

ON N -WIDTHS OF HOLOMORPHIC FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT. We consider the classes of holomorphic functions whose radial derivative of order r lies in the unit ball of the Hardy space $H_2(B_n)$ or the Bergman space $A_2(B_n)$. For these classes we calculate the linear and Gel'fand N -widths in $C(S_\rho)$, where S_ρ is the sphere in \mathbb{C}^n of radius $0 < \rho < 1$. Some results are obtained for analogous problems in polydiscs and for 2π -periodic functions of one variable holomorphic in a strip.

INTRODUCTION

Let A be a subset of a normed linear space X . The Kolmogorov N -width is defined by

$$d_N(A, X) := \inf_{X_N} \sup_{x \in A} \sup_{y \in X_N} \|x - y\|,$$

where X_N runs over all N -dimensional subspaces of X . Denote by $\mathcal{L}(H, X)$ the class of all continuous linear operators from H to X , where H and X are normed linear spaces. Let BH be the closed unit ball of H . For $T \in \mathcal{L}(H, X)$ set

$$d_N(T) := d_N(T(BH), X).$$

The linear N -width is given by

$$\lambda_N(A, X) := \inf_{P_N} \sup_{x \in A} \|x - P_N x\|,$$

where P_N runs over all bounded linear operators mapping X into X , whose range has dimension N or less. Assume that $0 \in A$. The Gel'fand N -width is defined by

$$d^N(A, X) := \inf_{X_N} \sup_{x \in A \cap X_N} \|x\|,$$

where the infimum is taken over all subspaces X^N of X of codimension N . Various properties of these N -widths (and others) may be found in [1].

Let B_n be the unit ball of \mathbb{C}^n

$$B_n := \left\{ z := (z_1, \dots, z_n) \in \mathbb{C}^n : |z|^2 := \sum_{k=1}^n |z_k|^2 < 1 \right\},$$

and S_ρ the sphere of radius ρ

$$S_\rho := \{z \in \mathbb{C}^n : |z| = \rho\}$$

(if $\rho = 1$ we write S). The Hardy space $H_p(B_n)$ is the set of holomorphic functions in B_n which satisfy

$$\|f\|_{H_p(B_n)} := \sup_{0 < r < 1} \left(\int_S |f(rz)|^p d\sigma(z) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_{H_\infty(B_n)} := \sup_{z \in B_n} |f(z)|,$$

where σ is the probability measure on the sphere S which is invariant with respect to the orthogonal group $O(2n)$. The Bergman space $A_p(B_n)$ is the set of holomorphic functions in B_n which satisfy the condition

$$\|f\|_{A_p(B_n)} := \left(\int_{B_n} |f(z)|^p d\nu(z) \right)^{1/p} < \infty,$$

where ν is the normalized Lebesgue measure in B_n ($A_\infty(B_n) = H_\infty(B_n)$).

Let $f(z)$ be a holomorphic function in B_n and

$$f(z) = \sum_{s=0}^{\infty} F_s(z)$$

be a homogeneous decomposition of f . The radial derivative of order r is defined by

$$\mathcal{R}^r f(z) := \sum_{s=r}^{\infty} \frac{s!}{(s-r)!} F_s(z)$$

(for $r = 1$ see [2, Chap. 6]). Let BX be the closed unit ball of a normed linear space X . We denote by $H\mathcal{R}_p^r(B_n)$ and $A\mathcal{R}_p^r(B_n)$ the classes of holomorphic functions in B_n for which $\mathcal{R}^r f$ lie in $BH_p(B_n)$ and $BA_p(B_n)$, respectively.

The exact values of $d_N(H\mathcal{R}_p^r(B_n), L_p(S_\rho))$ were obtained in [3]. When $n = 1$, $1 \leq q \leq p \leq \infty$ and E is a compact subset of B_1 , the values of $d_N(BH_p(B_1), L_q(E))$ were determined in [4] (for $E = S_\rho$ see also [5]).

The first result for the classes of holomorphic functions concerning the case when $p < q$ appeared in [6] where the values of $d^N(BH_2(B_n), C(S_\rho))$ and $\lambda_N(BH_2(B_n), C(S_\rho))$ were obtained (more precisely, for some subsequence of \mathbb{N}). The method of proof, as noted by V. M. Tikhomirov, was very similar to the one used in Ismagilov's Theorem [7] (see also [1]). In Section 1 we prove a theorem dual to the Ismagilov Theorem. Using this result, in Section 2 we obtain the values of the linear and Gel'fand N -widths of the classes $H\mathcal{R}_2^r(B_n)$ and $A\mathcal{R}_2^r(B_n)$ in $C(S_\rho)$.

Section 3 is devoted to analogous problems in polydiscs. Finally in Section 4 we calculate the N -widths of holomorphic functions in the annulus

$$\Delta_R := \{z \in \mathbb{C} : R^{-1} < |z| < R\}, \quad R > 1,$$

and 2π -periodic functions holomorphic in the strip

$$D_H := \{z \in \mathbb{C} : |\operatorname{Im} z| < H\}.$$

1. A THEOREM DUAL TO ISMAGILOV'S THEOREM

Let E be a compact set, μ a positive probability measure defined on E and $T \in \mathcal{L}(H, C(E))$. Denote by T_0 the operator T regarded as an operator from H into $L_2(E, \mu)$. Assume that

$$T_0' T_0 \phi_j = \lambda_j \phi_j, \quad j = 1, 2, \dots,$$

where $\lambda_1 \geq \lambda_2 \geq \dots > 0$, and that ϕ_1, ϕ_2, \dots is a complete orthonormal basis for the range of $T_0' T_0$ (a sufficient condition is that T_0 be a compact operator).

Theorem 1. *For T as above*

$$\begin{aligned} \sqrt{\sum_{j=N+1}^{\infty} \lambda_j} &\leq d^N(T(BH), C(E)) = \lambda_N(T(BH), C(E)) \\ &\leq \sup_{z \in E} \sqrt{\sum_{j=N+1}^{\infty} |(T\phi_j)(z)|^2}. \end{aligned}$$

Proof. Since $\operatorname{Ker} T_0' T_0 = \operatorname{Ker} T_0 = \operatorname{Ker} T$ we shall assume, without loss of generality, that ϕ_1, ϕ_2, \dots is a complete orthonormal basis for H . Set $\psi_j := T\phi_j$. Let us show that for all $z \in E$

$$(1) \quad \sum_{j=1}^{\infty} |\psi_j(z)|^2 \leq \|T\|^2 := \left(\sup_{\|h\|_H \leq 1} \|Th\|_{\infty} \right)^2$$

(we denote by $\|\cdot\|_{\infty}$ the norm in $C(E)$ and by $\|\cdot\|_H$ the norm in H). Let $z \in E$ and $m \in \mathbb{N}$. Then for $h := \sum_{j=1}^m \overline{\psi_j(z)} \phi_j \in H$ we have

$$\|Th\|_{\infty} = \sup_{z \in E} \left| \sum_{j=1}^m \overline{\psi_j(z)} \psi_j(s) \right| \geq \sum_{j=1}^m |\psi_j(z)|^2 = \|h\|_H^2.$$

Thus for $h \neq 0$

$$\|h\|_H \leq \frac{\|Th\|_{\infty}}{\|h\|_H} \leq \|T\|.$$

Consequently for all $z \in E$ and all $m \in \mathbb{N}$ the inequality

$$\sum_{j=1}^m |\psi_j(z)|^2 \leq \|T\|^2$$

holds. So (1) is proved.

Set

$$h_z := \sum_{j=1}^{\infty} \overline{\psi_j(z)} \psi_j.$$

It is easy to check that for all $x \in H$ and all $z \in E$

$$(Tx)(z) = (x, h_z)_H.$$

Denote by $\varphi : E \rightarrow H$ the mapping

$$\varphi(z) := h_z.$$

Then

$$\begin{aligned} \int_E (\varphi(z), \varphi(y))_H \overline{\psi_j(y)} d\mu(y) &= \int_E (Th_z)(y) \overline{\psi_j(y)} d\mu(y) \\ &= (T_0 h_z, T_0 \psi_j)_{L_2(E, \mu)} = (h_z, T_0' T_0 \psi_j)_H = \lambda_j \overline{\psi_j(z)}. \end{aligned}$$

Furthermore

$$(\psi_j, \psi_k)_{L_2(E, \mu)} = \lambda_j \delta_{jk}.$$

By the Ismagilov Theorem we obtain

$$\sqrt{\sum_{j=N+1}^{\infty} \lambda_j} \leq d_N(T') \leq \sup_{z \in E} \sqrt{\sum_{j=N+1}^{\infty} |(T\phi_j)(z)|^2}.$$

From duality

$$d_N(T') = d^N(T) := \inf_{X^N} \sup_{h \in BH \cap X^N} \|Th\|_{\infty},$$

where the infimum is taken over all subspaces X^N of H of codimension N . Since H is a Hilbert space

$$d_N(T') = d^N(T(BH), C(E)) = \lambda_N(T(BH), C(E)).$$

The theorem is proved. \square

Corollary 1. *Assume that the conditions of Theorem 1 hold and X_r is any r -dimensional subspace of $C(E)$ such that $X_r \perp T_0(H)$ in $L_2(E, \mu)$. Then*

$$\begin{aligned} \sqrt{\sum_{j=N+1}^{\infty} \lambda_j} &\leq d^{N+r}(T(BH) + X_r, C(E)) = \lambda_{N+r}(T(BH) + X_r, C(E)) \\ &\leq \sup_{z \in E} \sqrt{\sum_{j=N+1}^{\infty} |(T\phi_j)(z)|^2}. \end{aligned}$$

Proof. Let e_1, \dots, e_r be an orthonormal basis for X_r in $L_2(E, \mu)$. Denote by $H_{r,\varepsilon}$ the Hilbert space of elements $\{f, g\}$, $f \in H$, $g \in X_r$ with inner product

$$(\{f_1, g_1\}, \{f_2, g_2\})_{H_{r,\varepsilon}} := (f_1, f_2)_H + \varepsilon \sum_{j=1}^r c_j \bar{d}_j, \quad \varepsilon > 0,$$

where

$$g_1 = \sum_{j=1}^r c_j e_j, \quad g_2 = \sum_{j=1}^r d_j e_j.$$

Put $L(f, g) := Tf + g$. Denote by L_0 the operator L as an operator from $H_{r,\varepsilon}$ into $L_2(E, \mu)$. Then

$$L'_0 L_0 \{f, g\} = \{T'_0 T_0 f, \varepsilon^{-1} g\}.$$

Set

$$\varphi_j := \{0, \varepsilon^{-1/2} e_j\}, \quad j = 1, \dots, r, \quad \varphi_j := \{\phi_{j-r}, 0\}, \quad j = r+1, \dots.$$

The elements $\varphi_1, \varphi_2, \dots$ form a complete orthonormal basis for the range of $L'_0 L_0$ and

$$L'_0 L_0 \varphi_j = \varepsilon^{-1} \varphi_j, \quad j = 1, \dots, r, \quad L'_0 L_0 \varphi_j = \lambda_{j-r} \varphi_j, \quad j = r+1, \dots.$$

From Theorem 1 for $\varepsilon \leq \lambda_1^{-1}$ we have

$$d^{N+r}(L(BH_{r,\varepsilon}), C(E)) \geq \sqrt{\sum_{j=N+1}^{\infty} \lambda_j}.$$

Since $T(BH) + X_r \subset L(BH_{r,\varepsilon})$

$$d^{N+r}(T(BH) + X_r, C(E)) \geq d^{N+r}(L(BH_{r,\varepsilon}), C(E)) \geq \sqrt{\sum_{j=N+1}^{\infty} \lambda_j}.$$

The equality

$$d^{N+r}(T(BH) + X_r, C(E)) = \lambda_{N+r}(T(BH) + X_r, C(E))$$

follows from the fact that H is a Hilbert space (compare with Proposition 8.8 [1, p. 33]). It is easy to show that

$$\lambda_{N+r}(T(BH) + X_r, C(E)) \leq \lambda_N(T(BH), C(E)).$$

Now the upper bound follows directly from Theorem 1. The corollary is proved. \square

Let H be a Hilbert space of functions defined on some set Ω . A function $K(z, w)$ defined on $\Omega \times \Omega$ is called a reproducing kernel of H if for each $w \in \Omega$ and for all $f \in H$

$$f(w) = (f(\cdot), K(\cdot, w))_H.$$

It is easily seen that

$$K(z, w) = \overline{K(w, z)}.$$

Let $E \subset \Omega$ be a compact with positive probability measure μ . Suppose that $Tf := f|_E$ is a bounded linear operator from H to $C(E)$.

Theorem 2. *Let H and E be as above. Assume that $\varphi_1, \varphi_2, \dots$ is a complete orthonormal basis for H and X_r is any r -dimensional subspace of $C(E)$ such that $X_r \perp H$ in $L_2(E, \mu)$. If $\varphi_1, \varphi_2, \dots$ is an orthogonal system in $L_2(E, \mu)$ and $\lambda_j := \|\varphi_j\|_{L_2(E, \mu)}^2$ form a non-increasing sequence, then*

$$\begin{aligned} \sqrt{\sum_{j=N+1}^{\infty} \lambda_j} &\leq d^{N+r}(BH + X_r, C(E)) = \lambda_{N+r}(BH + X_r, C(E)) \\ &\leq \sup_{z \in E} \sqrt{\sum_{j=N+1}^{\infty} |\varphi_j(z)|^2}. \end{aligned}$$

Proof. Put $T_0f := f|_E$. Let us consider T_0 as an operator from H into $L_2(E, \mu)$. For all $g \in L_2(E, \mu)$ we have

$$\begin{aligned} (T_0'g)(w) &= ((T_0'g)(\cdot), K(\cdot, w))_H = (g(\cdot), T_0K(\cdot, w))_{L_2(E, \mu)} \\ &= \int_E g(z) \overline{K(z, w)} d\mu(z) = \int_E K(w, z) g(z) d\mu(z). \end{aligned}$$

Thus the eigenvalue-eigenfunction problem

$$T_0'T_0f = \lambda f$$

takes the form

$$(2) \quad \int_E K(w, z) g(z) d\mu(z) = \lambda f(w).$$

Since $\varphi_1, \varphi_2, \dots$ is a complete orthonormal basis for H the representation

$$K(z, w) = \sum_{j=1}^{\infty} \varphi_j(z) \overline{\varphi_j(w)}$$

holds. In view of the orthogonality of the system $\varphi_1, \varphi_2, \dots$ in $L_2(E, \mu)$ we have

$$\int_E K(w, z) \varphi_j(z) d\mu(z) = \lambda_j \varphi_j(w).$$

Thus λ_j is an eigenvalue and φ_j is an eigenfunction for Eq. (2). Now the theorem follows from Corollary 1. \square

2. N -WIDTHS OF $HR_2^r(B_n)$ AND $AR_2^r(B_n)$

Set $N_m := \sum_{s=0}^{m-1} \binom{n+s-1}{n-1}$. Note that $N_m = \dim \mathcal{P}_{m-1}^n$, where \mathcal{P}_m^n is the space of n -variable polynomials of degree m or less.

Theorem 3.

(i) For all $0 < \rho < 1$ and all $m \geq r \geq 0$

$$\begin{aligned} d^{N_m} (H\mathcal{R}_2^r(B_n), C(S_\rho)) &= \lambda_{N_m} (H\mathcal{R}_2^r(B_n), C(S_\rho)) \\ &= \rho^m \left(\frac{1}{(n-1)!} \sum_{s=0}^{\infty} \frac{((m-r+s)!)^2 (n+m-1+s)!}{((m+s)!)^3} \rho^{2s} \right)^{1/2}. \end{aligned}$$

(ii) For all $0 < \rho < 1$ and all $m \geq r \geq 1$

$$\begin{aligned} d^{N_m} (A\mathcal{R}_2^r(B_n), C(S_\rho)) &= \lambda_{N_m} (A\mathcal{R}_2^r(B_n), C(S_\rho)) \\ &= \rho^m \left(\frac{1}{n!} \sum_{s=0}^{\infty} \frac{((m-r+s)!)^2 (n+m+s)!}{((m+s)!)^3} \rho^{2s} \right)^{1/2}. \end{aligned}$$

(iii) For all

$$0 < \rho \leq \left(\frac{n}{n+m} \right)^{1/(2m)}$$

$$\begin{aligned} (3) \quad d^{N_m} (BA_2(B_n), C(S_\rho)) &= \lambda_{N_m} (BA_2(B_n), C(S_\rho)) \\ &= \rho^m \left(\frac{1}{n!} \sum_{s=0}^{\infty} \frac{(n+m+s)!}{(m+s)!} \rho^{2s} \right)^{1/2} \\ &= \frac{\rho^m}{(1-\rho^2)^{(n+1)/2}} \left(\binom{n+m}{n} \sum_{s=0}^n \frac{(-1)^s}{1+s/m} \binom{n}{s} \rho^{2s} \right)^{1/2}. \end{aligned}$$

Proof. For multiindex $\alpha := (\alpha_1, \dots, \alpha_n)$ and $z \in \mathbb{C}^n$ set

$$\begin{aligned} z^\alpha &:= z_1^{\alpha_1} \dots z_n^{\alpha_n}, \quad |\alpha| := \alpha_1 + \dots + \alpha_n, \quad \alpha! := \alpha_1! \dots \alpha_n!, \\ D_j &:= \partial/\partial z_j, \quad D^\alpha := D_1^{\alpha_1} \dots D_n^{\alpha_n}. \end{aligned}$$

Denote by \mathcal{H}_0 the space of holomorphic functions in B_n for which $(D^\alpha f)(0) = 0$, $|\alpha| = 0, \dots, r-1$, and $\mathcal{R}^r f \in H_2(B_n)$. It is known (see [2]) that functions from $H_2(B_n)$ have finite boundary values almost everywhere. Moreover $H_2(B_n)$ can be considered as a Hilbert space with inner product

$$(f, g)_{H_2(B_n)} := \int_S f(z) \overline{g(z)} d\sigma(z).$$

Thus \mathcal{H}_0 is a Hilbert space with inner product

$$(f, g) := (\mathcal{R}^r f, \mathcal{R}^r g)_{H_2(B_n)}.$$

Let $f, g \in \mathcal{H}_0$ and

$$f(z) = \sum_{|\alpha|=r}^{\infty} c_\alpha z^\alpha, \quad g(z) = \sum_{|\alpha|=r}^{\infty} d_\alpha z^\alpha.$$

Since monomials are orthogonal in $H_2(B_n)$ and

$$\|z^\alpha\|_{H_2(B_n)}^2 = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}$$

we have

$$(f, g) = \sum_{|\alpha|=r}^{\infty} \left(\frac{|\alpha|!}{(|\alpha|-r)!} \right)^2 \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!} c_\alpha \bar{d}_\alpha.$$

It is easily verified that

$$K(z, w) = \sum_{|\alpha|=r}^{\infty} \left(\frac{(|\alpha|-r)!}{|\alpha|!} \right)^2 \frac{(n-1+|\alpha|)!}{(n-1)!\alpha!} \bar{w}^\alpha z^\alpha$$

is the reproducing kernel of \mathcal{H}_0 .

Let us consider the space $L_2(S_\rho, \sigma_\rho)$, where σ_ρ is the probability measure on S_ρ , which is invariant with respect to the orthogonal group $O(2n)$. Set for $|\alpha| \geq r$

$$\varphi_\alpha(z) := \frac{(|\alpha|-r)!}{|\alpha|!} \left(\frac{(n-1+|\alpha|)!}{(n-1)!\alpha!} \right)^{1/2} z^\alpha.$$

The functions $\varphi_\alpha(z)$ form a complete orthonormal basis for \mathcal{H}_0 . Moreover these functions are orthogonal in $L_2(S_\rho, \sigma_\rho)$ and

$$\begin{aligned} \|\varphi_\alpha\|_{L_2(S_\rho, \sigma_\rho)}^2 &= \int_{S_\rho} |\varphi_\alpha(z)|^2 d\sigma_\rho(z) = \int_S |\varphi_\alpha(\rho\xi)|^2 d\sigma(\xi) \\ &= \left(\frac{(|\alpha|-r)!}{|\alpha|!} \right)^2 \rho^{2|\alpha|}. \end{aligned}$$

The number of different monomials z^α with $|\alpha| = s$ is equal to $\binom{n+s-1}{n-1}$. As $H\mathcal{R}_2^r(B_n) = B\mathcal{H}_0 + \mathcal{P}_r$, $\mathcal{H}_0 \perp \mathcal{P}_r$ in $L_2(S_\rho, \sigma_\rho)$, and $\dim \mathcal{P}_r = N_r$, we have by Theorem 2

$$\begin{aligned} &\left(\sum_{s=m}^{\infty} \left(\frac{(s-r)!}{s!} \right)^2 \binom{n+s-1}{n-1} \rho^{2s} \right)^{1/2} \\ &\leq d^{N_m}(H\mathcal{R}_2^r(B_n), C(S_\rho)) = \lambda_{N_m}(H\mathcal{R}_2^r(B_n), C(S_\rho)) \\ &\leq \sup_{z \in S_\rho} \left(\sum_{|\alpha| \geq m} \left(\frac{(|\alpha|-r)!}{|\alpha|!} \right)^2 \frac{(n-1+|\alpha|)!}{(n-1)!\alpha!} |z^{2\alpha}| \right)^{1/2}. \end{aligned}$$

Using the equation

$$\sum_{|\alpha|=s} \frac{s!}{\alpha!} |z^{2\alpha}| = |z|^{2s},$$

we obtain

$$\begin{aligned} d^{N_m}(H\mathcal{R}_2^r(B_n), C(S_\rho)) &= \lambda_{N_m}(H\mathcal{R}_2^r(B_n), C(S_\rho)) \\ &= \left(\sum_{s=m}^{\infty} \left(\frac{(s-r)!}{s!} \right)^2 \binom{n+s-1}{n-1} \rho^{2s} \right)^{1/2} \\ &= \rho^m \left(\frac{1}{(n-1)!} \sum_{s=m}^{\infty} \frac{((m-r+s)!)^2 (n+m-1+s)!}{((m+s)!)^3} \rho^{2s} \right)^{1/2}. \end{aligned}$$

To prove (ii) and (iii) we consider the space \mathcal{A}_0 of holomorphic functions in B_n for which $(D^\alpha f)(0) = 0$, $|\alpha| = 0, \dots, r-1$, and $\mathcal{R}^r f \in A_2(B_n)$. \mathcal{A}_0 is a Hilbert space with inner product

$$(f, g) := (\mathcal{R}^r f, \mathcal{R}^r g)_{A_2(B_n)} = \int_{B_n} \mathcal{R}^r f(z) \overline{\mathcal{R}^r g(z)} d\nu(z).$$

Analogous to the previous case, we can show that the functions

$$\psi_\alpha(z) := \sqrt{\frac{n+|\alpha|}{n}} \varphi_\alpha(z)$$

form a complete orthonormal basis for \mathcal{A}_0 in $L_2(S_\rho, \sigma_\rho)$. Furthermore

$$\|\psi_\alpha(z)\|_{L_2(S_\rho, \sigma_\rho)}^2 = \frac{n+|\alpha|}{n} \left(\frac{(|\alpha|-r)!}{|\alpha|!} \right)^2 \rho^{2|\alpha|} =: \lambda_{|\alpha|}.$$

Let $r \geq 1$ and $s \geq r$. Then

$$(n+s+1) \left(\frac{s+1-r}{s+1} \right)^2 \leq (n+s+1) \left(\frac{s}{s+1} \right)^2 \leq \frac{n+s+1}{s+1} s < n+s.$$

Thus

$$\begin{aligned} \lambda_{s+1} &= \frac{n+s+1}{n} \left(\frac{(s+1-r)!}{(s+1)!} \right)^2 \rho^{2(s+1)} \leq \frac{n+s}{n} \left(\frac{(s-r)!}{s!} \right)^2 \rho^{2s} \\ &= \lambda_s. \end{aligned}$$

If $r = 0$ (in this case $A\mathcal{R}_2^r(B_n) = BA_2(B_n)$), then $\{\lambda_j\}$ is not in general a non-increasing sequence. But if $((n+m)/n)\rho^{2m} \leq 1$ then for all $s \geq m$ and all $q < m$, $\lambda_q \geq \lambda_s$. Now (ii) and the first two equations of (3) follow from Theorem 2 in the same way as in the case of (i). Denote by

$$\Phi_n(m, \rho) := \sum_{s=0}^{\infty} \binom{n+m+s}{n} \rho^{2s}.$$

It easily verified that

$$\Phi_n(m, \rho) = \frac{1}{(1-\rho^2)^{n+1}} \binom{n+m}{n} \sum_{s=0}^n \frac{(-1)^s}{1+s/m} \binom{n}{s} \rho^{2s}.$$

So (iii) is proved. \square

Remark. The referee informed me that in the case $n = 1$ the exact values of N -widths of the Bergman classes were obtained in [8].

For $n = 1$ the class $H\mathcal{R}_2^r(B_1)$ coincides with the class BH_2^r , defined as the set of all holomorphic functions in B_1 for which $f^{(r)}(z) \in BH_2(B_1)$. The set of all holomorphic functions in B_1 , for which $f^{(r)}(z) \in BA_2(B_1)$ we denote by BA_2^r . If $r \geq 1$ the classes BA_2^r and $A\mathcal{R}_2^r(B_1)$ are different. Nevertheless the method of Theorem 2 can be applied. Thus we obtain the following result.

Theorem 4. *Let $0 < \rho < 1$. Then:*

(i) *for all $N \geq r \geq 0$*

$$\begin{aligned} d^N(BH_2^r, C(S_\rho)) &= \lambda_N(BH_2^r, C(S_\rho)) \\ &= \rho^N \left(\sum_{s=0}^{\infty} \left(\frac{(N-r+s)!}{(N+s)!} \right)^2 \rho^{2s} \right)^{1/2}; \end{aligned}$$

(ii) *for all $N \geq r \geq 1$*

$$\begin{aligned} d^N(BA_2^r, C(S_\rho)) &= \lambda_N(BA_2^r, C(S_\rho)) \\ &= \rho^N \left(\sum_{s=0}^{\infty} \left(\frac{(N-r+s)!}{(N+s)!} \right)^2 (N+s+1) \rho^{2s} \right)^{1/2}. \end{aligned}$$

3. THE N -WIDTHS FOR HARDY AND BERGMAN CLASSES IN POLYDISCS

Set

$$\begin{aligned} U^n &:= \{z \in \mathbb{C}^n : |z_1| < 1, \dots, |z_n| < 1\}, \\ T^n &:= \{z \in \mathbb{C}^n : |z_1| = 1, \dots, |z_n| = 1\}, \\ T_\rho^n &:= \{z \in \mathbb{C}^n : |z_1| = \rho_1, \dots, |z_n| = \rho_n\}, \end{aligned}$$

where $\rho = (\rho_1, \dots, \rho_n)$ and $0 \leq \rho_j < 1$, $j = 1, \dots, n$. Denote by $H_2(U^n)$ the set of all holomorphic functions in U^n for which

$$\|f\|_{H_2(U^n)} := \sup_{0 < r < 1} \left(\int_{T^n} |f(rz)|^2 d\mu(z) \right)^{1/2} < \infty,$$

where $\mu(z)$ is the normalized Lebesgue measure in T^n . We shall denote by $A_2(U^n)$ the set of all holomorphic functions in U^n for which

$$\|f\|_{A_2(U^n)} := \left(\int_{U^n} |f(z)|^2 d\omega(z) \right)^{1/2} < \infty,$$

where $\omega(z)$ is the normalized Lebesgue measure in U^n . The spaces $H_2(U^n)$ and $A_2(U^n)$ are Hilbert spaces with the reproducing kernels

$$K_H(z, w) := \begin{cases} (1 - z_1 \bar{w}_1)^{-1} \dots (1 - z_n \bar{w}_n)^{-1}, & H = H_2(U^n), \\ (1 - z_1 \bar{w}_1)^{-2} \dots (1 - z_n \bar{w}_n)^{-2}, & H = A_2(U^n) \end{cases}$$

(the details can be found in [9]).

Theorem 5. *Let $\rho = (\rho_1, \dots, \rho_n)$, $0 \leq \rho_j < 1$.*

(i) *Assume that $\alpha^{(1)}, \dots, \alpha^{(N)}$ are the N largest terms of the sequence $\{\rho^{2\alpha}\}$. Then*

$$(4) \quad d^N (BH_2(U^n), C(T_\rho^n)) = \lambda_N (BH_2(U^n), C(T_\rho^n)) \\ = \left((1 - \rho_1^2)^{-1} \dots (1 - \rho_n^2)^{-1} - \sum_{s=1}^N \rho^{2\alpha^{(s)}} \right)^{1/2}.$$

(ii) *Assume that $\alpha^{(1)}, \dots, \alpha^{(N)}$ are the N largest terms of the sequence $\{k_\alpha \rho^{2\alpha}\}$, where $k_\alpha := (\alpha_1 + 1) \dots (\alpha_n + 1)$. Then*

$$d^N (BA_2(U^n), C(T_\rho^n)) = \lambda_N (BA_2(U^n), C(T_\rho^n)) \\ = \left((1 - \rho_1^2)^{-2} \dots (1 - \rho_n^2)^{-2} - \sum_{s=1}^N k_{\alpha^{(s)}} \rho^{2\alpha^{(s)}} \right)^{1/2}.$$

Proof. Let us prove (i). The monomials z^α form a complete orthonormal basis in $H_2(U^n)$. They are also an orthogonal system in $L_2(T_\rho^n, \mu_\rho)$, where μ_ρ is the normalized Lebesgue measure in T_ρ^n . Moreover

$$\|z^\alpha\|_{L_2(T_\rho^n, \mu_\rho)}^2 = \rho^{2\alpha}$$

and for $z \in T_\rho^n$, $|z^\alpha| = \rho^{2\alpha}$. From Theorem 2 we have

$$d^N (BH_2(U^n), C(T_\rho^n)) = \lambda_N (BH_2(U^n), C(T_\rho^n)) = \left(\sum_{\alpha \notin \tau} \rho^{2\alpha} \right)^{1/2},$$

where $\tau := \{\alpha^{(1)}, \dots, \alpha^{(N)}\}$. Now (i) follows from the representation

$$(1 - \rho_1^2)^{-1} \dots (1 - \rho_n^2)^{-1} = \sum_{|\alpha| \geq 0} \rho^{2\alpha}.$$

Using the representation

$$(5) \quad (1 - \rho_1^2)^{-2} \dots (1 - \rho_n^2)^{-2} = \sum_{|\alpha| \geq 0} k_\alpha \rho^{2\alpha},$$

a similar argument proves (ii). □

We can obtain a more precise result in the case $\rho_1 = \dots = \rho_n$.

Theorem 6. *Let $\rho_1 = \dots = \rho_n = \rho$ and $0 < \rho < 1$. Then:*

(i) for $N_{m-1} < N \leq N_m$

$$\begin{aligned} d^N (BH_2(U^n), C(T_\rho^n)) &= \lambda_N (BH_2(U^n), C(T_\rho^n)) \\ &= \rho^{m-1} \left(N_m - N + \binom{n+m-1}{n-1} (1-\rho^2)^{-n} \right. \\ &\quad \left. \times \sum_{s=0}^{n-1} \frac{(-1)^s}{1+s/m} \binom{n-1}{s} \rho^{2(s+1)} \right)^{1/2}; \end{aligned}$$

(ii) for $n \geq 2$ and

$$(6) \quad 0 < \rho \leq m^{1/2}(m/n+1)^{-n/2}$$

$$\begin{aligned} d^N (BA_2(U^n), C(T_\rho^n)) &= \lambda_N (BA_2(U^n), C(T_\rho^n)) \\ &= \frac{\rho^m}{(1-\rho^2)^n} \left(\binom{2n+m-1}{2n-1} \sum_{s=0}^{2n-1} \frac{(-1)^s}{1+s/m} \binom{2n-1}{s} \rho^{2s} \right)^{1/2}. \end{aligned}$$

Proof. The sequence $\rho^{2|\alpha|}$ is a non-increasing sequence for $|\alpha| \rightarrow \infty$. The number of different multiindexes α with $|\alpha| = s$ is equal to $\binom{n+s-1}{n-1}$. By (4) we have for $N_{m-1} < N \leq N_m$

$$\begin{aligned} d^N (BH_2(U^n), C(T_\rho^n)) &= \lambda_N (BH_2(U^n), C(T_\rho^n)) \\ &= \left((1-\rho^2)^{-n} - \sum_{s=0}^{m-2} \binom{n+s-1}{n-1} \rho^{2s} - (N - N_{m-1}) \rho^{2(m-1)} \right)^{1/2} \\ &= \left((N_m - N) \rho^{2(m-1)} + \sum_{s=m}^{\infty} \binom{n+s-1}{n-1} \rho^{2s} \right)^{1/2}. \end{aligned}$$

Now (i) follows from equations

$$\begin{aligned} \sum_{s=m}^{\infty} \binom{n+s-1}{n-1} \rho^{2s} &= \rho^{2m} \sum_{s=0}^{\infty} \binom{n+m+s-1}{n-1} \rho^{2s} \\ &= \rho^{2m} \Phi_{n-1}(m, \rho) \\ &= \rho^{2m} (1-\rho^2)^{-n} \binom{n+m-1}{n-1} \sum_{s=0}^{n-1} \frac{(-1)^s}{1+s/m} \binom{n-1}{s} \rho^{2s}. \end{aligned}$$

To prove (ii) we will first prove that if the condition (6) holds, then for all $|\beta| \geq m$ and all $|\alpha| < m$

$$(7) \quad k_\beta \rho^{2|\beta|} \leq k_\alpha \rho^{2|\alpha|}.$$

In view of the monotone decreasing property of $y(x) := x(x/n+1)^{-n}$ for $x \geq 2$ and $n \geq 2$ we have

$$\rho^2 \leq \max\{y(1), y(2)\} \leq 1/2.$$

Consequently for all $s \geq 1$

$$(s+1)\rho^{2s} \leq s\rho^{2s-2}.$$

Thus for each $|\beta| \geq m$ choosing any $\beta_j \geq 1$ we will have

$$k_\beta \rho^{2|\beta|} \leq k_{\beta'} \rho^{2|\beta'|},$$

where $\beta' = (\beta_1, \dots, \beta_j - 1, \dots, \beta_n)$. Continuing this process we will find β^* with $|\beta^*| = m$ for which

$$k_\beta \rho^{2|\beta|} \leq k_{\beta^*} \rho^{2|\beta^*|} \leq (m/n + 1)^n \rho^{2m}.$$

On the other hand, if $|\alpha| < m$ then in view of the monotone decreasing of the sequence $\{s\rho^{2s-2}\}_1^\infty$ and by (6) we obtain

$$k_\alpha \rho^{2|\alpha|} \geq (|\alpha| + 1)\rho^{2|\alpha|} \geq m\rho^{2m-2} \geq (m/n + 1)^n \rho^{2m}.$$

So (7) is proved.

From Theorem 5 it follows that

$$\begin{aligned} d^{N_m}(BA_2(U^n), C(T_\rho^n)) &= \lambda_{N_m}(BA_2(U^n), C(T_\rho^n)) \\ &= \left((1 - \rho^2)^{-2n} - \sum_{|\alpha|=0}^{m-1} k_\alpha \rho^{2|\alpha|} \right)^{1/2} =: d. \end{aligned}$$

By (5)

$$\sum_{|\alpha|=s} k_\alpha = \binom{2n + s - 1}{2n - 1}.$$

Therefore

$$\begin{aligned} d^2 &= \sum_{s=m}^{\infty} \binom{2n + s - 1}{2n - 1} \rho^{2s} = \rho^{2m} \sum_{s=0}^{\infty} \binom{2n + s + m - 1}{2n - 1} \rho^{2s} \\ &= \rho^{2m} \Phi_{2n-1}(m, \rho) \\ &= \rho^{2m} (1 - \rho^2)^{-2n} \binom{2n + m - 1}{2n - 1} \sum_{s=0}^{2n-1} \frac{(-1)^s}{1 + s/m} \binom{2n - 1}{s} \rho^{2s}. \end{aligned}$$

□

4. N-WIDTHS OF HOLOMORPHIC FUNCTIONS OF ONE VARIABLE

Denote by H_γ the space of holomorphic functions in Δ_R

$$f(z) = \sum_{s=-\infty}^{+\infty} a_s z^s$$

which satisfy the condition

$$\sum_{s=-\infty}^{+\infty} \gamma_s |a_s|^2 < \infty,$$

where $\{\gamma_s\}$ is a sequence of non-negative numbers such that $\liminf_{s \rightarrow \mp\infty} \gamma_s^{1/|s|} \geq R^2$. Set $\Gamma := \{s : \gamma_s = 0\}$ and $r := \text{card } \Gamma$.

The space

$$H_\gamma^0 := \left\{ f(z) = \sum_{s=-\infty}^{+\infty} a_s z^s \in H_\gamma : a_j = 0, j \in \Gamma \right\}$$

is a Hilbert space with inner product

$$(f, g) = \sum_{s=-\infty}^{+\infty} \gamma_s a_s \bar{b}_s,$$

where

$$f(z) = \sum_{s=-\infty}^{+\infty} a_s z^s, \quad g(z) = \sum_{s=-\infty}^{+\infty} b_s z^s.$$

Moreover the space H_γ^0 has the reproducing kernel

$$K(z, w) := \sum_{s \notin \Gamma} \gamma_s^{-1} z^s \bar{w}^s.$$

Set $BH_\gamma := BH_\gamma^0 + \mathcal{P}_r$, where $\mathcal{P}_r := \{\sum_{s \in \Gamma} a_s z^s\}$. This convenient form for generalization of certain classes in the case of the unit disk was proposed by Fisher and Micchelli [10].

For $1 \leq \rho < R$ and $k \geq r$ set $\sigma_k(\rho) := \{s_1, \dots, s_{k-r}\} \cap \Gamma$, where $\{s_1, \dots, s_{k-r}\}$ are the $k-r$ largest terms of the sequence

$$\left\{ \gamma_s^{-1} \frac{\rho^{2s} + \rho^{-2s}}{2} \right\}_{s \notin \Gamma}.$$

Theorem 7. *Assume that for all $s \in \mathbb{N}$ $\gamma_s = \gamma_{-s}$.*

(i) *If $N \geq (r+1)/2$ and $0 \in \sigma_{2N-1}(\rho)$, then*

$$\begin{aligned} d^{2N-1}(BH_\gamma, C(\Delta_\rho)) &= \lambda_{2N-1}(BH_\gamma, C(\Delta_\rho)) \\ &= \left(\sum_{s \notin \sigma_{2N-1}(\rho)} \gamma_s^{-1} \frac{\rho^{2s} + \rho^{-2s}}{2} \right)^{1/2}. \end{aligned}$$

(ii) *If $N \geq r/2$ and $0 \notin \sigma_{2N}(\rho)$, then*

$$\begin{aligned} d^{2N}(BH_\gamma, C(\Delta_\rho)) &= \lambda_{2N}(BH_\gamma, C(\Delta_\rho)) \\ &= \left(\gamma_0 + \sum_{s \notin \sigma_{2N}(\rho)} \gamma_s^{-1} \frac{\rho^{2s} + \rho^{-2s}}{2} \right)^{1/2}. \end{aligned}$$

Proof. Let us prove (i). The functions

$$\varphi_s(z) := \gamma_s^{-1/2} z^s, \quad s \notin \Gamma,$$

form a complete orthonormal basis for H_γ^0 . Denote by $L_2(\partial\Delta_\rho)$ a Hilbert space of functions defined on the boundary of Δ_ρ with inner product

$$(f, g) := \frac{1}{4\pi} \int_0^{2\pi} \left[f(\rho e^{i\theta}) \overline{g(\rho e^{i\theta})} + f(\rho^{-1} e^{i\theta}) \overline{g(\rho^{-1} e^{i\theta})} \right] d\theta.$$

It is easily seen that φ_s form an orthogonal system in $L_2(\partial\Delta_\rho)$ and

$$\|\varphi_s\|_{L_2(\partial\Delta_\rho)}^2 = \gamma_s^{-1} \frac{\rho^{2s} + \rho^{-2s}}{2}.$$

From Theorem 2 follows

$$\begin{aligned} \left(\sum_{s \notin \sigma_{2N-1}(\rho)} \gamma_s^{-1} \frac{\rho^{2s} + \rho^{-2s}}{2} \right)^{1/2} &\leq d^{2N-1} (BH_\gamma, C(\Delta_\rho)) \\ &= \lambda_{2N-1} (BH_\gamma, C(\Delta_\rho)) \leq \sup_{z \in \partial\Delta_\rho} \left(\frac{1}{2} \sum_{s \notin \sigma_{2N-1}(\rho)} \gamma_s^{-1} (|z|^s + |z|^{-s}) \right)^{1/2} \\ &= \left(\sum_{s \notin \sigma_{2N-1}(\rho)} \gamma_s^{-1} \frac{\rho^{2s} + \rho^{-2s}}{2} \right)^{1/2}. \end{aligned}$$

Part (ii) is proved in a similar way. \square

For $\rho = 1$ the analogous application of Theorem 2 gives

Theorem 8. *For all $N \geq r$*

$$d^N (BH_\gamma, C(\Delta_1)) = \lambda_N (BH_\gamma, C(\Delta_1)) = \left(\sum_{s \notin \sigma_N(1)} \gamma_s^{-1} \right)^{1/2}.$$

Now we consider some examples of the spaces H_γ . Denote by $H_2(\Delta_R)$ the class of holomorphic functions in Δ_R for which

$$\|f\|_{H_2(\Delta_R)} := \sup_{1 < \rho < R} \left(\frac{1}{4\pi} \int_0^{2\pi} [|f(\rho e^{i\theta})|^2 + |f(\rho^{-1} e^{i\theta})|^2] d\theta \right)^{1/2} < \infty.$$

Let $A_2(\Delta_R)$ be the class of holomorphic functions in Δ_R for which

$$\|f\|_{A_2(\Delta_R)} := \left(\int_{\Delta_R} |f(z)|^2 d\eta(z) \right)^{1/2} < \infty,$$

where $\eta(z)$ is normalized Lebesgue measure in Δ_R . Let us consider the classes $BH_2^r(\Delta_R)$ and $BA_2^r(\Delta_R)$, which are the sets of holomorphic functions in Δ_R such that $f^{(r)}(z)$ lies in $BH_2(\Delta_R)$ and $BA_2(\Delta_R)$, respectively.

It can be easily shown that the class $BH_2^r(\Delta_R)$ coincides with BH_γ for

$$\gamma_s = (s(s-1) \dots (s-r+1))^2 \frac{R^{2(s-r)} + R^{-2(s-r)}}{2},$$

and $BA_2^r(\Delta_R)$ coincides with BH_γ , where for $r \geq 1$

$$\gamma_s = (s(s-1) \dots (s-r+2))^2 (s-r+1) \frac{R^{2(s-r+1)} + R^{-2(s-r+1)}}{R^2 - R^{-2}}$$

and for $r = 0$ (that is for $BA_2(\Delta_R)$)

$$\gamma_s = (s+1)^{-1} \frac{R^{2(s+1)} + R^{-2(s+1)}}{R^2 - R^{-2}}, \quad s \neq -1, \quad \gamma_{-1} = \frac{4 \log R}{R^2 - R^{-2}}.$$

We give some more examples of the classes BH_γ . Let $H_2(D_H)$ and $A_2(D_H)$ be the sets of all 2π -periodic holomorphic functions in D_H which satisfy the conditions

$$\|f\|_{H_2(D_H)} := \sup_{0 < h < H} \left(\frac{1}{4\pi} \int_0^{2\pi} [|f(x+ih)|^2 + |f(x-ih)|^2] dx \right)^{1/2} < \infty$$

and

$$\|f\|_{A_2(D_H)} := \left(\frac{1}{4\pi H} \int_0^{2\pi} \int_{-H}^H |f(x+iy)|^2 dx dy \right)^{1/2} < \infty,$$

respectively. Denote by $BH_2^r(D_H)$ and $BA_2^r(D_H)$ the sets of all 2π -periodic holomorphic functions in D_H for which $f^{(r)}(z)$ lie in $BH_2(D_H)$ and $BA_2(D_H)$, respectively.

To find the linear and Gel'fand N -widths of $BH_2^r(D_H)$ and $BA_2^r(D_H)$ in the space $C(D_h)$, $0 \leq h < H$, we use the map $z = (1/i) \log w$. Then the original problem reduces to the one for BH_γ with $R = e^{2\theta}$ and the space $C(\Delta_\rho)$ with $\rho = e^h$, where

$$\gamma_s = s^{2r} \cosh(2sH)$$

in the case of $BH_2^r(D_H)$ and

$$\gamma_s = \frac{1}{2H} s^{2r-1} \sinh(2sH)$$

in the case of $BA_2^r(D_H)$.

By Theorems 7 and 8 we obtain the following result.

Theorem 9. *Let $r \geq 0$.*

(i) *For all $0 \leq h < H$*

$$\begin{aligned} d^{2N-1}(BH_2^r(D_H), C(D_h)) &= \lambda_{2N-1}(BH_2^r(D_H), C(D_h)) \\ &= \left(2 \sum_{s=N}^{\infty} \frac{\cosh(2sh)}{s^{2r} \cosh(2sH)} \right)^{1/2}, \end{aligned}$$

$$\begin{aligned} d^{2N-1}(BA_2^r(D_H), C(D_h)) &= \lambda_{2N-1}(BA_2^r(D_H), C(D_h)) \\ &= 2H^{1/2} \left(\sum_{s=N}^{\infty} \frac{\cosh(2sh)}{s^{2r-1} \sinh(2sH)} \right)^{1/2}. \end{aligned}$$

(ii) For all $H > 0$

$$\begin{aligned} d^{2N}(BH_2^r(D_H), C[0, 2\pi]) &= \lambda_{2N}(BH_2^r(D_H), C[0, 2\pi]) \\ &= \left(\frac{1}{N^{2r} \cosh(2NH)} + 2 \sum_{s=N+1}^{\infty} \frac{1}{s^{2r} \cosh(2sH)} \right)^{1/2}, \end{aligned}$$

$$\begin{aligned} d^{2N}(BA_2^r(D_H), C[0, 2\pi]) &= \lambda_{2N}(BA_2^r(D_H), C[0, 2\pi]) \\ &= H^{1/2} \left(\frac{2}{N^{2r-1} \sinh(2NH)} + 4 \sum_{s=N+1}^{\infty} \frac{1}{s^{2r-1} \sinh(2sH)} \right)^{1/2}. \end{aligned}$$

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