic n -widths (see [12]).

Consider the operator (7) as an operator from l_∞ into l_2 . Here we assume that $\{\lambda_k\}_1^\infty \in l_2$. Galeev [13] proved the equality

$$b_n(T) = \min_{0 \leq m < n+1} \left(\frac{\sum_{k=m+1}^\infty \lambda_k^2}{n-m+1} \right)^{1/2}.$$

We will show how the upper bound can be obtained from Theorem 4.

Let $0 < \varepsilon < 1$ and $0 \leq m < n+1$. Define the probability measure ν_m on \mathbb{N} as

$$\nu_m(\{j\}) := \begin{cases} (1-\varepsilon) \frac{\lambda_j^2}{\sum_{k=m+1}^\infty \lambda_k^2}, & j > m, \\ \frac{\varepsilon}{m}, & j \leq m. \end{cases}$$

Denote $T_0: l_2(\nu_m) \rightarrow l_2$ by the equation $T_0 x := T x$. It is easy to obtain that

$$T_0' T_0 e_j = s_j^2 e_j, \quad j = 1, 2, \dots,$$

where

$$s_j^2 = \begin{cases} (1-\varepsilon)^{-1} \sum_{k=m+1}^\infty \lambda_k^2, & j > m, \\ \varepsilon^{-1} \lambda_j^2 m, & j \leq m. \end{cases}$$

From Theorem 4

$$b_n(T) \leq \left((1-\varepsilon) \frac{n-m+1}{\sum_{k=m+1}^\infty \lambda_k^2} + \frac{\varepsilon}{m} \sum_{j=1}^m \lambda_j^{-2} \right)^{-1/2}.$$

Letting $\varepsilon \rightarrow 0$, we obtain

$$b_n(T) \leq \min_{0 \leq m < n+1} \left(\frac{\sum_{k=m+1}^{\infty} \lambda_k^2}{n-m+1} \right)^{1/2}.$$

5. BERNSTEIN n -WIDTHS OF HARDY-SOBOLEV CLASSES

Let A be a closed, convex, centrally symmetric subset of a normed linear space Y . The Bernstein n -width of A is defined by

$$b_n(A, Y) := \sup_{Y_{n+1}} \sup \{ \lambda : \lambda B Y_{n+1} \subset A \},$$

where Y_{n+1} is any $(n+1)$ -dimensional subspace of Y . If $\text{Ker } T = 0$, then it is easily shown that

$$b_n(T) = b_n(T(BX), Y).$$

We need the following simple property of b_n .

Lemma 2. *Let H be a Hilbert space, A a closed, convex, centrally symmetric subset of H , and H_r an r -dimensional subspace of H such that $A \perp H_r$. Then*

$$(9) \quad b_{n+r}(A + H_r, H) = b_n(A, H).$$

Proof. Assume that $H_{n+1} \subset H$, $\dim H_{n+1} = n+1$, and

$$\sup \{ \lambda : \lambda B H_{n+1} \subset A \} = \mu > 0.$$

Put $H_{n+r+1} := H_{n+1} + H_r$. Since $A \perp H_r$ it follows that $H_{n+1} \perp H_r$ and $\dim H_{n+r+1} = n+r+1$. If $x \in H_{n+r+1}$, $\|x\|_H \leq \mu$, then $x = x_1 + x_2$, where $x_1 \in H_{n+1}$, $x_2 \in H_r$, and

$$\|x_1\|_H^2 \leq \|x_1\|_H^2 + \|x_2\|_H^2 = \|x\|_H^2 \leq \mu^2.$$

Thus $x_1 \in A$. Consequently, $x \in A + H_r$. We have

$$\sup \{ \lambda : \lambda B H_{n+r+1} \subset A + H_r \} \geq \mu.$$

So we proved that

$$(10) \quad b_{n+r}(A + H_r, H) \geq b_n(A, H).$$

If $b_{n+r}(A + H_r, H) = 0$, then (9) follows from (10). Suppose that $b_{n+r}(A + H_r, H) > 0$. Let $H_{n+r+1} \subset H$, $\dim H_{n+r+1} = n+r+1$, and

$$\sup \{ \lambda : \lambda B H_{n+r+1} \subset A + H_r \} = \mu > 0.$$

Since $H_{n+r+1} \subset \text{span } A + H_r$, $\dim(H_{n+r+1} \cap \text{span } A) \geq n+1$. Hence there exists a subspace $H_{n+1} \subset H_{n+r+1} \cap \text{span } A$ with $\dim H_{n+1} = n+1$. Let $x \in H_{n+1}$ and $\|x\|_H \leq \mu$. Then $x \in A + H_r$. In addition $x \in \text{span } A$. Thus $x \in A$ and

$$\sup \{ \lambda : \lambda B H_{n+1} \subset A \} \geq \mu.$$

Therefore,

$$b_n(A, H) \geq b_{n+r}(A + H_r, H).$$

The lemma is proved. \square

Theorem 5. *Let H , E , ν , H_1 , and X_E be as in Theorem 4. Suppose that the Hilbert space H_1 has a reproducing kernel. Let $\{\varphi_k\}$ be an orthonormal basis in H_1 and $T_0: H_1 \rightarrow H$ a bounded linear operator. Assume that $\{T_0\varphi_k\}$ is an orthogonal system in H , $\varphi_k \in X_E$, $k = 1, 2, \dots$ and $s_k := \|T_0\varphi_k\|_H$ form a non-increasing sequence. Define $T: X_E \rightarrow H$ by the equation $Tf := T_0f$. Then the inequalities (6) hold.*

Proof. Denote by $K(z, w)$ the reproducing kernel of H_1 . Since $\{\varphi_k\}$ is an orthonormal basis in H_1 the representation

$$K(z, w) = \sum_{k=1}^{\infty} \varphi_k(z) \overline{\varphi_k(w)}$$

holds. We have

$$(T'_0 T_0 f)(w) = ((T'_0 T_0 f)(\cdot), K(\cdot, w))_{H_1} = ((T_0 f)(\cdot), (T_0 K)(\cdot, w))_H.$$

Thus

$$(T'_0 T_0 \varphi_j)(w) = ((T_0 \varphi_j)(\cdot), \sum_{k=1}^{\infty} \overline{\varphi_k(w)} (T_0 \varphi_k)(\cdot))_H = s_j^2 \varphi_j(w).$$

Now it suffices to apply Theorem 4. The theorem is proved. \square

Let B_ρ be the ball of \mathbb{C}^n of radius ρ

$$B_\rho := \{z := (z_1, \dots, z_n) \in \mathbb{C}^n : |z|^2 := \sum_{k=1}^n |z_k|^2 < \rho\},$$

$S_\rho := \partial B_\rho$, σ_ρ the probability measure on the sphere S_ρ which is invariant with respect to the orthogonal group $O(2n)$, and ν_ρ the normalized Lebesgue measure in B_ρ (if $\rho = 1$ we will write B , S , σ , and ν).

The Hardy space $H_p(B)$ (H_p) is the set of holomorphic functions in B which satisfy

$$\|f\|_{H_p} := \sup_{0 < \rho < 1} \left(\int_S |f(z)|^p d\sigma(z) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_{H_\infty} := \sup_{z \in B} |f(z)|.$$

Let $f(z)$ be a holomorphic function in B and let

$$f(z) = \sum_{k=0}^{\infty} F_k(z)$$

be a homogeneous decomposition of f . The radial derivative of order r is defined by

$$\mathcal{R}^r f(z) := \sum_{k=r}^{\infty} \frac{k!}{(k-r)!} F_k(z)$$

(for $r = 1$ see [14, Chap. 6]). Denote by $H\mathcal{R}_\infty^r(B)$ ($H\mathcal{R}_\infty^r$) the class of holomorphic functions in B for which $\mathcal{R}^r f \in BH_\infty$.

Set

$$N_m := \sum_{k=0}^{m-1} \binom{n+k-1}{n-1}.$$

Note that $N_m = \dim \mathcal{P}_{m-1}^n$, where \mathcal{P}_m^n is the space of n -variable polynomials of degree m or less.

Theorem 6. *For all $0 < \rho \leq 1$ and all $m \geq r + 1$*

(11)

$$b_{N_m-1}(H\mathcal{R}_\infty^r, L_2(S_\rho, \sigma_\rho)) = \left(\frac{1}{(n-1)!} \sum_{k=r}^{m-1} \frac{k!(n+k-1)!}{((k-r)!)^2} \rho^{-2k} \right)^{-1/2},$$

(12)

$$b_{N_m-1}(H\mathcal{R}_\infty^r, L_2(B_\rho, \nu_\rho)) = \left(\frac{1}{n!} \sum_{k=r}^{m-1} \frac{k!(n+k)!}{((k-r)!)^2} \rho^{-2k} \right)^{-1/2}.$$

Proof. For multiindex $\alpha := (\alpha_1, \dots, \alpha_n)$ and $z \in \mathbb{C}^n$ set

$$z^\alpha := z_1^{\alpha_1} \dots z_n^{\alpha_n}, \quad |\alpha| := \alpha_1 + \dots + \alpha_n, \quad \alpha! := \alpha_1! \dots \alpha_n!,$$

$$D_j := \frac{\partial}{\partial z_j}, \quad D^\alpha := D_1^{\alpha_1} \dots D_n^{\alpha_n}.$$

Denote by H_p^0 the space of all functions $f \in H_p$ for which $(D^\alpha f)(0) = 0$, $|\alpha| = 0, \dots, r-1$. It is known (see [14]) that functions from H_p , $1 \leq p \leq \infty$, have finite boundary values almost everywhere. Moreover, H_2 is a Hilbert space with the inner product

$$(f, g)_{H_2} := \int_S f(z) \overline{g(z)} d\sigma(z).$$

The space H_2 has the reproducing kernel

$$K(z, w) = \left(1 - \sum_{k=1}^n z_k \overline{w_k} \right)^{-n}.$$

Define $T_0: H_2^0 \rightarrow L_2(S_\rho, \sigma_\rho)$ and $T: H_\infty^0 \rightarrow L_2(S_\rho, \sigma_\rho)$ by the equations

$$(13) \quad (T_0 f)(z) := \sum_{k=r}^{\infty} \frac{(k-r)!}{k!} F_k(z), \quad T f := T_0 f,$$

where

$$f(z) = \sum_{k=r}^{\infty} F_k(z).$$

It is easy to see that

$$H\mathcal{R}_\infty^r = T(BH_\infty^0) + \mathcal{P}_{r-1}^n.$$

Since monomials z^α are orthogonal in $L_2(S_\rho, \sigma_\rho)$ we obtain from Lemma 2

$$(14) \quad b_{N_m-1}(H\mathcal{R}_\infty^r, L_2(S_\rho, \sigma_\rho)) = b_{N_m-N_r-1}(T).$$

For every $0 < \rho \leq 1$ (see [14])

$$\|z^\alpha\|_{L_2(S_\rho, \sigma_\rho)}^2 = \frac{(n-1)!\alpha!}{(n+|\alpha|-1)!} \rho^{2|\alpha|}.$$

Thus the functions

$$\varphi_\alpha(z) := \left(\frac{(n+|\alpha|-1)!}{(n-1)!\alpha!} \right)^{1/2} z^\alpha, \quad |\alpha| \geq r,$$

form a complete orthonormal basis in H_2^0 . We have

$$\|T_0 \varphi_\alpha\|_{L_2(S_\rho, \sigma_\rho)}^2 = \left(\frac{(|\alpha|-r)!}{|\alpha!|} \right)^2 \rho^{2|\alpha|}.$$

The number of different monomials z^α with $|\alpha| = k$ equals $\binom{n+k-1}{n-1}$. By Theorem 5

$$\begin{aligned} \left(\sup_{z \in S} \sum_{|\alpha|=r}^{m-1} \left(\frac{|\alpha!|}{(|\alpha|-r)!} \right)^2 \rho^{-2|\alpha|} |\varphi_\alpha(z)|^2 \right)^{-1/2} &\leq b_{N_m - N_{r-1}}(T) \\ &\leq \left(\sum_{k=r}^{m-1} \left(\frac{k!}{(k-r)!} \right)^2 \binom{n+k-1}{n-1} \rho^{-2k} \right)^{-1/2}. \end{aligned}$$

Using the equation

$$\sum_{|\alpha|=k} \frac{|z^{2\alpha}|}{\alpha!} = \frac{|z|^{2k}}{k!},$$

we obtain

$$b_{N_m - N_{r-1}}(T) = \left(\frac{1}{(n-1)!} \sum_{k=r}^{m-1} \frac{k!(n+k-1)!}{((k-r)!)^2} \rho^{-2k} \right)^{-1/2}.$$

Now (11) follows from (14).

The proof of (12) is almost the same. The difference is that we have to consider the operators $T_0: H_2^0 \rightarrow L_2(B_\rho, \nu_\rho)$ and $T: H_\infty^0 \rightarrow L_2(B_\rho, \nu_\rho)$ defined by (13). Then we use

$$\|T_0 \varphi_\alpha\|_{L_2(B_\rho, \nu_\rho)}^2 = \left(\frac{(|\alpha|-r)!}{|\alpha!|} \right)^2 \frac{n}{n+|\alpha|} \rho^{2|\alpha|}.$$

The theorem is proved. \square

For $n = 1$ the class $H\mathcal{R}_\infty^r$ coincides with the class BH_∞^r defined as the set of all holomorphic functions in B for which $f^{(r)}(z) \in BH_\infty$. From Theorem 6 we obtain the following result.

Corollary 1. *For all $0 < \rho \leq 1$ and all $m \geq r$*

$$b_m(BH_\infty^r, L_2(S_\rho, \sigma_\rho)) = \left(\sum_{k=r}^m \left(\frac{k!}{(k-r)!} \right)^2 \rho^{-2k} \right)^{-1/2},$$

$$b_m(BH_\infty^r, L_2(B_\rho, \nu_\rho)) = \left(\sum_{k=r}^m \frac{k!(k+1)!}{((k-r)!)^2} \rho^{-2k} \right)^{-1/2}.$$

In particular, we have for $r = 0$, $m \geq 0$, and $0 < \rho < 1$

$$(15) \quad b_m(BH_\infty, L_2(S_\rho, \sigma_\rho)) = \rho^m \left(\frac{1 - \rho^2}{1 - \rho^{2m+2}} \right)^{1/2},$$

$$(16) \quad b_m(BH_\infty, L_2(B_\rho, \nu_\rho)) = \rho^m \frac{1 - \rho^2}{\sqrt{(m+1)(1 - \rho^2) - \rho^2(1 - \rho^{2m+2})}}.$$

The values (11) for $r = 0$ and (15) were calculated in [7].

Let us compare (15) and (16) with the exact values of the Kolmogorov, linear, and Gel'fand n -widths. From [15] and [16] it follows that

$$d_m(BH_\infty, X) = \lambda_m(BH_\infty, X) = d^m(BH_\infty, X) = \begin{cases} \rho^m, & X = L_2(S_\rho, \sigma_\rho), \\ \frac{\rho^m}{\sqrt{m+1}}, & X = L_2(B_\rho, \nu_\rho). \end{cases}$$

Finally, we will determine exact values of the Bernstein n -widths for some classes of periodic holomorphic functions. Let $D_\beta := \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$. Denote by $\tilde{H}_{p,\beta}$ the set of all 2π -periodic holomorphic functions in D_β which satisfy the conditions

$$\|f\|_{\tilde{H}_{p,\beta}} := \sup_{0 < h < \beta} \left(\frac{1}{4\pi} \int_0^{2\pi} (|f(x+ih)|^p + |f(x-ih)|^p) dx \right)^{1/p} < \infty,$$

$$1 \leq p < \infty,$$

$$\|f\|_{\tilde{H}_{\infty,\beta}} := \sup_{z \in D_\beta} |f(z)| < \infty.$$

Let $B\tilde{H}_{p,\beta}^r$ be the set of all 2π -periodic holomorphic functions in D_β for which $f^{(r)}(z) \in \tilde{H}_{p,\beta}$. Denote by L_2 the periodic complex-valued Lebesgue space on the real axis with the norm

$$\|f\|_{L_2} := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx \right)^{1/2}.$$

Theorem 7. (1) For all $n \geq 1$ and $r \geq 1$

$$b_{2n}(B\tilde{H}_{\infty,\beta}^r, L_2) = \left(2 \sum_{k=1}^n k^{2r} \cosh 2k\beta \right)^{-1/2}.$$

(2) For all $n \geq 0$

$$(17) \quad b_{2n}(B\tilde{H}_{\infty,\beta}, L_2) = \left(\frac{\sinh \beta}{\sinh(2n+1)\beta} \right)^{1/2}.$$

Proof. The space $\tilde{H}_{2,\beta}$ is the Hilbert space with the inner product

$$(f, g)_{\tilde{H}_{2,\beta}} := \frac{1}{4\pi} \int_0^{2\pi} \left(f(x+i\beta)\overline{g(x+i\beta)} + f(x-i\beta)\overline{g(x-i\beta)} \right) dx.$$

The functions

$$\varphi_k(z) := \frac{e^{ikz}}{\sqrt{\cosh 2k\beta}}, \quad k \in \mathbb{Z},$$

form an orthonormal basis in $\tilde{H}_{2,\beta}$. The space $\tilde{H}_{2,\beta}$ has the reproducing kernel

$$K(z, w) = \sum_{k \in \mathbb{Z}} \varphi_k(z)\overline{\varphi_k(w)} = 1 + 2 \sum_{k=1}^{\infty} \frac{\cos k(z-\bar{w})}{\cosh 2k\beta}.$$

Let $r \geq 1$. Denote by $\tilde{H}_{p,\beta}^0$ the space of functions $f \in \tilde{H}_{p,\beta}$ for which

$$\int_0^{2\pi} f(x) dx = 0.$$

Define $T_0: \tilde{H}_{2,\beta}^0 \rightarrow L_2$ and $T: \tilde{H}_{\infty,\beta}^0 \rightarrow L_2$ by the equations

$$(T_0 f)(z) := \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \frac{c_k}{(ik)^r} e^{ikz}, \quad T f := T_0 f,$$

where

$$f(z) = \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} c_k e^{ikz}.$$

It is easily seen that

$$B\tilde{H}_{\infty,\beta}^r = T(B\tilde{H}_{\infty,\beta}^0) + \mathbb{C}.$$

By Lemma 2 we obtain

$$b_{2n}(B\tilde{H}_{\infty,\beta}^r, L_2) = b_{2n-1}(T).$$

The functions $\varphi_k(z)$, $k = \pm 1, \pm 2, \dots$, form a complete orthonormal basis in $\tilde{H}_{2,\beta}^0$ and

$$\|T_0 \varphi_k\|_{L_2}^2 = \frac{1}{k^{2r} \cosh 2k\beta}.$$

Since for all $z \in \partial D_\beta$

$$|\varphi_k(z)|^2 + |\varphi_{-k}(z)|^2 = 2$$

we have by Theorem 5

$$b_{2n-1}(T) = \left(2 \sum_{k=1}^n k^{2r} \cosh 2k\beta \right)^{-1/2}.$$

To obtain (17) we use the same scheme and the equality

$$1 + 2 \sum_{k=1}^n \cosh 2k\beta = \frac{\sinh(2n+1)\beta}{\sinh \beta}.$$

The theorem is proved. \square

Denote by $B\tilde{H}_{\infty,\beta}^{\mathbb{R}}$ the set of functions from $B\tilde{H}_{\infty,\beta}$ that are real-valued on \mathbb{R} . For even n the exact values of the Kolmogorov, linear, and Gel'fand n -widths of $B\tilde{H}_{\infty,\beta}^{\mathbb{R}}$ in L_q , $1 \leq q \leq \infty$, were determined in [17]. In particular, for $q = 2$

$$\begin{aligned} d_{2n}(B\tilde{H}_{\infty,\beta}^{\mathbb{R}}, L_2) &= \lambda_{2n}(B\tilde{H}_{\infty,\beta}^{\mathbb{R}}, L_2) = d^{2n}(B\tilde{H}_{\infty,\beta}^{\mathbb{R}}, L_2) \\ &= \left(\frac{\lambda}{\Lambda} \int_0^1 \frac{t^2 dt}{\sqrt{(1-t^2)(1-\lambda^2 t^2)}} \right)^{1/2} = \sqrt{2}e^{-\beta n} + O(e^{-5\beta n}), \end{aligned}$$

where Λ is the complete elliptic integral of the first kind with modulus

$$\lambda = 4e^{-2\beta n} \left(\sum_{k=0}^{\infty} e^{-4\beta n k(k+1)} \right)^2 \left(1 + 2 \sum_{k=1}^{\infty} e^{-4\beta n k^2} \right)^{-2}.$$

By Theorem 5 it can be shown that (17) also holds in the real case for the class $B\tilde{H}_{\infty,\beta}^{\mathbb{R}}$. Thus,

$$b_{2n}(B\tilde{H}_{\infty,\beta}^{\mathbb{R}}, L_2) = \sqrt{1 - e^{-2\beta}} e^{-\beta n} + O(e^{-5\beta n}).$$

REFERENCES

- [1] Osipenko K. Yu., Stessin M. I. *On n -widths of the Hardy class H^2 in the unit ball of \mathbb{C}^n* , Uspekhi Mat. Nauk **45** (1990), 193–194; English transl. in Russian Math. Surveys **45** (1990), 235–236.
- [2] Ismagilov R. S. *On n -dimensional diameters of compacts in a Hilbert space*, Funktsional. Anal. i Prilozhen. **2** (1968), 32–39; English transl. in Functional Anal. Appl. **2** (1968), 125–132.
- [3] Pinkus A. *n -Widths in Approximation Theory*, Springer-Verlag, Berlin, 1980.
- [4] Fisher S. D. and Stessin M. I. *On n -widths of classes of holomorphic functions with reproducing kernels*, Illinois J. Math. **38** (1994), 589–615.
- [5] Osipenko K. Yu. *On N -widths of holomorphic functions of several variables*, J. Approx. Theory **82** (1995), 135–155.
- [6] Parfenov O. G. *Gel'fand n -widths of some classes of analytic functions*, Mat. Sb. **185** (1994), 105–112; English transl. in Russian Acad. Sci. Sb. Math. **82** (1995), 223–229.
- [7] Parfenov O. G. *Bernstein n -widths of some classes of analytic functions*, Mat. Zametki **56** (1994), 123–131; English transl. in Math. Notes **56** (1994), 1075–1081.

- [8] Gohberg I. C. and Krein M. G. *Introduction to the Theory of Linear Non-selfadjoint Operators*, Nauka, Moscow, 1965; English transl in Transl. Math. Monographs, vol. **18**, Amer. Math. Soc., Providence, R. I., 1969.
- [9] Smolyak S. A. *On optimal restoration of functions and functionals of them*, Candidate Dissertation, Moscow State University, Moscow, 1965. (Russian)
- [10] Sofman L. B. *Diameters of octahedra*, Mat. Zametki **5** (1969), 429–436; English transl. in Math. Notes **5** (1969), 258–262.
- [11] Hutton C. V., Morrell J. S. and Retherford J. R. *Diagonal operators, approximation numbers, and Kolmogoroff diameters*, J. Approx. Theory **16**(1976), 48–80.
- [12] Novak E. *Optimal linear randomized methods for linear operators in Hilbert spaces*, J. Complexity **8** (1992), 22–36.
- [13] Galeev E. M. *Bernstein diameters of infinite dimensional ellipsoids*, in Complex Analysis. Differential and Integral Equations, Elista, (1990), pp. 14–20. (Russian)
- [14] Rudin W. *Function Theory in the Unit Ball of \mathbb{C}^n* , Springer-Verlag, New-York, 1980.
- [15] Parfenov O. G. *Gel'fand diameters of the unit ball of the Hardy class H^p in weight spaces*, Mat. Zametki **37** (1985), 171–175; English transl. in Math. Notes **37** (1985), 96–98.
- [16] Farkov Yu. A. *Widths of Hardy and Bergman classes in a ball in \mathbb{C}^n* , Uspekhi Mat. Nauk **45** (1990), 197–198; English transl. in Russian Math. Surveys **45** (1990), 229–231.
- [17] Osipenko K. Yu. *On the n -widths, optimal quadrature formulae and optimal recovery of functions which are analytic in a strip*, Izv. Ross. Akad. Nauk Ser. Mat. bf58 (1994), 55–79; English transl. in Russian Acad. Sci. Izv. Math. **45** (1995), 55–78.

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