HADAMARD TYPE EXTREMAL PROBLEMS AND OPTIMAL RECOVERY OF ANALYTIC FUNCTIONS

K. YU. OSIPENKO

The well-known Hadamard three-circle theorem states that if $f(z)$ is a holomorphic function on the annulus $r_1 \le |z| \le r_2$ and

$$M(r) = \max_{|z|=r} |f(z)|,$$

then

$$M(\rho) \le M(r_1)^{\frac{\log r_2/r}{\log r_2/r_1}} M(r_2)^{\frac{\log r/r_1}{\log r_2/r_1}}$$

for any three concentric circles of radii $r_1 < \rho < r_2$.

For functions f from the Hardy space $H^2(\mathbb{B}^n)$ we consider the analogous extremal problem

$$\|f(\rho z)\|_{H^2(\mathbb{B}^n)} \to \max, \quad \|f(r_1 z)\|_{H^2(\mathbb{B}^n)} \le \delta_1, \quad \|f(r_2 z)\|_{H^2(\mathbb{B}^n)} \le \delta_2.$$

This problem is closely connected with the problem of optimal recovery of f on the sphere of radius ρ from the information about traces on the spheres of radii r_1 and r_2 given with errors. The optimal error of such recovery is defined as follows

$$\begin{aligned} &E_\rho(r_1, r_2, \delta_1, \delta_2) \\ &\quad = \inf_m \sup_{\substack{f \in H^2(\mathbb{B}^n),\ y_j \in L_2(\sigma_{r_j}),\ j=1,2 \\ \|f(r_j z) - y_j(r_j z)\|_{L_2(\sigma)} \le \delta_j,\ j=1,2}} \|f(\rho z) - m(y_1, y_2)(\rho z)\|_{L_2(\sigma)}, \end{aligned}$$

where the lower bound is taken over all maps (methods) $m\colon L_2(\sigma_{r_1}) \times L_2(\sigma_{r_2}) \to L_2(\sigma_\rho)$ and $d\sigma_r(z)$ are the positive normalized rotationally invariant measures on the spheres $r\mathbb{S}^{n-1}$ ($\sigma = \sigma_1$). Any method $\widehat{m}$ for which the lower bound is attained is called an optimal recovery method.

Let

$$(\lambda_1, \lambda_2) = \left(\frac{r_2^2 - \rho^2}{r_2^2 - r_1^2} \left(\frac{\rho}{r_1} \right)^{2s}, \frac{\rho^2 - r_1^2}{r_2^2 - r_1^2} \left(\frac{\rho}{r_2} \right)^{2s} \right),$$

if

$$\left(\frac{r_1}{r_2} \right)^{s+1} \le \frac{\delta_1}{\delta_2} < \left(\frac{r_1}{r_2} \right)^{s}, \quad s \in \mathbb{Z}_+,$$

and $(\lambda_1, \lambda_2) = (0, 1)$, if $\delta_1 \ge \delta_2$.

Theorem 1 ([1])**.** *The error of optimal recovery is given by*

$$E_\rho(r_1, r_2, \delta_1, \delta_2) = \sqrt{\lambda_1 \delta_1^2 + \lambda_2 \delta_2^2}$$

and the method

$$\widehat{m}(y_1,y_2)(z)=\sum_{k=0}^{\infty}\frac{1}{\lambda_1 r_1^{2k}+\lambda_2 r_2^{2k}}\sum_{|\alpha|=k}(\lambda_1 r_1^k c_\alpha^{(1)}+\lambda_2 r_2^k c_\alpha^{(2)})z^\alpha,$$

where

$$c_\alpha^{(j)}=\frac{(n+|\alpha|-1)!}{n!\alpha!}\int_{\mathbb{S}^{n-1}} y_j(r_j z)\overline{z}^\alpha\,d\sigma(z),\quad j=1,2,$$

is optimal.

It appears that it is possible to construct a collection of optimal recovery methods.

Theorem 2. *For all β_k, $k=0,1,\ldots$, such that*

$$\lambda_2\left(\frac{\rho}{r_1}\right)^{2k}|\beta_k|^2+\lambda_1\left(\frac{\rho}{r_2}\right)^{2k}|1-\beta_k|^2\le\lambda_1\lambda_2 \tag{1}$$

all methods

$$\widehat{m}(y_1,y_2)(z)=\sum_{k=0}^{\infty}\sum_{|\alpha|=k}\left(\frac{\beta_k}{r_1^k}c_\alpha^{(1)}+\frac{1-\beta_k}{r_2^k}c_\alpha^{(2)}\right)z^\alpha$$

are optimal.

Assume that $\delta_1<\delta_2$. Let $K_1=\max\{\,k\in\mathbb{Z}_+:\rho^{2k}\le\lambda_1 r_1^{2k}\,\}$, $K_2=\min\{\,k\in\mathbb{Z}_+:\rho^{2k}\le\lambda_2 r_2^{2k}\,\}$.

From Theorem 2 we have

Corollary 1. *For all $0\le k_1\le K_1$, $k_2\ge K_2$ and β_k, $k=k_1+1,\ldots,k_2-1$, such that (1) holds all methods*

$$\begin{aligned}m(y_1,y_2)(z)&=\sum_{k=0}^{k_1}\sum_{|\alpha|=k}\frac{c_\alpha^{(1)}}{r_1^k}z^\alpha\\&\quad+\sum_{k=k_1+1}^{k_2-1}\sum_{|\alpha|=k}\left(\frac{\beta_k}{r_1^k}c_\alpha^{(1)}+\frac{1-\beta_k}{r_2^k}c_\alpha^{(2)}\right)z^\alpha+\sum_{k=k_2}^{\infty}\sum_{|\alpha|=k}\frac{c_\alpha^{(2)}}{r_2^k}z^\alpha\end{aligned}$$

are optimal.

References

[1] *Osipenko K. Yu., Stessin M. I.* Hadamard and Schwarz type theorems and optimal recovery in spaces of analytic functions, *Constr. Approx.*, 31 (2010), 31–67.

Moscow State University