

# Optimal Recovery of Operators and Multidimensional Carlson Type Inequalities

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## Abstract

The paper is concerned with recovery problems of linear multiplier-type operators from noisy information on weighted classes of functions. Optimal methods of recovery are constructed. The dual extremal problem is closely connected with Carlson type inequalities.

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## 1. General Setting

Let  $T$  be a nonempty set,  $\Sigma$  be the  $\sigma$ -algebra of subsets of  $T$ , and  $\mu$  be a nonnegative  $\sigma$ -additive measure on  $\Sigma$ . We denote by  $L_p(T, \Sigma, \mu)$  (or simply  $L_p(T, \mu)$ ) the set of all  $\Sigma$ -measurable functions with values in  $\mathbb{R}$  or in  $\mathbb{C}$  for which

$$\|x(\cdot)\|_{L_p(T, \mu)} = \left( \int_T |x(t)|^p d\mu(t) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$
$$\|x(\cdot)\|_{L_\infty(T, \mu)} = \operatorname{ess\,sup}_{t \in T} |x(t)| < \infty, \quad p = \infty.$$

Put

$$\mathcal{W} = \{x(\cdot) \in L_p(T, \mu) : \|\varphi(\cdot)x(\cdot)\|_{L_r(T, \mu)} < \infty\},$$
$$W = \{x(\cdot) \in \mathcal{W} : \|\varphi(\cdot)x(\cdot)\|_{L_r(T, \mu)} \leq 1\},$$

where  $1 \leq p, r \leq \infty$ , and  $\varphi(\cdot)$  is a measurable function on  $T$ . Consider the problem of recovery of operator  $\Lambda: \mathcal{W} \rightarrow L_q(T, \mu)$ ,  $1 \leq q \leq \infty$ , defined by equality  $\Lambda x(\cdot) = \psi(\cdot)x(\cdot)$ , where  $\psi(\cdot)$  is a measurable function on  $T$ , on the

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class  $W$  by the information about functions  $x(\cdot) \in W$  given inaccurately. More precisely, we assume that for any function  $x(\cdot) \in W$  we know  $y(\cdot) \in L_p(T_0, \mu)$ , where  $T_0$  is not empty  $\mu$ -measurable subset of  $T$ , such that  $\|x(\cdot) - y(\cdot)\|_{L_p(T_0, \mu)} \leq \delta$ ,  $\delta \geq 0$ . We want to approximate the value  $\Lambda x(\cdot)$  knowing  $y(\cdot)$ .

As recovery methods we consider all possible mappings

$$m: L_p(T_0, \mu) \rightarrow L_q(T, \mu).$$

The error of a method  $m$  is defined as

$$e(p, q, r, m) = \sup_{\substack{x(\cdot) \in W, y(\cdot) \in L_p(T_0, \mu) \\ \|x(\cdot) - y(\cdot)\|_{L_p(T_0, \mu)} \leq \delta}} \|\Lambda x(\cdot) - m(y)(\cdot)\|_{L_q(T, \mu)}.$$

The quantity

$$E(p, q, r) = \inf_{m: L_p(T_0, \mu) \rightarrow L_q(T, \mu)} e(p, q, r, m) \quad (1)$$

is known as the optimal recovery error, and a method on which this infimum is attained is called optimal.

Various settings of optimal recovery theory and examples of such problems may be found in [11], [12], [18], [17], [15], [13]. Much of them are devoted to optimal recovery of linear functionals. There are not so many results about optimal recovery of linear operators when non-Euclidean metrics is used ([12, Theorem 12 on p. 45], [10], [14]). In [14] we considered problem (1) when any two of  $p$ ,  $q$ , and  $r$  coincide. Here we analyze the case when all metrics can be different and  $1 \leq q < p, r < \infty$ . We construct optimal method of recovery, find its error, and apply this result to obtain exact constants in Carlson type inequalities. The case  $p = \infty$  and/or  $r = \infty$  requires a slightly different approach. Some particular results of such kind may be found in [7] ( $T = \mathbb{Z}$ ) and [8] ( $T = \mathbb{R}$ ).

## 2. Main results

Let  $\chi_0(\cdot)$  be the characteristic function of the set  $T_0$ :

$$\chi_0(t) = \begin{cases} 1, & t \in T_0, \\ 0, & t \notin T_0. \end{cases}$$

**Theorem 1.** *Let  $1 \leq q < p, r < \infty$ ,  $\lambda_1, \lambda_2 \geq 0$ ,  $\lambda_1 + \lambda_2 > 0$ ,  $\varphi(t) \neq 0$  for almost all  $t \in T \setminus T_0$ ,  $\hat{x}(t) = \hat{x}(t, \lambda_1, \lambda_2) \geq 0$  be a solution of equation*

$$-q|\psi(t)|^q + p\lambda_1 x^{p-q}(t)\chi_0(t) + r\lambda_2 |\varphi(t)|^r x^{r-q}(t) = 0, \quad (2)$$

and  $\lambda_1, \lambda_2$  such that

$$\begin{aligned} \int_{T_0} \hat{x}^p(t) d\mu(t) &\leq \delta^p, & \int_T |\varphi(t)|^r \hat{x}^r(t) d\mu(t) &\leq 1, \\ \lambda_1 \left( \int_{T_0} \hat{x}^p(t) d\mu(t) - \delta^p \right) &= 0, & \lambda_2 \left( \int_T |\varphi(t)|^r \hat{x}^r(t) d\mu(t) - 1 \right) &= 0, \end{aligned} \quad (3)$$

and  $\lambda_2 > 0$ , if  $T \setminus T_0 \neq \emptyset$ . Then

$$E(p, q, r) = \left( \frac{p\lambda_1\delta^p + r\lambda_2}{q} \right)^{1/q},$$

and the method

$$\widehat{m}(y)(t) = \begin{cases} q^{-1}p\lambda_1\widehat{x}^{p-q}(t)|\psi(t)|^{-q}\psi(t)y(t), & t \in T_0, \psi(t) \neq 0, \\ 0, & \text{otherwise,} \end{cases} \quad (4)$$

is optimal recovery method.

To prove this theorem we need some preliminary results.

**Lemma 1.**

$$E(p, q, r) \geq \sup_{\substack{x(\cdot) \in W \\ \|x(\cdot)\|_{L_p(T_0, \mu)} \leq \delta}} \|\Lambda x(\cdot)\|_{L_q(T, \mu)}. \quad (5)$$

The lower bound of type (5) is the well-known result which is usually applied to obtain the error of optimal recovery. In more or less general forms it was proved in many papers (see, for example, [14]).

The extremal problem which arises on the right-hand side of (5), known as the dual problem, may be written as

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T, \mu)} \rightarrow \max, \quad \|x(\cdot)\|_{L_p(T_0, \mu)} \leq \delta, \quad \|\varphi(\cdot)x(\cdot)\|_{L_r(T, \mu)} \leq 1. \quad (6)$$

For  $T_0 = T \subset \mathbb{R}^n$  and  $q = 1$  problem (6) was examined in [2] in connection with Stechkin's problem.

We give a straightforward result (resembling the sufficient conditions in the Kuhn-Tucker theorem), which we will require in solving dual problems similar to (6).

Let  $f_j: A \rightarrow \mathbb{R}$ ,  $j = 0, 1, \dots, n$ , be functions defined on some set  $A$ . Consider the extremal problem

$$f_0(x) \rightarrow \max, \quad f_j(x) \leq 0, \quad j = 1, \dots, n, \quad x \in A, \quad (7)$$

and write down its Lagrange function

$$\mathcal{L}(x, \lambda) = -f_0(x) + \sum_{j=1}^n \lambda_j f_j(x), \quad \lambda = (\lambda_1, \dots, \lambda_n).$$

**Lemma 2** ([14]). *Assume that there exist  $\widehat{\lambda}_j \geq 0$ ,  $j = 1, \dots, n$ , and an element  $\widehat{x} \in A$ , admissible for problem (7), such that*

$$(a) \quad \min_{x \in A} \mathcal{L}(x, \widehat{\lambda}) = \mathcal{L}(\widehat{x}, \widehat{\lambda}), \quad \widehat{\lambda} = (\widehat{\lambda}_1, \dots, \widehat{\lambda}_n),$$

$$(b) \quad \sum_{j=1}^n \widehat{\lambda}_j f_j(\widehat{x}) = 0.$$

Then  $\widehat{x}$  is an extremal element for problem (7).

Put

$$F(u, v, \alpha) = -((1 - \alpha)u + \alpha v)^q + av^p + bu^r, \quad u, v \geq 0, \quad \alpha \in [0, 1],$$

where  $a, b \geq 0$ , and  $1 \leq p, q, r < \infty$ .

**Lemma 3.** *For all  $a, b \geq 0$ ,  $a + b > 0$ , and all  $1 \leq q < p, r < \infty$ , there exists the unique solution  $\hat{u} > 0$  of the equation*

$$-q + pau^{p-q} + rbu^{r-q} = 0. \quad (8)$$

Moreover, for all  $u, v \geq 0$  and  $\alpha = q^{-1}pa\hat{u}^{p-q} = 1 - q^{-1}rb\hat{u}^{r-q}$

$$F(\hat{u}, \hat{u}, \alpha) \leq F(u, v, \alpha). \quad (9)$$

In particular, for all  $u \geq 0$

$$-\hat{u}^q + a\hat{u}^p + b\hat{u}^r \leq -u^q + au^p + bu^r.$$

*Proof.* The existence of the unique solution of (8) follows from the fact that the continuous function  $f(u) = pau^{p-q} + rbu^{r-q}$  increases monotonically from 0 to  $+\infty$ .

Let us prove (9). The cases  $a = 0$  or  $b = 0$  are easily obtained by finding the minimum of  $F(u, v, 0) = -u^q + bu^r$  if  $a = 0$  or  $F(u, v, 1) = -v^q + av^p$  if  $b = 0$ . Assume that  $a, b > 0$ . Then  $\alpha \in (0, 1)$ . Let

$$C > \max\{a^{-\frac{1}{p-q}}, b^{-\frac{1}{r-q}}\}.$$

Then for  $u \geq C$  and  $v \leq u$  we have

$$F(u, v, \alpha) \geq -u^q + bu^r = u^q(-1 + bu^{r-q}) > 0. \quad (10)$$

If  $v \geq C$  and  $v \geq u$ , then

$$F(u, v, \alpha) \geq -v^q + av^p = v^q(-1 + av^{p-q}) > 0. \quad (11)$$

Since  $F(0, 0, \alpha) = 0$  we obtain that

$$\inf_{(u,v) \in \mathbb{R}_+^2} F(u, v, \alpha) = \inf_{\substack{0 \leq u \leq C \\ 0 \leq v \leq C}} F(u, v, \alpha).$$

It follows from the Weierstrass extreme value theorem that there exist  $0 \leq u_0 \leq C$  and  $0 \leq v_0 \leq C$  such that

$$\inf_{(u,v) \in \mathbb{R}_+^2} F(u, v, \alpha) = F(u_0, v_0, \alpha).$$

In view of (10) and (11)  $u_0 < C$  and  $v_0 < C$ . We have

$$\begin{aligned} F_u(u, v, \alpha) &= -q((1 - \alpha)u + \alpha v)^{q-1}(1 - \alpha) + rbu^{r-1} \\ &= rb(-((1 - \alpha)u + \alpha v)^{q-1}\hat{u}^{r-q} + u^{r-1}). \end{aligned}$$

Thus, for any  $v_0 \geq 0$  and sufficiently small  $u > 0$   $F_u(u, v_0, \alpha) < 0$ . Consequently,

$$F(u, v_0, \alpha) < F(0, v_0, \alpha)$$

for sufficiently small  $u$ . It means that  $0 < u_0 < C$ . The similar arguments show that  $0 < v_0 < C$ . Hence

$$F_u(u_0, v_0, \alpha) = F_v(u_0, v_0, \alpha) = 0.$$

Since

$$\begin{aligned} F_v(u, v, \alpha) &= -q((1 - \alpha)u + \alpha v)^{q-1} \alpha + pav^{p-1} \\ &= pa(-((1 - \alpha)u + \alpha v)^{q-1} \widehat{u}^{p-q} + v^{p-1}) \end{aligned}$$

we have

$$-((1 - \alpha)u_0 + \alpha v_0)^{q-1} \widehat{u}^{r-q} + u_0^{r-1} = 0, \quad (12)$$

$$-((1 - \alpha)u_0 + \alpha v_0)^{q-1} \widehat{u}^{p-q} + v_0^{p-1} = 0. \quad (13)$$

Consequently,

$$\frac{u_0^{r-1}}{v_0^{p-1}} = \widehat{u}^{r-p}.$$

Suppose that  $p \leq r$ . Substituting

$$u_0 = \widehat{u}^{\frac{r-p}{r-1}} v_0^{\frac{p-1}{r-1}} \quad (14)$$

into (13), we obtain the equality

$$(\alpha v_0 + (1 - \alpha) \widehat{u}^{\frac{r-p}{r-1}} v_0^{\frac{p-1}{r-1}})^{q-1} \widehat{u}^{p-q} = v_0^{p-1}.$$

This equality may be rewritten in the form

$$(\alpha + (1 - \alpha)t^{\frac{p-r}{r-1}})^{q-1} = t^{p-q}, \quad (15)$$

where  $t = v_0 \widehat{u}^{-1}$ . It is easily seen that (15) has the unique solution  $t = 1$ . Consequently,  $v_0 = \widehat{u}$  and it follows by (14) that  $u_0 = \widehat{u}$ .

If  $p > r$ , then we substitute

$$v_0 = \widehat{u}^{\frac{p-r}{p-1}} u_0^{\frac{r-1}{p-1}}$$

into (12). Similar to the previous case we obtain the equality which may be written in the form

$$(\alpha s^{\frac{r-p}{p-1}} + 1 - \alpha)^{q-1} = s^{r-q}, \quad (16)$$

where  $s = u_0 \widehat{u}^{-1}$ . The unique solution of (16) is  $s = 1$ . Thus, for the case when  $p > r$  we have the same solution of (12), (13)  $u_0 = v_0 = \widehat{u}$ . Hence, for all  $u, v \geq 0$

$$F(u, v, \alpha) \geq \inf_{(u,v) \in \mathbb{R}_+^2} F(u, v, \alpha) = F(\widehat{u}, \widehat{u}, \alpha).$$

□

*Proof of Theorem 1.*

1. Lower estimate. The extremal problem (6) (for convenience, we raise the quantity to be maximized to the  $q$ -th power) is as follows:

$$\int_T |\psi(t)x(t)|^q d\mu(t) \rightarrow \max, \quad \int_{T_0} |x(t)|^p d\mu(t) \leq \delta^p, \\ \int_T |\varphi(t)x(t)|^r d\mu(t) \leq 1. \quad (17)$$

The Lagrange function for this problem reads as

$$\mathcal{L}(x(\cdot), \lambda_1, \lambda_2) = \int_T L(t, x(t), \lambda_1, \lambda_2) d\mu(t),$$

where

$$L(t, x, \lambda_1, \lambda_2) = -|\psi(t)x|^q + \lambda_1|x|^p\chi_{T_0}(t) + \lambda_2|\varphi(t)x|^r.$$

If  $t \in T$  such that  $\psi(t) = 0$ , then evidently  $\hat{x}(t) = 0$  and for those  $t$  for all  $x(\cdot) \in \mathcal{W}$

$$L(t, 0, \lambda_1, \lambda_2) \leq L(t, x(t), \lambda_1, \lambda_2).$$

Using this fact and Lemma 3, we obtain that there is the unique solution  $\hat{x}(\cdot)$  of (2) and, moreover, for almost all  $t \in T$  and all  $x(\cdot) \in \mathcal{W}$

$$L(t, \hat{x}(t), \lambda_1, \lambda_2) \leq L(t, x(t), \lambda_1, \lambda_2).$$

Consequently,

$$\mathcal{L}(\hat{x}(\cdot), \lambda_1, \lambda_2) \leq \mathcal{L}(x(\cdot), \lambda_1, \lambda_2).$$

Taking into account (3) we obtain by Lemma 2 that  $\hat{x}(\cdot)$  is the extremal function in (17). It follows by (5) that

$$E(p, q, r) \geq \left( \int_T |\psi(t)|^q \hat{x}^q(t) d\mu(t) \right)^{1/q}.$$

From (2) we have

$$|\psi(t)|^q \hat{x}^q(t) = q^{-1}p\lambda_1 \hat{x}^p(t)\chi_{T_0}(t) + q^{-1}r\lambda_2|\varphi(t)|^r \hat{x}^r(t).$$

Integrating this equality over the set  $T$ , we obtain

$$\int_T |\psi(t)|^q \hat{x}^q(t) d\mu(t) = \frac{p\lambda_1\delta^p + r\lambda_2}{q}. \quad (18)$$

Thus,

$$E(p, q, r) \geq \left( \frac{p\lambda_1\delta^p + r\lambda_2}{q} \right)^{1/q}.$$

2. Upper estimate. To estimate the error of method (4) we need to find the value of the extremal problem:

$$\int_{T_0} |\psi(t)x(t) - \psi(t)\alpha(t)y(t)|^q d\mu(t) + \int_{T \setminus T_0} |\psi(t)x(t)|^q d\mu(t) \rightarrow \max,$$

$$\int_{T_0} |x(t) - y(t)|^p d\mu(t) \leq \delta^p, \quad \int_T |\varphi(t)x(t)|^r d\mu(t) \leq 1, \quad (19)$$

where

$$\alpha(t) = \begin{cases} q^{-1} p \lambda_1 \widehat{x}^{p-q}(t) |\psi(t)|^{-q}, & t \in T_0, \psi(t) \neq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (20)$$

Taking

$$z(t) = \begin{cases} x(t) - y(t), & t \in T_0, \\ 0, & t \in T \setminus T_0, \end{cases}$$

we rewrite (19) as follows:

$$\int_T |\psi(t)|^q |(1 - \alpha(t))x(t) + \alpha(t)z(t)|^q d\mu(t) \rightarrow \max,$$

$$\int_{T_0} |z(t)|^p d\mu(t) \leq \delta^p, \quad \int_T |\varphi(t)x(t)|^r d\mu(t) \leq 1.$$

The value of this problem does not exceed the value of the problem

$$\int_T |\psi(t)|^q ((1 - \alpha(t))u(t) + \alpha(t)v(t))^q d\mu(t) \rightarrow \max,$$

$$\int_{T_0} v^p(t) d\mu(t) \leq \delta^p, \quad \int_T |\varphi(t)|^r u^r(t) d\mu(t) \leq 1,$$

$$u(t) \geq 0, \quad v(t) \geq 0 \quad \text{for almost all } t \in T. \quad (21)$$

The Lagrange function for this problem is

$$\mathcal{L}_1(u(\cdot), v(\cdot), \mu_1, \mu_2) = \int_T L_1(t, u(t), v(t), \mu_1, \mu_2) d\mu(t),$$

where

$$L_1(t, u, v, \mu_1, \mu_2) = -|\psi(t)|^q ((1 - \alpha(t))u + \alpha(t)v)^q + \mu_1 v^p \chi_0(t) + \mu_2 |\varphi(t)|^r u^r.$$

By Lemma 3 we have

$$L_1(t, \widehat{x}(t), \widehat{x}(t), \lambda_1, \lambda_2) \leq L_1(t, u(t), v(t), \lambda_1, \lambda_2).$$

Thus,

$$\mathcal{L}_1(\widehat{x}(\cdot), \widehat{x}(\cdot), \lambda_1, \lambda_2) \leq \mathcal{L}_1(u(\cdot), v(\cdot), \lambda_1, \lambda_2).$$

It follows by Lemma 2 that functions  $u(t) = v(t) = \widehat{x}(t)$  are extremal in (21). Consequently,

$$e(p, q, r, \widehat{m}) \leq \left( \int_T |\psi(t)|^q \widehat{x}^q(t) d\mu(t) \right)^{1/q} = \left( \frac{p\lambda_1\delta^p + r\lambda_2}{q} \right)^{1/q} \leq E(p, q, r).$$

It means that the method (4) is optimal and the optimal recovery error is as stated.  $\square$

Note that if conditions of Theorem 1 hold we proved the equality

$$E(p, q, r) = \sup_{\substack{\|x(\cdot)\|_{L_p(T_0, \mu)} \leq \delta \\ \|\varphi(\cdot)x(\cdot)\|_{L_r(T, \mu)} \leq 1}} \|\psi(\cdot)x(\cdot)\|_{L_q(T, \mu)}. \quad (22)$$

**Corollary 1.** *Let  $1 \leq q < p, r < \infty$ ,  $\varphi(t) \neq 0$  for almost all  $t \in T$ , and*

$$0 < \int_T \left| \frac{\psi(t)}{\varphi(t)} \right|^{\frac{qr}{r-q}} d\mu(t) < \infty, \quad \int_{T_0} \left( \frac{|\psi(t)|^q}{|\varphi(t)|^r} \right)^{\frac{p}{r-q}} d\mu(t) < \infty.$$

Then for all

$$\delta \geq \frac{\left( \int_{T_0} \left( \frac{|\psi(t)|^q}{|\varphi(t)|^r} \right)^{\frac{p}{r-q}} d\mu(t) \right)^{1/p}}{\left( \int_T \left| \frac{\psi(t)}{\varphi(t)} \right|^{\frac{qr}{r-q}} d\mu(t) \right)^{1/r}}$$

$$E(p, q, r) = \left( \int_T \left| \frac{\psi(t)}{\varphi(t)} \right|^{\frac{qr}{r-q}} d\mu(t) \right)^{\frac{r-q}{qr}},$$

and the method  $\widehat{m}(y)(t) = 0$  is optimal recovery method.

*Proof.* It suffices to check that  $\lambda_1 = 0$  and

$$\lambda_2 = \frac{q}{r} \left( \int_T \left| \frac{\psi(t)}{\varphi(t)} \right|^{\frac{qr}{r-q}} d\mu(t) \right)^{\frac{r-q}{r}}$$

satisfy the conditions of Theorem 1.  $\square$

**Corollary 2.** *Let  $1 \leq q < p, r < \infty$ ,  $T_0 = T$ , and*

$$0 < \int_T |\varphi(t)|^r \|\psi(t)\|^{\frac{qr}{p-q}} d\mu(t) < \infty, \quad \int_T |\psi(t)|^{\frac{qp}{p-q}} d\mu(t) < \infty.$$

Then for all

$$\delta \leq \frac{\left( \int_T |\psi(t)|^{\frac{qp}{p-q}} d\mu(t) \right)^{1/p}}{\left( \int_T |\varphi(t)|^r \|\psi(t)\|^{\frac{qr}{p-q}} d\mu(t) \right)^{1/r}}$$

$$E(p, q, r) = \delta \left( \int_T |\psi(t)|^{\frac{qp}{p-q}} d\mu(t) \right)^{\frac{p-q}{qp}},$$

and the method  $\widehat{m}(y)(t) = \psi(t)y(t)$  is optimal recovery method.

*Proof.* It suffices to check that

$$\lambda_1 = \frac{q}{p\delta^{p-q}} \left( \int_T |\psi(t)|^{\frac{qp}{p-q}} d\mu(t) \right)^{\frac{p-q}{p}}$$

and  $\lambda_2 = 0$  satisfy the conditions of Theorem 1.  $\square$

Note that assumption (3) need not be satisfied in all cases. For example, in the trivial case  $\delta = 0$ ,  $T_0 = T$ , and  $\psi(t) = 1$  there are no such  $\lambda_1$  and  $\lambda_2$  which satisfy (3).

Let us consider the problem of optimal recovery of the linear functional

$$Lx = \int_T \psi(t)x(t) d\mu(t)$$

on the class  $W$ , knowing  $y(\cdot) \in L_p(T_0, \mu)$ ,  $T_0 \subset T$ , such that  $\|x(\cdot) - y(\cdot)\|_{L_p(T_0, \mu)} \leq \delta$ ,  $\delta \geq 0$ . In this case as recovery methods we consider all possible mappings  $m: L_p(T_0, \mu) \rightarrow \mathbb{C}$  or  $\mathbb{R}$ . The error of a method  $m$  is defined as

$$e_1(p, r, m) = \sup_{\substack{x(\cdot) \in W, y(\cdot) \in L_p(T_0, \mu) \\ \|x(\cdot) - y(\cdot)\|_{L_p(T_0, \mu)} \leq \delta}} |Lx - m(y)|.$$

The quantity

$$E_1(p, r) = \inf_{m: L_p(T_0, \mu) \rightarrow \mathbb{C}(\mathbb{R})} e_1(p, r, m) \quad (23)$$

is optimal recovery error, and a method on which this infimum is attained is called optimal.

**Theorem 1'.** *Let  $1 < p, r < \infty$ ,  $\lambda_1, \lambda_2 \geq 0$ ,  $\lambda_1 + \lambda_2 > 0$ ,  $\varphi(t) \neq 0$  for almost all  $t \in T \setminus T_0$ ,  $\widehat{x}(t) = \widehat{x}(t, \lambda_1, \lambda_2) \geq 0$  be a solution of equation*

$$-|\psi(t)| + p\lambda_1 x^{p-1}(t)\chi_0(t) + r\lambda_2 |\varphi(t)|^r x^{r-1}(t) = 0,$$

and  $\lambda_1, \lambda_2$  such that conditions (3) are fulfilled, and  $\lambda_2 > 0$ , if  $T \setminus T_0 \neq \emptyset$ . Then

$$E_1(p, r) = p\lambda_1 \delta^p + r\lambda_2,$$

and the method

$$\widehat{m}(y) = p\lambda_1 \int_{T_0} \widehat{x}^{p-1}(t)\varepsilon(t)y(t) d\mu(t), \quad (24)$$

where

$$\varepsilon(t) = \begin{cases} \frac{\psi(t)}{|\psi(t)|}, & \psi(t) \neq 0, \\ 1, & \psi(t) = 0, \end{cases}$$

is optimal recovery method.

*Proof.* For the functional case it is known (see, for example, [6]) that

$$E_1(p, r) = \sup_{\substack{x(\cdot) \in W \\ \|x(\cdot)\|_{L_p(T_0, \mu)} \leq \delta}} \left| \int_T \psi(t)x(t) d\mu(t) \right|.$$

Put  $\tilde{x}(\cdot) = \overline{\varepsilon(\cdot)}\hat{x}(\cdot)$ . It follows by (3) that  $\tilde{x}(\cdot) \in W$  and  $\|\tilde{x}(\cdot)\|_{L_p(T_0, \mu)} \leq \delta$ . Taking into account (18), we obtain

$$E_1(p, r) \geq \left| \int_T \psi(t)\tilde{x}(t) d\mu(t) \right| = \int_T |\psi(t)|\tilde{x}(t) d\mu(t) = p\lambda_1\delta^p + r\lambda_2.$$

Now we estimate the error of method (24). We have

$$\begin{aligned} e_1(p, r, \hat{m}) &= \sup_{\substack{x(\cdot) \in W, y(\cdot) \in L_p(T_0, \mu) \\ \|x(\cdot) - y(\cdot)\|_{L_p(T_0, \mu)} \leq \delta}} \left| \int_T \psi(t)x(t) d\mu(t) - \hat{m}(y) \right| \\ &\leq \sup_{\substack{x(\cdot) \in W, z(\cdot) \in L_p(T_0, \mu) \\ \|z(\cdot)\|_{L_p(T_0, \mu)} \leq \delta}} \int_T |\psi(t)| |(1 - \alpha(t))x(t) + \alpha(t)z(t)| d\mu(t), \end{aligned}$$

where  $\alpha(\cdot)$  is defined by (20) for  $q = 1$ . It follows from the proof of Theorem 1 that

$$E_1(p, r) \leq e_1(p, r, \hat{m}) \leq \int_T |\psi(t)|\hat{x}(t) d\mu(t) = p\lambda_1\delta^p + r\lambda_2. \quad \square$$

One can easily obtain analogs of Corollaries 1 and 2 for problem (23).

### 3. The case of homogenous weight functions

Let  $T$  be a cone in a linear space,  $T_0 = T$ ,  $|\psi(\cdot)|$  and  $|\varphi(\cdot)|$  be homogenous functions of degrees  $\eta, \nu$ , respectively,  $\varphi(t) \neq 0$  and  $\psi(t) \neq 0$  for almost all  $t \in T$ , and  $\mu(\cdot)$  be a homogenous measure of degree  $d$ . We assume, again, that  $1 \leq p < q, r < \infty$ . For  $k \in [0, 1)$  the function  $k^{\frac{1}{p-q}}(1-k)^{-\frac{1}{r-q}}$  increases monotonically from 0 to  $+\infty$ . Consequently, for all  $z \in T$  such that  $\varphi(z) \neq 0$  and  $\psi(z) \neq 0$  (if  $p < r$ ), there exists  $k(z)$  for which

$$\frac{k^{\frac{1}{p-q}}(z)}{(1-k(z))^{\frac{1}{r-q}}} = \frac{|\psi(z)|^{\frac{q(p-r)}{(p-q)(r-q)}}}{|\varphi(z)|^{\frac{r}{r-q}}}. \quad (25)$$

Thus, the function  $k(z)$  is well defined for almost all  $z \in T$ .

**Theorem 2.** Let  $1 \leq q < p, r < \infty$ ,  $\varphi(t), \psi(t) \neq 0$  for almost all  $t \in T$ , and  $\nu + d(1/r - 1/p) \neq 0$ . Assume that

$$I_1 = \int_T |\psi(z)|^{\frac{qp}{p-q}} k^{\frac{p}{p-q}}(z) d\mu(z) < \infty,$$

$$I_2 = \int_T |\psi(z)|^{\frac{qr}{p-q}} |\varphi(z)|^r k^{\frac{r}{p-q}}(z) d\mu(z) < \infty.$$

Then

$$E(p, q, r) = \delta^\gamma I_1^{-\gamma/p} I_2^{-(1-\gamma)/r} (I_1 + I_2)^{1/q},$$

where

$$\gamma = \frac{\nu - \eta - d(1/q - 1/r)}{\nu + d(1/r - 1/p)}, \quad (26)$$

and the method

$$\widehat{m}(y)(t) = k(\xi t) \psi(t) y(t),$$

where

$$\xi = \left( \delta I_1^{-1/p} I_2^{1/r} \right)^{\frac{1}{\nu + d(1/r - 1/p)}}, \quad (27)$$

is optimal recovery method.

*Proof.* Put

$$\widehat{x}(t) = \left( \frac{q|\psi(t)|^q}{p\lambda_1} \right)^{\frac{1}{p-q}} k^{\frac{1}{p-q}}(\xi t),$$

where  $\lambda_1 > 0$  will be specified later. We show that  $\widehat{x}(\cdot)$  satisfies (2), where

$$\lambda_2 = r^{-1} q^{\frac{p-r}{p-q}} (p\lambda_1)^{\frac{r-q}{p-q}} \xi^{\nu r - \eta \frac{q(p-r)}{p-q}}. \quad (28)$$

We have

$$p\lambda_1 \widehat{x}^{p-q}(t) = q|\psi(t)|^q k(\xi t),$$

and further,

$$r\lambda_2 |\varphi(t)|^r \widehat{x}^{r-q}(t) = r\lambda_2 |\varphi(t)|^r \left( \frac{q|\psi(t)|^q}{p\lambda_1} \right)^{\frac{r-q}{p-q}} k^{\frac{r-q}{p-q}}(\xi t).$$

Since  $|\varphi(\cdot)|$  and  $|\psi(\cdot)|$  are homogenous it follows by (25) that

$$k^{\frac{r-q}{p-q}}(\xi t) = \frac{|\psi(\xi t)|^{\frac{q(p-r)}{p-q}}}{|\varphi(\xi t)|^r} (1 - k(\xi t)) = \xi^{\eta \frac{q(p-r)}{p-q} - \nu r} \frac{|\psi(t)|^{\frac{q(p-r)}{p-q}}}{|\varphi(t)|^r} (1 - k(\xi t)).$$

Thus,

$$\begin{aligned} r\lambda_2 |\varphi(t)|^r \widehat{x}^{r-q}(t) &= r\lambda_2 \left( \frac{q}{p\lambda_1} \right)^{\frac{r-q}{p-q}} \xi^{\eta \frac{q(p-r)}{p-q} - \nu r} |\psi(t)|^q (1 - k(\xi t)) \\ &= q|\psi(t)|^q (1 - k(\xi t)) = q|\psi(t)|^q - p\lambda_1 \widehat{x}^{p-q}(t). \end{aligned}$$

Now we show that for

$$\lambda_1 = \frac{q}{p} I_1^{\frac{p-q}{p}} \xi^{-\eta q - d \frac{p-q}{p}} \delta^{q-p} \quad (29)$$

the equalities

$$\int_T \widehat{x}^p(t) d\mu(t) = \delta^p, \quad \int_T |\varphi(t)|^r \widehat{x}^r(t) d\mu(t) = 1$$

hold. In view of the definition of  $\widehat{x}(\cdot)$  we need to check that

$$\begin{aligned} \int_T \left( \frac{q|\psi(t)|^q}{p\lambda_1} \right)^{\frac{p}{p-q}} k^{\frac{p}{p-q}}(\xi t) d\mu(t) &= \delta^p, \\ \int_T |\varphi(t)|^r \left( \frac{q|\psi(t)|^q}{p\lambda_1} \right)^{\frac{r}{p-q}} k^{\frac{r}{p-q}}(\xi t) d\mu(t) &= 1. \end{aligned}$$

Changing  $z = \xi t$  and taking into account that functions  $|\psi(\cdot)|$ ,  $|\varphi(\cdot)|$  with the measure  $\mu(\cdot)$  are homogenous, we obtain

$$\begin{aligned} \left( \frac{q}{p\lambda_1} \right)^{\frac{p}{p-q}} I_1 &= \delta^p \xi^{\frac{\eta q p}{p-q} + d}, \\ \left( \frac{q}{p\lambda_1} \right)^{\frac{r}{p-q}} I_2 &= \xi^{\frac{\eta q r}{p-q} + \nu r + d}. \end{aligned}$$

The validity of these equalities immediately follows from the definitions of  $\lambda_1$  and  $\xi$ .

It follows by Theorem 1, (29), (28), and (27) that

$$\begin{aligned} E^q(p, q, r) &= \frac{p\lambda_1 \delta^p + r\lambda_2}{q} = I_1^{\frac{p-q}{p}} \xi^{-\eta q - d \frac{p-q}{p}} \delta^q + \left( \frac{p\lambda_1}{q} \right)^{\frac{r-q}{p-q}} \xi^{\nu r - \eta \frac{q(p-r)}{p-q}} \\ &= \delta^{q\gamma} I_1^{-q\gamma/p} I_2^{-q(1-\gamma)/r} (I_1 + I_2). \end{aligned}$$

Moreover, the same theorem states that the method

$$\widehat{m}(y)(t) = q^{-1} p \lambda_1 \widehat{x}^{p-q}(t) |\psi(t)|^{-q} \psi(t) y(t) = k(\xi t) \psi(t) y(t)$$

is optimal. □

It follows by Theorem 2 and (22) that for all  $x(\cdot) \in \mathcal{W}$  such that  $\|\varphi(\cdot)x(\cdot)\|_{L_r(T, \mu)} \leq 1$  the exact inequality

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T, \mu)} \leq C \|x(\cdot)\|_{L_p(T, \mu)}^{\gamma} \quad (30)$$

holds, where

$$C = I_1^{-\gamma/p} I_2^{-(1-\gamma)/r} (I_1 + I_2)^{1/q}.$$

(Here and later the exactness means that  $C$  cannot be replaced by any other constant smaller than  $C$ ).

From (30) the following exact inequality can be easily obtained

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)} \leq C\|x(\cdot)\|_{L_p(T,\mu)}^\gamma \|\varphi(\cdot)x(\cdot)\|_{L_r(T,\mu)}^{1-\gamma}, \quad (31)$$

which holds for all  $x(\cdot) \in \mathcal{W}$ ,  $x(\cdot) \neq 0$ .

Let  $|w(\cdot)|$ ,  $|w_0(\cdot)|$ , and  $|w_1(\cdot)|$  be homogenous functions of degrees  $\theta$ ,  $\theta_0$ , and  $\theta_1$ , respectively. We assume that  $w(t), w_0(t), w_1(t) \neq 0$  for almost all  $t \in T$  and  $1 \leq q < p, r < \infty$ . Then for almost all  $z \in T$  such that  $w(z), w_0(z), w_1(z) \neq 0$  there exists  $\tilde{k}(z)$  satisfying

$$\frac{\tilde{k}^{\frac{1}{p-q}}(z)}{(1 - \tilde{k}(z))^{\frac{1}{r-q}}} = \left| \frac{w(z)}{w_1(z)} \right|^{\frac{r}{r-q}} \left| \frac{w_0(z)}{w(z)} \right|^{\frac{p}{p-q}}.$$

Put

$$\tilde{\theta} = \theta + d/q, \quad \tilde{\theta}_0 = \theta_0 + d/p, \quad \tilde{\theta}_1 = \theta_1 + d/r. \quad (32)$$

**Corollary 3.** *Let  $1 \leq q < p, r < \infty$ ,  $w(t), w_0(t), w_1(t) \neq 0$  for almost all  $t \in T$ , and  $\tilde{\theta}_0 \neq \tilde{\theta}_1$ . Assume that*

$$\begin{aligned} \tilde{I}_1 &= \int_T \left| \frac{w(z)}{w_0(z)} \right|^{\frac{qp}{p-q}} \tilde{k}^{\frac{p}{p-q}}(z) d\mu(z) < \infty, \\ \tilde{I}_2 &= \int_T \frac{|w(z)|^{\frac{qr}{p-q}}}{|w_0(z)|^{\frac{pr}{p-q}}} |w_1(z)|^r \tilde{k}^{\frac{r}{p-q}}(z) d\mu(z) < \infty. \end{aligned}$$

Then for all  $x(\cdot) \neq 0$  such that  $w_0(\cdot)x(\cdot) \in L_p(T, \mu)$  and  $w_1(\cdot)x(\cdot) \in L_r(T, \mu)$  the exact inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \leq \tilde{C} \|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu)}^{\tilde{\gamma}} \|w_1(\cdot)x(\cdot)\|_{L_r(T,\mu)}^{1-\tilde{\gamma}} \quad (33)$$

holds; here

$$\tilde{C} = \tilde{I}_1^{-\tilde{\gamma}/p} \tilde{I}_2^{-(1-\tilde{\gamma})/r} (\tilde{I}_1 + \tilde{I}_2)^{1/q}, \quad \tilde{\gamma} = \frac{\tilde{\theta}_1 - \tilde{\theta}}{\tilde{\theta}_1 - \tilde{\theta}_0}.$$

*Proof.* Put

$$\psi(x) = \frac{w(x)}{w_0(x)}, \quad \varphi(x) = \frac{w_1(x)}{w_0(x)}.$$

Then  $|\psi(\cdot)|$  and  $|\varphi(\cdot)|$  are homogeneous functions of degrees  $\eta = \theta - \theta_0$  and  $\nu = \theta_1 - \theta_0$ , respectively. It follows by (31) that for all  $x(\cdot) \in \mathcal{W}$ ,  $x(\cdot) \neq 0$ , the exact inequality

$$\|\psi(\cdot)x(\cdot)\|_{L_q(T,\mu)} \leq \tilde{C} \|x(\cdot)\|_{L_p(T,\mu)}^{\tilde{\gamma}} \|\varphi(\cdot)x(\cdot)\|_{L_r(T,\mu)}^{1-\tilde{\gamma}}$$

holds. Substituting  $x(\cdot) = w_0(\cdot)y(\cdot)$ , we obtain (33).  $\square$

The well-known Carlson inequality [4]

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \leq \sqrt{\pi} \|x(t)\|_{L_2(\mathbb{R}_+)}^{1/2} \|tx(t)\|_{L_2(\mathbb{R}_+)}^{1/2} \quad (34)$$

was generalized in many directions (see [5], [1], [3]). Inequality (33) is also a generalization of the Carlson inequality.

Let  $1 \leq p < q, r < \infty$ ,  $T$  be a cone in  $\mathbb{R}^d$ ,  $d\mu(t) = dt$ ,  $|\psi(\cdot)|$  and  $|\varphi(\cdot)|$  be homogenous functions of degrees  $\eta$ ,  $\nu$ , respectively,  $\varphi(t) \neq 0$  and  $\psi(t) \neq 0$  for almost all  $t \in T$ . Thus  $\mu(\cdot)$  is a homogeneous measure of degree  $d$ . Consider the polar transformation

$$\begin{aligned} x_1 &= \rho \cos \omega_1, \\ x_2 &= \rho \sin \omega_1 \cos \omega_2, \\ &\dots\dots\dots \\ x_{d-1} &= \rho \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \cos \omega_{d-1}, \\ x_d &= \rho \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \sin \omega_{d-1}. \end{aligned}$$

Set  $\omega = (\omega_1, \dots, \omega_{d-1})$ ,

$$\begin{aligned} \tilde{\psi}(\omega) &= \rho^{-\eta} |\psi(\rho \cos \omega_1, \dots, \rho \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \sin \omega_{d-1})|, \\ \tilde{\varphi}(\omega) &= \rho^{-\nu} |\varphi(\rho \cos \omega_1, \dots, \rho \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \sin \omega_{d-1})|. \end{aligned} \quad (35)$$

Denote by  $\Omega$  the range of  $\omega$ . Since  $T$  is a cone,  $\Omega$  does not depend on  $\rho$ . Put

$$J(\omega) = \sin^{d-2} \omega_1 \sin^{d-3} \omega_2 \dots \sin \omega_{d-2}.$$

By (25) we obtain the following equality for  $k(\cdot)$ :

$$\frac{k^{\frac{1}{p-q}}(\rho, \omega)}{(1 - k(\rho, \omega))^{\frac{1}{r-q}}} = \rho^{\frac{\eta q(p-r) - \nu r(p-q)}{(p-q)(r-q)}} \frac{\tilde{\psi}^{\frac{q(p-r)}{(p-q)(r-q)}}(\omega)}{\tilde{\varphi}^{\frac{r}{r-q}}(\omega)}. \quad (36)$$

Assume that  $\gamma \in (0, 1)$ , where  $\gamma$  is defined by (26). Put

$$\frac{1}{q^*} = \frac{1}{q} - \frac{\gamma}{p} - \frac{1-\gamma}{r}. \quad (37)$$

It is easy to verify that  $q^* > q \geq 1$ . Moreover,

$$q^* = \frac{pqr(\nu + d(1/r - 1/p))}{\nu r(p-q) - \eta q(p-r)}.$$

**Theorem 3.** *Let  $1 \leq q < p, r < \infty$ ,  $\gamma \in (0, 1)$ , and  $\tilde{\varphi}(\omega), \tilde{\psi}(\omega) \neq 0$  for almost all  $\omega \in \Omega$ . Assume that*

$$I = \int_{\Omega} \frac{\tilde{\psi}^{q^*}(\omega)}{\tilde{\varphi}^{q^*(1-\gamma)}(\omega)} J(\omega) d\omega < \infty.$$

Then

$$E(p, q, r) = C_1 \delta^\gamma,$$

where

$$C_1 = \gamma^{-\frac{\gamma}{p}}(1-\gamma)^{-\frac{1-\gamma}{r}} \left( \frac{B(q^*\gamma/p, q^*(1-\gamma)/r)I}{|\nu + d(1/r - 1/p)|(\gamma r + (1-\gamma)p)} \right)^{1/q^*},$$

where  $B(\cdot, \cdot)$  is the beta-function. Moreover, the method

$$\widehat{m}(y)(t) = k \left( \xi_1^{\frac{1}{\nu+d(1/r-1/p)}} t \right) \psi(t)y(t),$$

where

$$\xi_1 = \delta (\gamma^{q-r}(1-\gamma)^{p-q} C_1^{p-r})^{\frac{q^*}{pqr}},$$

is optimal recovery method.

*Proof.* Using Theorem 2, we obtain

$$\begin{aligned} I_1 &= \int_T |\psi(z)|^{\frac{qp}{p-q}} k^{\frac{p}{p-q}}(z) dz \\ &= \int_{\Omega} \widetilde{\psi}^{\frac{qp}{p-q}}(\omega) J(\omega) d\omega \int_0^{+\infty} \rho^{\frac{\eta qp}{p-q}+d-1} k^{\frac{p}{p-q}}(\rho, \omega) d\rho. \end{aligned}$$

By (36) we have

$$\rho^{\nu r(p-q)-\eta q(p-r)} = \frac{(1-k(\rho, \omega))^{p-q} \widetilde{\psi}^{q(p-r)}(\omega)}{k^{r-q}(\rho, \omega) \widetilde{\varphi}^{r(p-q)}(\omega)}. \quad (38)$$

Fixing  $\omega$ , we pass to  $k$

$$\begin{aligned} d\rho^{\frac{\eta qp}{p-q}+d} &= \left( \frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)} \right)^{\zeta} d \frac{(1-k)^{(p-q)\zeta}}{k^{(r-q)\zeta}} \\ &= -\zeta \left( \frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)} \right)^{\zeta} \frac{(1-k)^{(p-q)\zeta-1}}{k^{(r-q)\zeta+1}} (r-q + (p-r)k) dk, \end{aligned}$$

where

$$\zeta = \frac{\eta qp + d(p-q)}{(p-q)(\nu r(p-q) - \eta q(p-r))} = \frac{q^*(1-\gamma)}{r(p-q)}.$$

Consequently,

$$\begin{aligned} &\int_0^{+\infty} \rho^{\frac{\eta qp}{p-q}+d-1} k^{\frac{p}{p-q}}(\rho, \omega) d\rho \\ &= \frac{p-q}{\eta qp + d(p-q)} \int_0^{+\infty} k^{\frac{p}{p-q}}(\rho, \omega) d\rho^{\frac{\eta qp}{p-q}+d} \\ &= \frac{1}{|\nu r(p-q) - \eta q(p-r)|} \left( \frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)} \right)^{\zeta} (K_1 + K_2), \end{aligned}$$

where

$$\begin{aligned}
K_1 &= (r - q) \int_0^1 k^{\widehat{p}} (1 - k)^{\widehat{q}-1} dk = (r - q) B(\widehat{p} + 1, \widehat{q}), \\
K_2 &= (p - r) \int_0^1 k^{\widehat{p}+1} (1 - k)^{\widehat{q}-1} dk = (p - r) B(\widehat{p} + 2, \widehat{q}) \\
&= (p - r) \frac{\widehat{p} + 1}{\widehat{p} + \widehat{q} + 1} B(\widehat{p} + 1, \widehat{q}), \\
\widehat{p} &= \frac{qr(\nu - \eta) - d(r - q)}{\nu r(p - q) - \eta q(p - r)} = q^* \frac{\gamma}{p}, \quad \widehat{q} = \frac{\eta qp + d(p - q)}{\nu r(p - q) - \eta q(p - r)} = q^* \frac{1 - \gamma}{r}.
\end{aligned}$$

Thus,

$$\begin{aligned}
K_1 + K_2 &= p \frac{\nu r(p - q) - \eta q(p - r)}{\nu pr + d(p - r)} B(\widehat{p} + 1, \widehat{q}) = \frac{pq}{q^*} B(\widehat{p} + 1, \widehat{q}) \\
&= \frac{q\gamma}{q^*} \left( \frac{\gamma}{p} + \frac{1 - \gamma}{r} \right)^{-1} B(\widehat{p}, \widehat{q}).
\end{aligned}$$

The analogous calculations give

$$\begin{aligned}
I_2 &= \int_T |\psi(z)|^{\frac{qr}{p-q}} |\varphi(z)|^r k^{\frac{r}{p-q}}(z) d\mu(z) \\
&= \int_{\Omega} \widetilde{\psi}^{\frac{qr}{p-q}}(\omega) \widetilde{\varphi}^r(\omega) J(\omega) d\omega \int_0^{+\infty} \rho^{\frac{\eta qr}{p-q} + \nu r + d - 1} k^{\frac{r}{p-q}}(\rho, \omega) d\rho.
\end{aligned}$$

Fixing  $\omega$ , we pass to  $k$

$$\begin{aligned}
d\rho^{\frac{\eta qr}{p-q} + \nu r + d} &= \left( \frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)} \right)^{\zeta_1} d \frac{(1 - k)^{(p-q)\zeta_1}}{k^{(r-q)\zeta_1}} \\
&= -\zeta_1 \left( \frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)} \right)^{\zeta_1} \frac{(1 - k)^{(p-q)\zeta_1 - 1}}{k^{(r-q)\zeta_1 + 1}} (r - q + (p - r)k) dk,
\end{aligned}$$

where

$$\zeta_1 = \frac{\eta qr + (\nu r + d)(p - q)}{(p - q)(\nu r(p - q) - \eta q(p - r))} = \frac{q^*(1 - \gamma)}{r(p - q)} + \frac{1}{p - q}.$$

We have

$$\begin{aligned}
&\int_0^{+\infty} \rho^{\frac{\eta qr}{p-q} + \nu r + d - 1} k^{\frac{r}{p-q}}(\rho, \omega) d\rho \\
&= \frac{p - q}{\eta qr + (\nu r + d)(p - q)} \int_0^{+\infty} k^{\frac{r}{p-q}}(\rho, \omega) d\rho^{\frac{\eta qr}{p-q} + \nu r + d} \\
&= \frac{1}{|\nu r(p - q) - \eta q(p - r)|} \left( \frac{\widetilde{\psi}^{q(p-r)}(\omega)}{\widetilde{\varphi}^{r(p-q)}(\omega)} \right)^{\zeta_1} (L_1 + L_2),
\end{aligned}$$

where

$$\begin{aligned} L_1 &= (r - q) \int_0^1 k^{\widehat{p}-1} (1 - k)^{\widehat{q}} dk = (r - q) B(\widehat{p}, \widehat{q} + 1), \\ L_2 &= (p - r) \int_0^1 k^{\widehat{p}} (1 - k)^{\widehat{q}} dk = (p - r) B(\widehat{p} + 1, \widehat{q} + 1) \\ &= (p - r) \frac{\widehat{p}}{\widehat{p} + \widehat{q} + 1} B(\widehat{p}, \widehat{q} + 1). \end{aligned}$$

Thus,

$$\begin{aligned} L_1 + L_2 &= r \frac{\nu r(p - q) - \eta q(p - r)}{\nu pr + d(p - r)} B(\widehat{p}, \widehat{q} + 1) = \frac{qr}{q^*} B(\widehat{p}, \widehat{q} + 1) \\ &= \frac{q(1 - \gamma)}{q^*} \left( \frac{\gamma}{p} + \frac{1 - \gamma}{r} \right)^{-1} B(\widehat{p}, \widehat{q}). \end{aligned}$$

We obtain

$$\begin{aligned} I_1 &= \frac{\gamma}{pr|\nu + d(1/r - 1/p)|} \left( \frac{\gamma}{p} + \frac{1 - \gamma}{r} \right)^{-1} B(\widehat{p}, \widehat{q}) I, \\ I_2 &= \frac{1 - \gamma}{pr|\nu + d(1/r - 1/p)|} \left( \frac{\gamma}{p} + \frac{1 - \gamma}{r} \right)^{-1} B(\widehat{p}, \widehat{q}) I. \end{aligned}$$

It remains to apply Theorem 2.  $\square$

Note that for  $d = 1$  we have  $I = 1$  when  $T = \mathbb{R}_+$  and  $I = 2$  when  $T = \mathbb{R}$ .

Assume that  $|w(\cdot)|$ ,  $|w_0(\cdot)|$ , and  $|w_1(\cdot)|$  are homogenous functions of degrees  $\theta$ ,  $\theta_0$ , and  $\theta_1$ , respectively. Define  $\tilde{w}(\cdot)$ ,  $\tilde{w}_0(\cdot)$ ,  $\tilde{w}_1(\cdot)$  by the analogy with (35).

From Theorem 2 (analogously to Corollary 3) we immediately obtain

**Corollary 4** ([3]<sup>2</sup>). *Suppose that  $w(t), w_0(t), w_1(t) \neq 0$  for almost all  $t \in T$ ,  $1 \leq q < p, r < \infty, \tilde{\gamma} \in (0, 1)$ , where*

$$\tilde{\gamma} = \frac{\tilde{\theta}_1 - \tilde{\theta}}{\tilde{\theta}_1 - \tilde{\theta}_0},$$

and  $\tilde{\theta}, \tilde{\theta}_0$ , and  $\tilde{\theta}_1$  are defined by (32). Moreover, assume that

$$\tilde{I} = \int_{\Omega} \frac{\tilde{w}^{\tilde{q}}(\omega)}{\tilde{w}_0^{\tilde{q}\tilde{\gamma}}(\omega) \tilde{w}_1^{\tilde{q}(1-\tilde{\gamma})}(\omega)} J(\omega) d\omega < \infty,$$

where

$$\frac{1}{\tilde{q}} = \frac{1}{q} - \frac{\tilde{\gamma}}{p} - \frac{1 - \tilde{\gamma}}{r}.$$

<sup>2</sup>The exact constant in [3] (formula (10)) was given with a misprint.

Then the exact inequality

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \leq \tilde{C}_1 \|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu)}^{\tilde{\gamma}} \|w_1(\cdot)x(\cdot)\|_{L_r(T,\mu)}^{1-\tilde{\gamma}} \quad (39)$$

holds; here

$$\tilde{C}_1 = \tilde{\gamma}^{-\frac{\tilde{\gamma}}{p}} (1-\tilde{\gamma})^{-\frac{1-\tilde{\gamma}}{r}} \left( \frac{B(\tilde{q}\tilde{\gamma}/p, \tilde{q}(1-\tilde{\gamma})/r) \tilde{I}}{|\theta_1 - \theta_0|(\tilde{\gamma}r + (1-\tilde{\gamma})p)} \right)^{1/\tilde{q}}.$$

Put

$$w_0(t) = 1, \quad w_1(t) = t^{1-(\lambda+1)/p}, \quad w_2(t) = t^{1+(\mu-1)/q}.$$

From Corollary 4 we obtain

**Corollary 5** ([5]). *Let  $1 < p, q < \infty$  and  $\lambda, \mu > 0$ . Put*

$$\alpha = \frac{\mu}{p\mu + q\lambda}, \quad \beta = \frac{\lambda}{p\mu + q\lambda}.$$

Then the exact inequality

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \leq C \|t^{1-(\lambda+1)/p} x(t)\|_{L_p(\mathbb{R}_+)}^{p\alpha} \|t^{1+(\mu-1)/q} x(t)\|_{L_q(\mathbb{R}_+)}^{q\beta}$$

holds; here

$$C = \frac{1}{(p\alpha)^\alpha (q\beta)^\beta} \left( \frac{1}{\lambda + \mu} B\left(\frac{\alpha}{1-\alpha-\beta}, \frac{\beta}{1-\alpha-\beta}\right) \right)^{1-\alpha-\beta}.$$

Using Theorem 1' and calculations from the proofs of Theorems 2 and 3 we obtain

**Theorem 3'**. *Let  $1 < p, r < \infty$ ,  $\tilde{\varphi}(\omega), \tilde{\psi}(\omega) \neq 0$  for almost all  $\omega \in \Omega$  and  $\gamma, q^*, I, k(\cdot), C_1, \xi_1$  as above but for  $q = 1$ . Assume that  $\gamma \in (0, 1)$  and  $I < \infty$ . Then*

$$E_1(p, r) = C_1 \delta^\gamma.$$

Moreover, the method

$$\hat{m}(y) = \int_T k\left(\xi_1^{\frac{1}{\nu+d(1/r-1/p)}} t\right) \psi(t) y(t) d\mu(t)$$

is optimal recovery method.

#### 4. Optimal recovery of functions from a noisy Fourier transform

Let  $S$  be the Schwartz space of rapidly decreasing  $C^\infty$ -functions on  $\mathbb{R}$ ,  $S'$  the corresponding space of distributions, and let  $F: S' \rightarrow S'$  be the Fourier transform. We let  $\mathcal{F}_p$  denote the space of distribution  $x(\cdot)$  in  $S'$  for which

$$\|x(\cdot)\|_p = \left( \int_{\mathbb{R}} |Fx(t)|^p dt \right)^{1/p} < \infty, \quad 1 \leq p < \infty.$$

We set

$$\begin{aligned}\mathcal{F}_p^n &= \{x(\cdot) \in \mathcal{S}' : \|x^{(n)}(\cdot)\|_p < \infty\}, \\ F_p^n &= \{x(\cdot) \in \mathcal{F}_p^n : \|x^{(n)}(\cdot)\|_p \leq 1\}.\end{aligned}$$

Assume that the Fourier transform of a function  $x(\cdot) \in F_r^n \cap \mathcal{F}_p$  is known on  $\mathbb{R}$  to within  $\delta > 0$  in the metric of  $L_p(\mathbb{R})$ . In other words, we know a function  $y(\cdot) \in L_p(\mathbb{R})$  such that  $\|Fx(\cdot) - y(\cdot)\|_{L_p(\mathbb{R})} \leq \delta$ . How should we best use this information to recover the  $l$ th derivative of the function in the metric  $\mathcal{F}_q$ ,  $0 \leq l < n$ ? By recovery methods here we mean all possible mappings  $m: L_p(\mathbb{R}) \rightarrow \mathcal{F}_q$ . The error of a method is, by definition, the quantity

$$e_{p,q,r}(m) = \sup_{\substack{x(\cdot) \in F_r^n \cap \mathcal{F}_p, y(\cdot) \in L_p(\mathbb{R}) \\ \|Fx(\cdot) - y(\cdot)\|_{L_p(\Delta_\sigma)} \leq \delta}} \|x^{(l)}(\cdot) - m(y)(\cdot)\|_q.$$

The optimal recovery error is defined as follows:

$$E_{p,q,r} = \inf_{m: L_p(\mathbb{R}) \rightarrow \mathcal{F}_q} e_{p,q,r}(m).$$

A method on which this lower bound is attained is called optimal.

It is readily checked that this problem is a special case of the general problem (1) with  $T = T_0 = \mathbb{R}$ ,  $\psi(t) = (it)^l$ ,  $\varphi(t) = (it)^n$ .

The cases 1)  $1 \leq q = r < p < \infty$ , 2)  $1 \leq q = p < r < \infty$ , 3)  $1 \leq q = p = r < \infty$ , and 4)  $1 \leq q < p = r < \infty$  were studied in [14].

For the case  $1 \leq q < p, r < \infty$  we can apply Theorem 3. In this case

$$\frac{k^{\frac{1}{p-q}}(t)}{(1-k(t))^{\frac{1}{r-q}}} = |t|^{\frac{lq(p-r)-nr(p-q)}{(p-q)(r-q)}}, \quad \gamma = \frac{n-l-1/q+1/r}{n+1/r-1/p},$$

and  $I = 2$ . It is easy to verify that if  $n > l + 1/q - 1/r$ , then  $\gamma \in (0, 1)$ . Thus, it follows by Theorem 3

**Theorem 4.** *Let  $1 \leq q < p, r < \infty$  and  $n > l + 1/q - 1/r$ . Then*

$$E_{p,q,r} = C_1 \delta^\gamma, \tag{40}$$

where

$$C_1 = \gamma^{-\frac{\gamma}{p}} (1-\gamma)^{-\frac{1-\gamma}{r}} \left( \frac{2B(q^* \gamma/p, q^*(1-\gamma)/r)}{(n+1/r-1/p)(\gamma r + (1-\gamma)p)} \right)^{1/q^*}$$

and  $q^*$  is defined by (37). Moreover, the method  $\hat{m}(y)(\cdot) = F^{-1}Y_y(\cdot)$  is optimal, where

$$Y_y(t) = (it)^l k \left( \xi_1^{\frac{1}{n+1/r-1/p}} t \right) y(t), \quad \xi_1 = \delta (\gamma^{q-r} (1-\gamma)^{p-q} C_1^{p-r})^{\frac{q^*}{pqr}}.$$

Note that case 4) immediately follows from Theorem 4 for  $p = r$ . In cases 1)–3) the optimal recovery error coincides with the limits  $\lim_{r \rightarrow q} E_{p,q,r}$ ,  $\lim_{p \rightarrow q} E_{p,q,r}$ ,  $\lim_{p \rightarrow q} E_{p,q,p}$ , respectively, where  $E_{p,q,r}$  is given by (40).

## 5. Optimal recovery of derivatives and generalized Carlson-Levin-Taikov inequalities

For functions  $x(\cdot) \in L_2(\mathbb{R})$  whose  $(n-1)$ st derivative is locally absolutely continuous and  $0 \leq k \leq n-1$ , L. V. Taikov [16] obtained exact inequality

$$|x^{(k)}(0)| \leq K \|x(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2n-2k-1}{2n}} \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2k+1}{2n}},$$

where

$$K = \left( \frac{2k+1}{2n-2k-1} \right)^{\frac{2n-2k-1}{4n}} \left( (2k+1) \sin \frac{2k+1}{2n} \pi \right)^{-1/2}.$$

Passing to the Fourier transform we have the following equivalent inequality

$$\left| \frac{1}{2\pi} \int_{\mathbb{R}} t^k Fx(t) dt \right| \leq K \left( \frac{1}{2\pi} \int_{\mathbb{R}} |Fx(t)|^2 dt \right)^{\frac{2n-2k-1}{4n}} \times \left( \frac{1}{2\pi} \int_{\mathbb{R}} t^{2n} |Fx(t)|^2 dt \right)^{\frac{2k+1}{4n}}.$$

Set  $g(t) = t^k Fx(t)$ . Then we obtain the following inequality

$$\left| \int_{\mathbb{R}} g(t) dt \right| \leq K \sqrt{2\pi} \left( \int_{\mathbb{R}} t^{-2k} |g(t)|^2 dt \right)^{\frac{2n-2k-1}{4n}} \times \left( \int_{\mathbb{R}} t^{2(n-k)} |g(t)|^2 dt \right)^{\frac{2k+1}{4n}}.$$

Put  $p = q = 2$ ,  $\lambda = 2k+1$ , and  $\mu = 2n-2k-1$ . Then by Corollary 4 we have

$$\int_0^\infty |g(t)| dt \leq C \left( \int_0^\infty t^{-2k} |g(t)|^2 dt \right)^{\frac{2n-2k-1}{4n}} \times \left( \int_0^\infty t^{2(n-k)} |g(t)|^2 dt \right)^{\frac{2k+1}{4n}},$$

where

$$C = \left( \frac{2k+1}{2n-2k-1} \right)^{\frac{2n-2k-1}{4n}} (2k+1)^{-1/2} B^{1/2} \left( \frac{2n-2k-1}{2n}, \frac{2k+1}{2n} \right).$$

Since

$$B \left( 1 - \frac{2k+1}{2n}, \frac{2k+1}{2n} \right) = \frac{\pi}{\sin \frac{2k+1}{2n} \pi}$$

we have

$$C = \sqrt{\pi} \left( \frac{2k+1}{2n-2k-1} \right)^{\frac{2n-2k-1}{4n}} \left( (2k+1) \sin \frac{2k+1}{2n} \pi \right)^{-1/2}.$$

From the inequality

$$a_1 b_1 + a_2 b_2 \leq 2^{1-s-t} (a_1^{1/r} + a_2^{1/r})^r (b_1^{1/s} + b_2^{1/s})^s$$

it follows that

$$\begin{aligned} \int_{\mathbb{R}} |g(t)| dt &= \int_{-\infty}^0 |g(t)| dt + \int_0^{\infty} |g(t)| dt \\ &\leq C \left( \int_{-\infty}^0 t^{-2k} |g(t)|^2 dt \right)^{\frac{2n-2k-1}{4n}} \left( \int_{-\infty}^0 t^{2(n-k)} |g(t)|^2 dt \right)^{\frac{2k+1}{4n}} \\ &\quad + C \left( \int_0^{\infty} t^{-2k} |g(t)|^2 dt \right)^{\frac{2n-2k-1}{4n}} \left( \int_0^{\infty} t^{2(n-k)} |g(t)|^2 dt \right)^{\frac{2k+1}{4n}} \\ &\leq \sqrt{2} C \left( \int_{\mathbb{R}} t^{-2k} |g(t)|^2 dt \right)^{\frac{2n-2k-1}{4n}} \left( \int_{\mathbb{R}} t^{2(n-k)} |g(t)|^2 dt \right)^{\frac{2k+1}{4n}}. \end{aligned}$$

Thus Taikov's inequality follows from Levin's inequality.

This inequality is closely connected with the problem of optimal recovery of derivatives from inaccurate information about the Fourier transform (see [9]). We consider such problem in multidimensional case.

Consider linear operators  $D_1: L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d) \cap C(\mathbb{R}^d)$  and  $D_2: L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)$  ( $D_1$  and  $D_2$  are not necessary differentiation operators). Put

$$W = \{ x(\cdot) \in L_2(\mathbb{R}^d) : \|D_2 x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq 1 \}.$$

We consider the problem of optimal recovery of  $D_1 x(\tau)$ ,  $\tau \in \mathbb{R}^d$ , on the class  $W$  from the information about  $x(\cdot)$ , given inaccurately in  $L_2(\mathbb{R}^d)$ -metric.

As recovery methods we consider all possible mappings  $m: L_2(\mathbb{R}^d) \rightarrow \mathbb{C}$  or  $\mathbb{R}$ . The error of a method  $m$  is defined as

$$e(m) = \sup_{\substack{x(\cdot) \in W, y(\cdot) \in L_2(\mathbb{R}^d) \\ \|x(\cdot) - y(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \delta}} |D_1 x(\tau) - m(y)|.$$

The quantity

$$E = \inf_{m: L_2(\mathbb{R}^d) \rightarrow \mathbb{C}(\mathbb{R})} e(m) \quad (41)$$

is known as the optimal recovery error, and a method on which this infimum is attained is called optimal.

For the case when  $d = 1$ ,  $D_1 x(\cdot) = x^{(k)}(\cdot)$ , and  $D_2 x(\cdot) = x^{(n)}(\cdot)$ ,  $0 \leq k < n$ , similar problems were considered in [9].

Let  $d_1(t)$  and  $d_2(\cdot)$  be measurable functions on  $\mathbb{R}^d$ . Put

$$X = \{ x(\cdot) \in L_2(\mathbb{R}^d) : d_2(\cdot) F x(\cdot) \in L_2(\mathbb{R}^d) \}.$$

We define the operator  $D_2$  as follows

$$D_2 x(\cdot) = F^{-1}(d_2(\cdot) F x(\cdot))(\cdot).$$

Assume that  $d_1(\cdot)Fx(\cdot) \in L_2(\mathbb{R}^d)$  for all  $x(\cdot) \in X$  and the operator  $D_1$  which is defined by the equality

$$D_1x(\cdot) = F^{-1}(d_1(\cdot)Fx(\cdot))(\cdot)$$

maps  $X$  to  $L_2(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ .

Let  $|d_1(\cdot)|$  and  $|d_2(\cdot)|$  be homogenous functions of degrees  $k, n$ , respectively ( $k$  and  $n$  are not necessarily integer),  $d_j(t) \neq 0, j = 1, 2$ , for almost all  $t \in \mathbb{R}^d$ . Put

$$\begin{aligned}\tilde{d}_1(\omega) &= \rho^{-k} |d_1(\rho \cos \omega_1, \dots, \rho \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \sin \omega_{d-1})|, \\ \tilde{d}_2(\omega) &= \rho^{-n} |d_2(\rho \cos \omega_1, \dots, \rho \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \sin \omega_{d-1})|.\end{aligned}$$

By Plancherel's theorem we have

$$\begin{aligned}W &= \left\{ x(\cdot) \in L_2(\mathbb{R}^d) : \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |d_2(t)Fx(t)|^2 dt \leq 1 \right\}, \\ \|x(\cdot) - y(\cdot)\|_{L_2(\mathbb{R}^d)} &= \frac{1}{(2\pi)^{d/2}} \|Fx(\cdot) - Fy(\cdot)\|_{L_2(\mathbb{R}^d)}.\end{aligned}$$

Moreover,

$$D_1x(\tau) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d_1(t)Fx(t)e^{i\langle \tau, t \rangle} dt,$$

where  $\langle \tau, t \rangle = \tau_1 t_1 + \dots + \tau_d t_d$ . Thus we obtain problem (23) with  $p = r = 2$ ,  $\delta_1 = \delta(2\pi)^{d/2}$ ,

$$\psi(t) = \frac{1}{(2\pi)^d} d_1(t)e^{i\langle \tau, t \rangle}, \quad \varphi(t) = \frac{1}{(2\pi)^{d/2}} d_2(t).$$

By Theorem 3' we have

**Theorem 5.** *Let  $k \geq 0$  and  $n > k + d/2$ . Assume that*

$$I = \int_{\Pi_{d-1}} \frac{\tilde{d}_1^2(\omega)}{\tilde{d}_2^{\frac{2k+d}{n}}(\omega)} J(\omega) d\omega < \infty, \quad \Pi_{d-1} = [0, \pi]^{d-2} \times [0, 2\pi].$$

Then

$$E = \frac{(\pi I)^{1/2}}{(2\pi)^{d/2}} K_d(k, n) \delta^{\frac{2n-2k-d}{2n}},$$

where

$$K_d(k, n) = \left( \frac{2k+d}{2n-2k-d} \right)^{\frac{2n-2k-d}{4n}} \left( (2k+d) \sin \frac{2k+d}{2n} \pi \right)^{-1/2}.$$

Moreover, the method

$$\hat{m}(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d_1(t) \left( 1 + \frac{\delta^2(2k+d)}{(2\pi)^d(2n-2k-d)} \right)^{-1} y(t)e^{i\langle \tau, t \rangle} dt$$

is optimal recovery method.

By this theorem analogously to (31) we obtain the exact inequality

$$|D_1 x(\tau)| \leq \frac{(\pi I)^{1/2}}{(2\pi)^{d/2}} K_d(k, n) \|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2n-2k-d}{2n}} \|D_2 x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2k+d}{2n}}$$

or

$$\|D_1 x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \leq \frac{(\pi I)^{1/2}}{(2\pi)^{d/2}} K_d(k, n) \|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2n-2k-d}{2n}} \|D_2 x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2k+d}{2n}}. \quad (42)$$

Now we consider some examples. Define the operator  $(-\Delta)^{n/2}$ ,  $n \geq 0$ , as follows

$$(-\Delta)^{n/2} x(\cdot) = F^{-1}(|t|^n Fx(t))(\cdot).$$

Put  $d_1(t) = |t|^k$  and  $d_2(t) = |t|^n$ . Then problem (41) is the problem of optimal recovery of  $(-\Delta)^{k/2} x(\tau)$  on the class

$$W = \{x(\cdot) \in L_2(\mathbb{R}^d) : \|(-\Delta)^{n/2} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq 1\}$$

by the inaccurate information about  $x(\cdot)$ .

By Theorem 5 we obtain

**Corollary 6.** *Let  $n > k + d/2$ . Then*

$$E = C_d(k, n) \delta^{\frac{2n-2k-d}{2n}}, \quad C_d(k, n) = \frac{K_d(k, n)}{(2^{d-1} \pi^{d/2-1} \Gamma(d/2))^{1/2}},$$

and the method

$$\widehat{m}(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |t|^k \left(1 + \frac{\delta^2(2k+d)}{(2\pi)^d(2n-2k-d)}\right)^{-1} y(t) e^{i\langle \tau, t \rangle} dt$$

is optimal.

By (42) we get the exact inequality

$$\|(-\Delta)^{k/2} x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \leq C_d(k, n) \|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2n-2k-d}{2n}} \|(-\Delta)^{n/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2k+d}{2n}}.$$

Consider one more example. Let  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d$ . We define  $D^\alpha$  (the derivative of order  $\alpha$ ) as follows:

$$D^\alpha x(\cdot) = F^{-1}((it)^\alpha Fx(t))(\cdot),$$

where  $(it)^\alpha = (it_1)^{\alpha_1} \dots (it_d)^{\alpha_d}$ . Let  $D_1 = D^\alpha$  and  $D_2 = (-\Delta)^{n/2}$ . Then (41) is the problem of optimal recovery of  $D^\alpha x(\tau)$  on the class  $W$  by the inaccurate information about  $x(\cdot)$ .

From the well-known Dirichlet formula we have

$$\int_{\substack{x_1 \geq 0, \dots, x_d \geq 0 \\ x_1^2 + \dots + x_d^2 \leq 1}} x_1^{p_1-1} \dots x_d^{p_d-1} dx_1 \dots dx_d = \frac{\Gamma(p_1/2) \dots \Gamma(p_d/2)}{2^d \Gamma(p_1/2 + \dots + p_d/2 + 1)},$$

$p_1, \dots, p_d > 0$ . Using this formula and passing to the polar transformation we obtain

$$I(p_1, \dots, p_d) = \int_{\Pi_{d-1}} \Phi(\omega, p_1, \dots, p_d) J(\omega) d\omega = 2 \frac{\Gamma(p_1/2) \dots \Gamma(p_d/2)}{\Gamma(p_1/2 + \dots + p_d/2)},$$

where

$$\begin{aligned} \Phi(\omega, p_1, \dots, p_d) &= |\cos \omega_1|^{p_1-1} |\sin \omega_1 \cos \omega_2|^{p_2-1} \times \dots \\ &\times |\sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \cos \omega_{d-1}|^{p_{d-1}-1} \\ &\times |\sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-2} \sin \omega_{d-1}|^{p_d-1}. \end{aligned}$$

Thus for  $d_1(t) = (it)^\alpha$  and  $d_2(t) = |t|^n$  we have

$$I = I(2\alpha_1 + 1, \dots, 2\alpha_d + 1) = 2 \frac{\Gamma(\alpha_1 + 1/2) \dots \Gamma(\alpha_d + 1/2)}{\Gamma(|\alpha| + d/2)},$$

where  $|\alpha| = \alpha_1 + \dots + \alpha_d$ .

**Corollary 7.** *Let  $n > |\alpha| + d/2$ . Then*

$$E = C_{d,\alpha}(n) \delta^{\frac{2n-2|\alpha|-d}{2n}},$$

where

$$C_{d,\alpha}(n) = \frac{K_d(|\alpha|, n)}{(2\pi)^{d(d-1)/2}} \left( \frac{\Gamma(\alpha_1 + 1/2) \dots \Gamma(\alpha_d + 1/2)}{\Gamma(|\alpha| + d/2)} \right)^{1/2},$$

and the method

$$\widehat{m}(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (it)^\alpha \left( 1 + \frac{\delta^2(2|\alpha| + d)}{(2\pi)^d(2n - 2|\alpha| - d)} \right)^{-1} y(t) e^{i\langle \tau, t \rangle} dt$$

is optimal.

The exact inequality in this case has the form:

$$\|D^\alpha x(\cdot)\|_{L_\infty(\mathbb{R}^d)} \leq C_{d,\alpha}(n) \|x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2n-2|\alpha|-d}{2n}} \|(-\Delta)^{n/2} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\frac{2|\alpha|+d}{2n}}.$$

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