MINIMAL BLASHKE PRODUCTS AND OPTIMAL QUADRATURE FORMULAE IN H^∞

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We study the extremal problem

$$(1) \qquad \int_a^b s(x)|B(x,\overline{x})|^q\,dx \to \inf, \quad -1 < x_1 < \ldots < x_n < 1,$$

on the set of all Blashke products with fixed multiplicities ν_j of the zeros x_j,

$$B(x,\overline{x}) = \prod_{j=1}^{n} ((x-x_j)/(1-x_jx))^{\nu_j},$$

where $-1 \le a < b \le 1$, $1 \le q < \infty$, $\overline{x} = (x_1,\ldots,x_n)$, and $s(x) \not\equiv 0$ is nonnegative weight function, continuous in the interval (a,b).

Similar problems for $q=\infty$ connected with the optimal recovery of bounded analytic functions in the unit disk are considered in [1–5].

Theorem 1 ([6]). *For each fixed $1 \le q < \infty$ there exists a solution $\overline{x} = (x_1,\ldots,x_n)$ of the problem (1). Moreover, every such system of points satisfies $a < x_1 < \ldots < x_n < b$.*

The particular case $s(x) \equiv 1$ was studied before in [10].

If $-1 < a < b < 1$ then under the conformal mapping of the unit disk onto the disk of radius $k^{-1/2}$ the interval $[a,b]$ goes to $[-1,1]$ (k is uniquely defined by a and b) and the problem (1) transforms to the following one:

$$(2) \qquad \int_{-1}^{1} p(t)|Q(t,\overline{t},k)|^q\,dt \to \inf, \quad -1 < t_1 < \ldots < t_n < 1,$$

where $\overline{t} = (t_1,\ldots,t_n)$ and

$$Q(t,\overline{t},k) = \prod_{j=1}^{n} ((t-t_j)/(1-kt_jt))^{\nu_j}.$$

Note that in case $k=0$ the problem (2) is in fact the problem on polynomials of least deviation with fixed multiplicities of the zeros considered in [7].

The problem (2), and hence the problem (1), may has not a unique solution.

Theorem 2. *Let $n = 1$, $\nu q > 1$. The problem (2) has a unique solution for each weight function if and only if*

$$0 \le k \le \frac{\nu q - 1}{\nu q + 1}. \tag{3}$$

Proof. We set

$$\varphi(t_1) = \int_{-1}^{1} p(t) \left| \frac{t - t_1}{1 - kt_1 t} \right|^{\nu q} dt.$$

If $\varphi'(t_1) = 0$ then

$$\begin{aligned}
\varphi''(t_1) &= \varphi''(t_1) + \frac{2kt_1}{1 + kt_1^2} \varphi'(t_1) \\
&= \nu q \int_{-1}^{1} p(t) \left| \frac{t - t_1}{1 - kt_1 t} \right|^{\nu q} \frac{1 - kt^2}{(t - t_1)^2 (1 - kt_1 t)^2} \Bigg[\nu q (1 - kt^2) \\
&\qquad\qquad - \frac{1 - kt_1^2}{1 + kt_1^2} (1 + kt^2) \Bigg] dt \\
&\ge [\nu q - 1 + (\nu q + 1)k] \int_{-1}^{1} p(t) \left| \frac{t - t_1}{1 - kt_1 t} \right|^{\nu q} \frac{1 - kt^2}{(t - t_1)^2 (1 - kt_1 t)^2} \, dt.
\end{aligned}$$

Therefore, if inequalities (3) are satisfied, we have $\varphi''(t_1) > 0$ for all t_1 with $\varphi'(t_1) = 0$. This yields uniqueness of the optimal node t_1 for each weight function provided (3) is fulfilled.

Suppose now that $\dfrac{\nu q - 1}{\nu q + 1} < k \le 1$ and set $p(t) = |t|^\alpha$, $\alpha > 0$. The function $\varphi(t_1)$ is even in this case. So, if the optimal node t_1 is unique then $t_1 = 0$. It is not difficult to see that for sufficiently large α we obtain $\varphi(0) < 0$ and hence the function φ has not minimum at $t_1 = 0$. The theorem is proved. □

Let us denote

$$r = q \min_{1 \le j \le n} \nu_j, \quad N = \sum_{j=1}^{n} \nu_j, \quad \gamma_m(p, q) = \inf_{t_j} \int_{-1}^{1} p^*(t) \left| \prod_{j=1}^{n} (t - t_j) \right|^q dt,$$

where p^* is the normalized weight function

$$p^*(t) = p(t) \Big/ \int_{-1}^{1} p(t) \, dt.$$

Theorem 3 ([6]). *If $r > 1$ nd*

$$0 \le k \le (r - 1) / \left(9r - 7 + qN 2^{qN+1} \gamma_N^{-1}(p, q) \right) \tag{4}$$

the problem (2)has a unique solution.

Let us set

$$\begin{aligned} p_1(t) &= \left((1-t^2)(1-k^2t^2)\right)^{-1/2}, \\ p_2(t) &= p_1(t)\left((1-t^2)/(1-k^2t^2)\right)^{q/2}, \\ \overline{u}_1 &= \left\{\operatorname{sn}\left[\left(\frac{2j-1}{n}-1\right)K,k\right]\right\}_{j=1}^{n}, \\ \overline{u}_2 &= \left\{\operatorname{sn}\left[\left(\frac{2j}{n+1}-1\right)K,k\right]\right\}_{j=1}^{n}. \end{aligned}$$

Denote by K and Λ_m the complete elliptic integrals of the first kind with modulus k and λ_m, respectively.

Theorem 4 ([6])**.** *Suppose that $\nu_1 = \ldots = \nu_n = 1$, $1 < q < \infty$, and $k \in [0, k_i]$, where*

$$k_i = \frac{(q-1)\Gamma\left(\frac{q+1}{2}\right)}{(9q-7)\Gamma\left(\frac{q+1}{2}\right) + 2\sqrt{\pi}q\Gamma\left(\frac{q}{2}+1\right)n2^{q(2n-2+i)}}, \quad i = 1, 2.$$

Then

$$\inf_{t_j\in\mathbb{C}} \int_{-1}^{1} p_i(t)\left|\prod_{j=1}^{n}(t-t_j)/(1-k\bar{t}_jt)\right|^q dt = \int_{-1}^{1} p_i(t)\left|Q(t,\overline{u}_i,k)\right|^q dt = \frac{2d_{n+i-1}^q(k)K}{\Lambda_{n+i-1}}I_q(\lambda_{n+i-1}), \quad i = 1, 2, \tag{5}$$

$$d_m(k) = \prod_{j=1}^{[m/2]}\operatorname{sn}^2\left(\frac{2j-1}{m}K,k\right), \quad \lambda_m = k^m d_m^2(k),$$

$$I_q(\lambda) = \int_0^1 x^q(1-x^2)^{-1/2}(1-\lambda^2x^2)^{-1/2}\,dx.$$

The nodes $\overline{u}_1$ and $\overline{u}_2$ for which the infima (5) *are attained are unique.*

For $\nu_1 = \ldots = \nu_n = 1$, $k = 0$ the rational functions $Q(t, \overline{u}_i, k)$ coincide with the Chebyshev polynomials

$$\begin{aligned} Q(t,\overline{u}_1,k) &= 2^{1-n}\cos(n\arccos t), \\ Q(t,\overline{u}_2,k) &= 2^{-n}(1-t^2)^{-1/2}\sin((n+1)\arccos t). \end{aligned}$$

Corollary 1. *Suppose that* $\nu_1 = \ldots = \nu_n = 1$, $1 < q < \infty$. *If* $k \in [0, k_i)$ *then*

$$\inf_{z_j\in\mathbb{C}} \int_{-\sqrt{k}}^{\sqrt{k}} s_i(t) \left|\prod_{j=1}^{n}(z-z_j)/(1-\overline{z}_j z)\right|^q dz = \int_{-\sqrt{k}}^{\sqrt{k}} s_i(t)\, |B(z, Z_i)|^q\, dz \tag{6}$$

$$= \frac{2\lambda_{n+i-1}^{q/2}}{k^{(i-1)q/2}\Lambda_{n+i-1}} I_q(\lambda_{n+i-1}), \quad i = 1, 2,$$

where $Z_1 = k^{1/2}\overline{u}_1$, $Z_2 = k^{1/2}\overline{u}_2$, $s_1(z) = ((k-z^2)(1-kz^2))^{-1/2}$,

$$s_2(z) = s_1(z)\left((k-z^2)(1-kz^2)\right)^{q/2}.$$

The nodes Z_1 *and* Z_2 *for which the infima* (6) *are attained are unique.*

Note that the nodes $\overline{u}_1$ and $\overline{u}_2$ satisfy the necessary extremum conditions for all $k \in [0, 1)$. But the problem whether the equalities (5) (or (6)) hold for all $k \in [0, 1)$ remains still open.

Consider now the problem of optimal quadrature formula in the class $H^\infty(G)$ of functions analytic in the domain G and such that

$$\|f\|_\infty = \sup_{z\in G} |f(z)| < \infty.$$

We define the error of the optimal quadrature formula by

$$R(\mu, p, G) = \inf_{a\le x_1<\ldots<x_n\le b}\ \inf_{a_j}\ \sup_{\|f\|_\infty\le 1} \left|\int_a^b p(x)f(x)\,dx - \sum_{j=1}^{n}\sum_{m=0}^{\mu_j} a_{jm} f^{(m)}(x_j)\right|, \tag{7}$$

$\mu = (\mu_1, \ldots, \mu_n)$, $[a, b] \subset G$. The nodes for which the infimum (7) is attained are called *optimal nodes*. It is well known [8] (also [9]), that if $G = D := \{z \in \mathbb{C} : |z| < 1\}$ then

$$R(\mu, p, D) = \inf_{a\le x_1<\ldots<x_n\le b} \int_a^b p(x) \prod_{j=1}^{n} \left((x-x_j)/(1-x_j x)\right)^{\nu_j} dx,$$

$\nu_j = 2[(\mu_j + 1)/2]$. It follows from Theorem 3 that when $a = -b = k^{1/2}$ the optimal nodes are unique for sufficiently small k. Using Corollary 1 we find the optimal nodes for the weight functions $s_1(z)$ and $s_2(z)$.

Mapping conformally the unit disk D onto a domain G, so that the interval $[-\sqrt{k}, \sqrt{k}]$ goes to $[a, b] \subset G$, we obtain some results on the problem (7) for the corresponding weight functions. For example, let us map conformally the unit disk onto the ellipse E_c with foci at ±1 and sum of semi-axis c, so that the interval $[-\sqrt{k}, \sqrt{k}]$ transforms on $[-1, 1]$. If we denote by K' the complete elliptic integral of the first kind corresponding to the modulus $(1-k^2)^{1/2}$ we have

$$c=\exp\left(\frac{\pi K'}{4K}\right), \quad \sqrt{k}=\frac{2}{c}\cdot\left(\sum_{m=0}^{\infty}c^{-4m(m+1)}\right)\Big/\left(1+2\sum_{m=1}^{\infty}c^{-4m^2}\right).$$

Letting $c_i=\left(\dfrac{\pi K_i'}{4K_i}\right)$, $i=1,2$ (k_1, k_2 are defined as in Theorem 4),

$$p_1(x)=(1-x^2)^{-1/2}, \quad p_2(x)=p_1(x)\,\mathrm{sn}^q\left(\frac{2K}{\pi}\arccos x, k\right),$$

we get the following assertion.

Corollary 2. *Suppose that q is an even number and $q-1\le\mu_j\le q$, $j=1,\dots,n$. Then*

$$R(\mu,p_i,E_c)=\frac{\sqrt{\pi}\,\Gamma\left(\dfrac{q+1}{2}\right)2^q}{k^{(i-1)q/2}\Gamma\left(\dfrac{q}{2}+1\right)}c^{-(n+i-1)q}+O\left(c^{-n(q+4)}\right)$$

for all $c\ge c_i$, $i=1,2$, and

$$x_j=\begin{cases}\cos\dfrac{2j-1}{2n}\pi, & j=1,\dots,n, \text{ if } i=1,\\[2mm] \cos\dfrac{j}{n+1}\pi, & j=1,\dots,n, \text{ if } i=2,\end{cases}$$

are the unique optimal nodes.

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