

Optimal recovery of linear operators

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Let X be a linear space, Y_1, \dots, Y_n be linear spaces with semi-inner products, $I_j: X \rightarrow Y_j$, $j = 1, \dots, n$ be linear operators, and Z be a normed linear space. We consider the problem of optimal recovery of a linear operator $T: X \rightarrow Z$ on a set

$$W = \{x \in X : \|I_j x\|_{Y_j} \leq \delta_j, j = 1, \dots, k, k < n\}$$

by inaccurate information about values of the operators I_{k+1}, \dots, I_n . More precisely, we are interesting in the value

$$E(T, W, I, \delta) = \inf_{\varphi: Y_{k+1} \times \dots \times Y_n \rightarrow Z} \sup_{\substack{x \in W, (y_{k+1}, \dots, y_n) \in Y_{k+1} \times \dots \times Y_n \\ \|I_j x - y_j\|_{Y_j} \leq \delta_j, j = k+1, \dots, n}} \|Tx - \varphi(y)\|_Z,$$

and in a method $\widehat{\varphi}$ for which this infimum is attained (we call it an optimal method of recovery).

Consider the following extremal problem

$$(1) \quad \|Tx\|_Z^2 \rightarrow \max, \quad \|I_j x\|_{Y_j} \leq \delta_j^2, j = 1, \dots, n, x \in X.$$

Denote by

$$\mathcal{L}(x, \lambda) = -\|Tx\|_Z^2 + \sum_{j=1}^n \lambda_j \|I_j x\|_{Y_j}^2$$

the Lagrange function of this problem.

Theorem 1. *Suppose that there exist nonnegative $\widehat{\lambda}_j$, $j = 1, \dots, n$ such that $\mathcal{L}(x, \widehat{\lambda}) \geq 0$ for all $x \in X$. Let $\{x_m\}$ be a sequence of admissible elements in (1) such that*

$$(a) \quad \lim_{m \rightarrow \infty} \mathcal{L}(x_m, \widehat{\lambda}) = 0,$$

$$(b) \quad \lim_{m \rightarrow \infty} \sum_{j=1}^n \widehat{\lambda}_j (\|I_j x_m\|_{Y_j}^2 - \delta_j^2) = 0.$$

If for all $y = (y_{k+1}, \dots, y_n) \in Y_{k+1} \times \dots \times Y_n$ there exists an element x_y which is a solution of the problem

$$\sum_{j=1}^k \widehat{\lambda}_j \|I_j x\|_{Y_j}^2 + \sum_{j=k+1}^n \widehat{\lambda}_j \|I_j x - y_j\|_{Y_j}^2 \rightarrow \min, \quad x \in X,$$

then the method

$$\widehat{\varphi}(y) = Tx_y$$

is optimal and

$$E(T, W, I, \delta) = \sqrt{\sum_{j=1}^n \widehat{\lambda}_j \delta_j^2}.$$

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