ON EXACT VALUES OF n-WIDTHS FOR CLASSES DEFINED BY NONLINEAR CYCLIC VARIATION DIMINISHING OPERATORS

K. YU. OSIPENKO

Denote by h_∞^β (H_∞^β) the class of real-valued, 2π-periodic functions which are analytic in the strip $S_\beta := \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$ and satisfy the condition

$$|\operatorname{Re} f(z)| \le 1 \quad (|f(z)| \le 1), \quad z \in S_\beta.$$

Let $P_n(D)$, $D = \dfrac{d}{dx}$, be a differential polynomial with real coefficients. Denote by $h_\infty^{Q,\beta}$ $(H_\infty^{Q,\beta})$ the class of real-valued, 2π-periodic functions which are analytic in S_β and satisfy

$$Q(D)f \in h_\infty^\beta \quad (Q(D)f \in H_\infty^\beta), \quad z \in S_\beta.$$

Set

$$y(Q) := \max\{\operatorname{Im} z : Q(z) = 0\}.$$

For the Kolmogorov (d_n), linear (δ_n), and Gel'fand (d_n) n-widths we prove that for all $n > 2y(Q)$

$$\begin{aligned} d_{2n}(W, C(\mathbb{T})) &= \delta_{2n}(W, C(\mathbb{T})) = d^{2n}(W, C(\mathbb{T}) = d_{2n-1}(W, C(\mathbb{T})) \\ &= \delta_{2n-1}(W, C(\mathbb{T})) = d^{2n-1}(W, C(\mathbb{T})) = \|\Omega_Q * \varphi(K_\beta * h_n)\|_\infty, \end{aligned}$$

where

$$\Omega_Q(t) = \sum_{\substack{k=-\infty \\ Q(ik)\neq 0}}^{+\infty} \frac{e^{ikt}}{Q(ik)}, \quad K_\beta(t) = 1 + 2\sum_{k=1}^{\infty} \frac{\cos kt}{\operatorname{ch} k\beta},$$

$$h_n(t) = (-1)^{j+1}, \quad \frac{(j-1)\pi}{n} \le t < \frac{j\pi}{n}, \quad j = 1, \dots, 2n,$$

$$\varphi(t) = \begin{cases} 1, & W = h_\infty^{Q,\beta}, \\ \tan \dfrac{\pi}{4} t, & W = H_\infty^{Q,\beta}. \end{cases}$$

To obtain this result we introduce special classes of functions defined by cyclic variation diminishing operators which are not necessarily linear.

We also prove the analogous result for information n-widths

$$i_n(W, C(\mathbb{T})) := \inf_{l_1,\dots,l_n} \inf_{S\colon \mathbb{R}^n \to C(\mathbb{T})} \sup_{f \in W} \|f - S(l_1 f, \dots, l_n f)\|_\infty,$$

where $l_1, \dots, l_n$ are any continuous linear functionals. Any continuous linear functionals $l_1^*, \dots, l_n^*$ for which the infimum is attained are called optimal. We show that the first $2n-1$ Fourier coefficients are optimal for i_{2n} and i_{2n-1} in the case $W = h_\infty^{Q,\beta}$ or $W = H_\infty^{Q,\beta}$.