MINIMAL BLASHKE PRODUCTS AND OPTIMAL QUADRATURES IN H^∞

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We consider the problem of minimization Blashke products with real nodes x_j of multiplicities ν_j in integral metric

$$(1)\quad \int_a^b \left|\prod_{j=1}^n \left(\frac{x-x_j}{1-x_jx}\right)^{\nu_j}\right|^q s(x)\,dx \to \inf, \quad -1 < x_1 < \ldots < x_n < 1,$$

where $-1 \le a < b \le 1$, $1 \le q < \infty$, and $s(x)$ is nonnegative weight function which is continuous in (a, b). In [1] it was proved that the solution of the problem (1) existed but it was not unique in general case (for $s(x) = 1$ the existence was also proved in [2]).

The problem (1) can be transformed to the follows:

$$\int_{-1}^1 \left|\prod_{j=1}^n \left(\frac{t-t_j}{1-t_jt}\right)^{\nu_j}\right|^q p(t)\,dt \to \inf, \quad -1 < t_1 < \ldots < t_n < 1.$$

This problem has a unique solution for all k sufficiently small. We find it for some weight functions.

Theorem. *Let $\nu_1 = \ldots = \nu_n = 1$, $q > 1$. Then for the weight functions*

$$p_1(t) = \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad p_2(t) = p_1(t)\left(\frac{1-t^2}{1-k^2t^2}\right)^{q/2},$$

and all k sufficiently small the unique system of optimal nodes are

$$\bar{t}_1 = \left\{\operatorname{sn}\left[\left(\frac{2j-1}{n} - 1\right)K, k\right]\right\}_{j=1}^n,$$

$$\bar{t}_2 = \left\{\operatorname{sn}\left[\left(\frac{2j}{n+1} - 1\right)K, k\right]\right\}_{j=1}^n,$$

correspondingly; here

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

The received results are applied to the problem of finding optimal quadrature formulae in $H^\infty(G)$, the set of bounded analytic functions in G with the norm

$$\|f\|_\infty = \sup_{z\in G} |f(z)|$$

(cf. [3]). Set

$$R(\mu, p, G) = \inf_{a \le x_1 < \ldots < x_n \le b} \inf_{a_{jm}} \sup_{\|f\|_\infty \le 1} \left| \int_a^b f(x)p(x)\,dx - \sum_{j=1}^{n} \sum_{m=0}^{\mu_j - 1} a_{jm} f^{(m)}(x_j) \right|,$$

$[a, b] \subset G$. Let $Э_c$, $c > 1$, be the interior of the ellipse given by the equations $2x = (c + c^{-1})\cos\theta$, $2y = (c - c^{-1})\sin\theta$, $0 \le \theta \le 2\pi$.

Corollary. *Let $[a, b] = [-1, 1]$ and q is an even number. Then for $q - 1 \le \mu_j \le q$, $j = 1, \ldots, n$, and for all c sufficiently large the asymptotic equation*

$$R\left(\mu, \frac{1}{\sqrt{1 - x^2}}, Э_c\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{q+1}{2}\right) 2^q}{\Gamma\left(\frac{q}{2} + 1\right)} c^{-nq} + O\left(c^{-q(n+4)}\right)$$

holds. The only system of optimal nodes is

$$x_j = \cos\frac{2j - 1}{2n}\pi, \quad j = 1, \ldots, n.$$

References

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