# EXACT INEQUALITIES AND OPTIMAL RECOVERY BY INACCURATE INFORMATION

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Abstract. The paper considers a multidimensional problem of optimal recovery of an operator whose action is represented by multiplying the original function by a weight function of a special type, based on inaccurately specified information about the values of operators of a similar type. An exact inequality for the norms of such operators is obtained. The problem under consideration is a generalization of the problem of optimal recovery of a derivative based on other inaccurately specified derivatives in the space $\mathbb{R}^d$ and the problem of an exact inequality, which is an analogue of the Hardy–Littlewood–Polya inequality.

## 1. General setting

Let $X$ be a linear space, $Y_0, Y_1, \dots, Y_N$ be normed linear spaces, and $\Lambda_j\colon X \to Y_j$, $j = 0, 1, \dots, N$, be linear operators. Consider the problem of optimal recovery of $\Lambda_0$ on the set

$$W = \{\, x \in X : \|\Lambda_j x\|_{Y_j} \le \delta_j,\ \delta_j > 0,\ j = m+1, \dots, N \,\},$$

where $1 \le m \le N$ (if $m = N$, then $W = X$), by values of operators $\Lambda_1, \dots, \Lambda_m$ given with errors. More precisely, we will assume that for each $x \in W$ we know $y = (y_1, \dots, y_m) \in Y_1 \times \ldots \times Y_m$ such that $\|\Lambda_j x - y_j\|_{Y_j} \le \delta_j$, $\delta_j > 0$, $j = 1, \dots, m$.

Any recovery method by known information $y = (y_1, \dots, y_m)$ should give an element from $Y_0$ that is taken as an approximate value of $\Lambda_0 x$. Thus, every recovery method is a mapping $\Phi\colon Y_1 \times \ldots \times Y_m \to Y_0$. The error of a method $\Phi$ is defined as

$$e(\Lambda_0, W, \delta, \Phi) = \sup_{\substack{x \in W,\ y=(y_1,\dots,y_m) \in Y_1 \times \ldots \times Y_m \\ \|\Lambda_j x - y_j\|_{Y_j} \le \delta_j,\ j=1,\dots,m}} \|\Lambda_0 x - \Phi(y)\|_{Y_0},$$

where $\delta = (\delta_1, \dots, \delta_m)$.

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We are interested in those methods for which the error is minimal, i.e. those methods $\widehat{\Phi}$ for which

$$e(\Lambda_0, W, \delta, \widehat{\Phi}) = \inf_{\Phi\colon Y_1\times\ldots\times Y_m\to Y_0} e(\Lambda_0, W, \delta, \Phi). \tag{1}$$

We will call such methods *optimal recovery methods*. The quantity on the right-hand side of (1) will be called *the error of optimal recovery* and denoted by $E(\Lambda_0, W, \delta)$.

Let

$$\begin{gathered} \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k_+, \quad \varphi(\cdot) = (\varphi_1(\cdot), \ldots, \varphi_k(\cdot)), \\ \varphi^{\alpha}(\cdot) = \varphi_1^{\alpha_1}(\cdot)\ldots\varphi_k^{\alpha_k}(\cdot), \end{gathered} \tag{2}$$

where $\varphi_j(\cdot)$, $j = 1, \ldots, k$ are continuous (generally speaking, complex-valued) functions on $\mathbb{R}^d$. Set

$$\mathcal{W}_p^{\mathcal{A}}(\mathbb{R}^d) = \left\{\, x(\cdot) \in L_p(\mathbb{R}^d) : \varphi^{\alpha^j}(\cdot)x(\cdot) \in L_p(\mathbb{R}^d),\ j = 1, \ldots, N \,\right\},$$

where $1 \le p \le \infty$, $\alpha^j \in \mathbb{R}^d_+$, $j = 1, \ldots, N$, and $\mathcal{A} = \{\alpha^1, \ldots, \alpha^N\}$. We define operators $\Lambda_j\colon \mathcal{W}_p^{\mathcal{A}}(\mathbb{R}^d) \to L_p(\mathbb{R}^d)$ as follows

$$\Lambda_j x(\cdot) = \varphi^{\alpha^j}(\cdot)x(\cdot), \quad j = 0, 1, \ldots, N.$$

For these operators we consider problem (1), in which $X = \mathcal{W}_p^{\mathcal{A}}(\mathbb{R}^d)$, $Y_0 = Y_1 = \ldots = Y_N = L_p(\mathbb{R}^d)$. The corresponding set of functions $W$ is denoted by $W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta})$ where $\overline{\delta} = (\delta_{m+1}, \ldots, \delta_N)$. The case when $\varphi(\xi) = i\xi$ was considered in [1].

The consideration of the problem posed is connected with the desire to generalize the recovery problem for functions of many variables from inaccurately given values of derivatives (see [2, p. 249]). In addition, as a consequence of the solution of the problem under study, a generalization of the exact inequality of the Hardy–Littlewood–Polya type is obtained (see [3]).

Note that the idea of considering recovery problems for a whole family of operators was used in [4–7].

## 2. General result

Set

$$Q = \operatorname{co}\{(\alpha^1, \ln 1/\delta_1), \ldots, (\alpha^N, \ln 1/\delta_N)\},$$

where $\operatorname{co} A$ is the convex hall of A. Define the function $S(\cdot)$ on $\mathbb{R}^k$ by the equality

$$S(\alpha) = \max\{\, z \in \mathbb{R} : (\alpha, z) \in Q \,\} \tag{3}$$

$(S(\alpha) = -\infty$, if $(\alpha, z) \notin Q$ for all $z)$.

Let $\alpha^0 \in \operatorname{co}\mathcal{A}$ and let $z = \langle \alpha, \widehat{\eta}\rangle + \widehat{a}$, where $\widehat{\eta} = (\widehat{\eta}_1, \ldots, \widehat{\eta}_k) \in \mathbb{R}^k$, be a support hyperplane to the graph of $S(\cdot)$ at $\alpha^0$. By Caratheodory's theorem there exist points in this hyperplane $(\alpha^{j_s}, \ln 1/\delta_{j_s})$, $s = 1, \ldots, l$, $l \le k+1$, such that

$$\alpha^0 = \sum_{s=1}^{l} \theta_{j_s}\alpha^{j_s}, \quad \theta_{j_s} > 0,\ s = 1, \ldots, l, \quad \sum_{s=1}^{l} \theta_{j_s} = 1. \tag{4}$$

Set

$$M = \{j_1, \ldots, j_l\} \cap \{1, \ldots, m\}.$$

**Theorem 1.** *Let $\alpha^0 \in \operatorname{co}\mathcal{A}$. Assume that for any $a_1, \ldots, a_k > 0$ there exists $\widehat{\xi} \in \mathbb{R}^d$ such that $|\varphi_j(\widehat{\xi})| = a_j$, $j = 1, \ldots, k$. Then*

$$E(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta) = e^{-S(\alpha^0)}. \tag{5}$$

*If $M \ne \emptyset$, then all methods*

$$\Phi(y(\cdot))(\cdot) = \sum_{j \in M} a_j(\cdot) y_j(\cdot), \tag{6}$$

*where functions $a_{j_s}(\cdot)$, $s = 1, \ldots, l$, satisfy the conditions*

$$\sum_{s=1}^{l} \varphi^{\alpha^{j_s}}(\xi) a_{j_s}(\xi) = \varphi^{\alpha^0}(\xi), \tag{7}$$

$$\begin{cases} \displaystyle\sum_{s=1}^{l} \frac{\delta_{j_s}^{p'} |a_{j_s}(\xi)|^{p'}}{\theta_{j_s}^{p'/p}} \le e^{-p'S(\alpha^0)}, & 1 < p < \infty, \quad \dfrac{1}{p} + \dfrac{1}{p'} = 1, \\ \displaystyle\max_{1 \le s \le l} \frac{\delta_{j_s} |a_{j_s}(\xi)|}{\theta_{j_s}} \le e^{-S(\alpha^0)}, & p = 1, \\ \displaystyle\sum_{s=1}^{l} |a_{j_s}(\xi)| \delta_{j_s} \le e^{-S(\alpha^0)}, & p = \infty, \end{cases} \tag{8}$$

*for almost all $\xi \in \mathbb{R}^d$, are optimal.*

*If $M = \emptyset$ then the method $\Phi(y(\cdot))(\cdot) = 0$ is optimal.*

*Proof.* We prove that

$$E(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta) \ge \sup_{\substack{x(\cdot) \in W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}) \\ \|\Lambda_j x(\cdot)\|_{L_p(\mathbb{R}^d)} \le \delta_j,\ j = 1, \ldots, m}} \|\Lambda_0 x(\cdot)\|_{L_p(\mathbb{R}^d)}. \tag{9}$$

Indeed, for any method $\Phi\colon (L_2(\mathbb{R}^d))^m \to L_2(\mathbb{R}^d)$ and any function $x(\cdot) \in W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta})$ such that $\|\Lambda_j x(\cdot)\|_{L_p(\mathbb{R}^d)} \le \delta_j$, $j = 1, \dots, m$, we have

$$\begin{aligned}2\|\Lambda_0 x(\cdot)\|_{L_p(\mathbb{R}^d)} &= \|\Lambda_0 x(\cdot) - \Phi(0)(\cdot) - (\Lambda_0(-x(\cdot)) - \Phi(0)(\cdot))\|_{L_p(\mathbb{R}^d)} \\ &\le \|\Lambda_0 x(\cdot) - \Phi(0)(\cdot)\|_{L_p(\mathbb{R}^d)} + \|\Lambda_0(-x(\cdot)) - \Phi(0)(\cdot)\|_{L_p(\mathbb{R}^d)} \\ &\le 2e(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta, \Phi).\end{aligned}$$

Due to the arbitrariness of $\Phi$ it follows that

$$\|\Lambda_0 x(\cdot)\|_{L_p(\mathbb{R}^d)} \le E(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta).$$

Taking the upper bound over all functions $x(\cdot)$ satisfying the given conditions, we obtain inequality (9).

The extremal problem in the right-hand side of (9) may be written in the form

(10)
$$\||\varphi(\cdot)|^{\alpha^0} x(\cdot)\|_{L_p(\mathbb{R}^d)} \to \max, \quad \||\varphi(\cdot)|^{\alpha^j} x(\cdot)\|_{L_p(\mathbb{R}^d)} \le \delta_j, \quad j = 1, \dots, N,$$

where $|\varphi(\xi)|^\alpha = |\varphi_1(\xi)|^{\alpha_1} \dots |\varphi_k(\xi)|^{\alpha_k}$ for $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}_+^k$.

Let $\widehat{\xi}_{\widehat{\eta}} \in \mathbb{R}^d$ be such that $|\varphi_j(\widehat{\xi}_{\widehat{\eta}})| = e^{-\widehat{\eta}_j}$, $j = 1, \dots, k$. Consider the case when $1 \le p < \infty$. Due to the continuity of the functions $\varphi_s(\cdot)$, $s = 1, \dots, k$, for any $\varepsilon > 0$ and any $j \in \{0, 1, \dots, N\}$ there exist $\widetilde{\delta}_j$ such that

$$\Big| |\varphi(\xi)|^{p\alpha^j} - |\varphi(\widehat{\xi}_{\widehat{\eta}})|^{p\alpha^j} \Big| \le \varepsilon \tag{11}$$

for all $\xi \in B_{\widetilde{\delta}_j}(\widehat{\xi}_{\widehat{\eta}})$ where

$$B_\delta(\xi_0) = \{\, \xi \in \mathbb{R}^d : |\xi - \xi_0| < \delta \,\}.$$

Set $\widetilde{\delta} = \min\{\widetilde{\delta}_1, \dots, \widetilde{\delta}_N\}$. Then for all $\xi \in B_{\widetilde{\delta}}(\widehat{\xi}_{\widehat{\eta}})$ and all $j \in \{0, 1, \dots, N\}$ inequalities (11) hold.

Put $\widehat{A} = e^{-2\widehat{a}}$ and

$$x_{\varepsilon,p}(\xi) = \begin{cases} \left( \dfrac{\widehat{A}}{|B_{\widetilde{\delta}}(\widehat{\xi}_{\widehat{\eta}})| \gamma_\varepsilon} \right)^{1/p}, & \xi \in B_{\widetilde{\delta}}(\widehat{\xi}_{\widehat{\eta}}), \\ 0, & \xi \notin B_{\widetilde{\delta}}(\widehat{\xi}_{\widehat{\eta}}), \end{cases}$$

where

$$\gamma_\varepsilon = 1 + \varepsilon \frac{\widehat{A}}{\min_{1 \le j \le N} \delta_j^p}$$

($|B_\delta(\widehat{\xi}_{\widehat{\eta}})|$ denotes the volume of the ball $B_\delta(\widehat{\xi}_{\widehat{\eta}})$).

We have

$$\int_{\mathbb{R}^d} |\varphi(\xi)|^{p\alpha^j} |x_{\varepsilon,p}(\xi)|^p \, d\xi \le \frac{\widehat{A}}{\gamma_\varepsilon} \left( |\varphi(\widehat{\xi}_{\widehat{\eta}})|^{p\alpha^j} + \varepsilon \right) = \frac{e^{-p(\langle \alpha^j, \widehat{\eta}\rangle + \widehat{a})} + \widehat{A}\varepsilon}{\gamma_\varepsilon}.$$

Since $z = \langle \alpha, \widehat{\eta}\rangle + \widehat{a}$ is the equation of support hyperplane of $Q$, we get

$$\langle \alpha^j, \widehat{\eta}\rangle + \widehat{a} \ge \ln 1/\delta_j.$$

Consequently,

$$\int_{\mathbb{R}^d} |\varphi(\xi)|^{p\alpha^j} |x_{\varepsilon,p}(\xi)|^p \, d\xi \le \frac{\delta_j^p + \widehat{A}\varepsilon}{\gamma_\varepsilon} \le \delta_j^p.$$

Thus, $x_{\varepsilon,p}(\cdot)$ is an admissible function for extremal problem (10). Therefore, taking into account (9) and (11), we obtain

$$E^p(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta) \ge \int_{\mathbb{R}^d} |\varphi(\xi)|^{p\alpha^0} |x_\varepsilon(\xi)|^p \, d\xi \ge \frac{\widehat{A}}{\gamma_\varepsilon} \left( |\varphi(\widehat{\xi}_{\widehat{\eta}})|^{p\alpha^0} - \varepsilon \right) = \frac{e^{-p(\langle \alpha^0, \widehat{\eta}\rangle + \widehat{a})} - \widehat{A}\varepsilon}{\gamma_\varepsilon}.$$

Making $\varepsilon$ tends to zero, we have

$$E(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta) \ge e^{-(\langle \alpha^0, \widehat{\eta}\rangle + \widehat{a})} = e^{-S(\alpha^0)}.$$

Now let $p = \infty$. Similar to the previous reasoning, for any $\varepsilon > 0$ there exists $\widetilde{\delta} > 0$ such that for all $\xi \in B_{\widetilde{\delta}}(\widehat{\xi}_{\widehat{\eta}})$ and all $j \in \{0, 1, \dots, N\}$ inequalities

$$\left| |\varphi(\xi)|^{\alpha^j} - |\varphi(\widehat{\xi}_{\widehat{\eta}})|^{\alpha^j} \right| \le \varepsilon$$

hold.

Put

$$x_{\varepsilon,\infty}(\xi) = \begin{cases} \widehat{A}\widetilde{\gamma}_\varepsilon^{-1}, & \xi \in B_{\widetilde{\delta}}(\widehat{\xi}_{\widehat{\eta}}), \\ 0, & \xi \notin B_{\widetilde{\delta}}(\widehat{\xi}_{\widehat{\eta}}), \end{cases}$$

where

$$\widetilde{\gamma}_\varepsilon = 1 + \varepsilon \frac{\widehat{A}}{\min_{1 \le j \le N} \delta_j}$$

We have

$$\| |\varphi(\cdot)|^{\alpha^j} x_{\varepsilon,\infty}(\cdot) \|_{L_\infty(\mathbb{R}^d)} \le \frac{\widehat{A}}{\widetilde{\gamma}_\varepsilon} \left( |\varphi(\widehat{\xi}_{\widehat{\eta}})|^{p\alpha^j} + \varepsilon \right) = \frac{e^{-(\langle \alpha^j, \widehat{\eta}\rangle + \widehat{a})} + \widehat{A}\varepsilon}{\widetilde{\gamma}_\varepsilon} \le \delta_j.$$

Consequently, $x_{\varepsilon,\infty}(\cdot)$ is an admissible function for extremal problem (10). Thus,

$$E(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta) \geq \||\varphi(\cdot)|^{\alpha^0} x_{\varepsilon,\infty}(\cdot)\|_{L_\infty(\mathbb{R}^d)} \geq \frac{\widehat{A}}{\widetilde{\gamma}_\varepsilon}\Big(|\varphi(\widehat{\xi}_{\widehat{\eta}})|^{\alpha^0} - \varepsilon\Big) = \frac{e^{-(\langle\alpha^0,\widehat{\eta}\rangle + \widehat{a})} - \widehat{A}\varepsilon}{\widetilde{\gamma}_\varepsilon}.$$

Making $\varepsilon$ tends to zero, we have

$$E(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta) \geq e^{-(\langle\alpha^0,\widehat{\eta}\rangle + \widehat{a})} = e^{-S(\alpha^0)}. \tag{12}$$

Now we prove the optimality of the recovery methods (6). To estimate the error of methods (6) we consider the following extremal problem

$$\Big\|\varphi^{\alpha^0}(\cdot)x(\cdot) - \sum_{j\in M} a_j(\cdot)y_j(\cdot)\Big\|_{L_p(\mathbb{R}^d)} \to \max, \tag{13}$$
$$\|\varphi^{\alpha^j}(\cdot)x(\cdot) - y_j(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta_j,\ j = 1,\dots,m,$$
$$\|\varphi^{\alpha^j}(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta_j,\ j = m+1,\dots,N.$$

For $M = \emptyset$, the corresponding sums are considered equal to zero.

Put $z_j(\xi) = \varphi^{\alpha^j}(\xi)x(\xi) - y_j(\xi)$, $j = 1,\dots,m$, and

$$\omega(\xi) = \varphi^{\alpha^0}(\xi) - \sum_{j\in M}\varphi^{\alpha^j}(\xi)a_j(\xi).$$

Then (13) may be written in the form

$$\Big\|\omega(\xi)x(\xi) + \sum_{j\in M} a_j(\xi)z_j(\xi)\Big\|_{L_p(\mathbb{R}^d)} \to \max,$$
$$\|z_j(\xi)\|_{L_p(\mathbb{R}^d)} \leq \delta_j,\ j = 1,\dots,m,$$
$$\|\varphi^{\alpha^j}(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta_j,\ j = m+1,\dots,N.$$

It follows from (7) that

$$\omega(\xi) = \sum_{j\in\{j_1,\dots,j_l\}\setminus M}\varphi^{\alpha^j}(\xi)a_j(\xi)x(\xi).$$

Thus, we have to estimate the value of the following problem

$$\Big\|\sum_{j\in\{j_1,\dots,j_l\}\setminus M}\varphi^{\alpha^j}(\xi)a_j(\xi)x(\xi)+\sum_{j\in M}a_j(\xi)z_j(\xi)\Big\|_{L_p(\mathbb{R}^d)}\to\max,$$
$$\|z_j(\xi)\|_{L_p(\mathbb{R}^d)}\le\delta_j,\ j=1,\dots,m,$$
$$\|\varphi^{\alpha^j}(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)}\le\delta_j,\ j=m+1,\dots,N.$$

Let $1\le p<\infty$. Set

$$\widehat\lambda_{j_s}=\theta_{j_s}e^{p\langle\alpha^{j_s}-\alpha^0,\,\widehat\eta\rangle},\quad s=1,\dots,l,$$

Consider the case when $1<p<\infty$. Then by Hölder's inequality we have

$$\begin{aligned}\Big|\sum_{j\in\{j_1,\dots,j_l\}\setminus M}&\varphi^{\alpha^j}(\xi)a_j(\xi)x(\xi)+\sum_{j\in M}a_j(\xi)z_j(\xi)\Big|\\&=\Big|\sum_{j\in\{j_1,\dots,j_l\}\setminus M}\frac{a_j(\xi)}{\widehat\lambda_j^{1/p}}\widehat\lambda_j^{1/p}\varphi^{\alpha^j}(\xi)x(\xi)+\sum_{j\in M}\frac{a_j(\xi)}{\widehat\lambda_j^{1/p}}\widehat\lambda_j^{1/p}z_j(\xi)\Big|\\&\le Q_p(\xi)\Big(\sum_{j\in\{j_1,\dots,j_l\}\setminus M}\widehat\lambda_j|\varphi(\xi)|^{p\alpha^j}|x(\xi)|^p+\widehat\lambda_j|z_j(\xi)|^p\Big)^{1/p},\end{aligned}$$

where

$$Q_p(\xi)=\left(\sum_{s=1}^l\frac{|a_{j_s}(\xi)|^{p'}}{\widehat\lambda_{j_s}^{p'/p}}\right)^{1/p'}.$$

In view of equalities

$$\ln 1/\delta_{j_s}=\langle\alpha^{j_s},\widehat\eta\rangle+\widehat a,\quad s=1,\dots,l,\quad S(\alpha^0)=\langle\alpha^0,\widehat\eta\rangle+\widehat a$$

we obtain

$$\widehat\lambda_{j_s}=\theta_{j_s}e^{p\langle\alpha^{j_s}-\alpha^0,\,\widehat\eta\rangle}=\theta_{j_s}e^{p(\ln 1/\delta_{j_s}-S(\alpha^0))}=\frac{\theta_{j_s}}{\delta_{j_s}^p}e^{-pS(\alpha^0)}.$$

Hence,

$$Q_p(\xi)=\left(e^{p'S(\alpha^0)}\sum_{s=1}^l\frac{\delta_{j_s}^{p'}|a_{j_s}(\xi)|^{p'}}{\theta_{j_s}^{p'/p}}\right)^{1/p'}.\tag{14}$$

It follows from (8) that $Q_p(\xi)\le 1$ almost for all $\xi\in\mathbb{R}^d$. Therefore, we get

$$\begin{aligned}\Big\|\sum_{j\in\{j_1,\dots,j_l\}\setminus M}&\varphi^{\alpha^j}(\xi)a_j(\xi)x(\xi)+\sum_{j\in M}a_j(\xi)z_j(\xi)\Big\|^p_{L_p(\mathbb{R}^d)}\\ &\le\sum_{j\in\{j_1,\dots,j_l\}\setminus M}\widehat{\lambda}_j\int_{\mathbb{R}^d}\varphi|(\xi)|^{p\alpha^j}|x(\xi)|^p\,d\xi+\sum_{j\in M}\widehat{\lambda}_j\int_{\mathbb{R}^d}|z_j(\xi)|^p\,d\xi\\ &\le\sum_{s=1}^l\widehat{\lambda}_{j_s}\delta_{j_s}^p=\sum_{s=1}^l\theta_{j_s}e^{-pS(\alpha^0)}=e^{-pS(\alpha^0)}.\end{aligned}$$

Consequently,

$$e(\Lambda_0,W_p^{\mathcal{A}}(\mathbb{R}^d,\overline{\delta}),\delta,\Phi)\le e^{-S(\alpha^0)}.$$

Taking into account (12), it means that methods (6) are optimal and equality (5) holds.

If $p=1$ we use the inequality

$$\begin{aligned}\Big|\sum_{j\in\{j_1,\dots,j_l\}\setminus M}&\varphi^{\alpha^j}(\xi)a_j(\xi)x(\xi)+\sum_{j\in M}a_j(\xi)z_j(\xi)\Big|\\ &\le Q_1(\xi)\Big(\sum_{j\in\{j_1,\dots,j_l\}\setminus M}\widehat{\lambda}_j|\varphi(\xi)|^{\alpha^j}|x(\xi)|+\widehat{\lambda}_j|z_j(\xi)|\Big),\end{aligned}$$

where

$$Q_1(\xi)=\max_{1\le s\le l}\frac{|a_{j_s}(\xi)|}{\widehat{\lambda}_{j_s}}.$$

By analogy with (14) we get

$$Q_1(\xi)=e^{S(\alpha^0)}\max_{1\le s\le l}\frac{\delta_{j_s}|a_{j_s}(\xi)|}{\theta_{j_s}}.$$

Using the same arguments as for the case $1<p<\infty$, we obtain the assertion of the theorem in the case under consideration.

For $p=\infty$ we use the inequality

$$\begin{aligned}\Big\|\sum_{j\in\{j_1,\dots,j_l\}\setminus M}&\varphi^{\alpha^j}(\xi)a_j(\xi)x(\xi)+\sum_{j\in M}a_j(\xi)z_j(\xi)\Big\|_{L_\infty(\mathbb{R}^d)}\\ &\le\Big\|\sum_{s=1}^l|a_{j_s}(\xi)|\delta_{j_s}\Big\|_{L_\infty(\mathbb{R}^d)}.\end{aligned}$$

It follows from (8) that

$$e(\Lambda_0,W_\infty^{\mathcal{A}}(\mathbb{R}^d,\overline{\delta}),\delta,\Phi)\le e^{-S(\alpha^0)}.$$

Consequently, using (12), we obtain that methods (6) are optimal and (5) holds.

It remains to show that the set of functions $\alpha_{j_s}(\cdot)$, $s=1,\dots,l$, satisfying conditions (7) and (8) is nonempty. Let $1\le p<\infty$. Consider the function

$$f(\eta)=-1+\sum_{s=1}^{l}\widehat{\lambda}_{j_s}e^{-p\langle\alpha^{j_s}-\alpha^0,\,\eta\rangle},\quad \eta\in\mathbb{R}^k.$$

This is obviously a convex function, and it is easy to verify that $f(\widehat{\eta})=0$ and the derivative of this function at the point $\widehat{\eta}$ is also zero. It follows that $f(\eta)\ge 0$ for all $\eta\in\mathbb{R}^k$. Consequently,

$$f_1(\eta)=e^{-p\langle\alpha^0,\,\eta\rangle}f(\eta)\ge 0$$

for all $\eta\in\mathbb{R}^k$. Putting $e^{-\eta_j}=|\varphi_j(\xi)|$, $j=1,\dots,k$, we obtain that

$$g(\xi)=-|\varphi(\xi)|^{p\alpha^0}+\sum_{s=1}^{l}\widehat{\lambda}_{j_s}|\varphi(\xi)|^{p\alpha^{j_s}}\ge 0$$

for all $\xi\in\mathbb{R}^d$. Thus,

$$\frac{|\varphi(\xi)|^{p\alpha^0}}{\sum_{s=1}^{l}\widehat{\lambda}_{j_s}|\varphi(\xi)|^{p\alpha^{j_s}}}\le 1.$$

Set

$$a_{j_s}(\xi)=\varphi^{\alpha^0}(\xi)\frac{\widehat{\lambda}_{j_s}\varphi^{(p/2-1)\alpha^{j_s}}(\xi)\overline{\varphi}^{p/2\alpha^{j_s}}(\xi)}{\sum_{s=1}^{l}\widehat{\lambda}_{j_s}|\varphi(\xi)|^{p\alpha^{j_s}}},\quad s=1,\dots,l.$$

It is easy to check that condition (7) is valid. If $1<p<\infty$, then

$$\sum_{s=1}^{l}\frac{\delta_{j_s}^{p'}|a_{j_s}(\xi)|^{p'}}{\theta_{j_s}^{p'/p}}=e^{-p'S(\alpha^0)}\sum_{s=1}^{l}\frac{|a_{j_s}(\xi)|^{p'}}{\lambda_{j_s}^{p'/p}}$$
$$=e^{-p'S(\alpha^0)}\left(\frac{|\varphi(\xi)|^{p\alpha^0}}{\sum_{s=1}^{l}\widehat{\lambda}_{j_s}|\varphi(\xi)|^{p\alpha^{j_s}}}\right)^{p'-1}\le e^{-p'S(\alpha^0)}.$$

If $p=1$, then

$$\frac{\delta_{j_s}|a_{j_s}(\xi)|}{\theta_{j_s}}=e^{-S(\alpha^0)}\frac{|\varphi(\xi)|^{\alpha^0}}{\sum_{s=1}^{l}\widehat{\lambda}_{j_s}|\varphi(\xi)|^{\alpha^{j_s}}}\le e^{-S(\alpha^0)},\quad s=1,\dots,l.$$

If $p=\infty$, we set

$$a_{j_s}(\xi)=\varphi^{\alpha^0}(\xi)\frac{\widehat{\lambda}_{j_s}e^{-i\arg\{\varphi^{\alpha^{j_s}}(\xi)\}}}{\sum_{s=1}^{l}\widehat{\lambda}_{j_s}|\varphi(\xi)|^{\alpha^{j_s}}},\quad s=1,\dots,l,$$

where

$$\widehat{\lambda}_{j_s} = \frac{\theta_{j_s}}{\delta_{j_s}} e^{-S(\alpha^0)}, \quad s = 1, \dots, l.$$

Then condition (7) is obviously satisfied, and

$$\sum_{s=1}^{l} |a_{j_s}(\xi)|\delta_{j_s} = e^{-S(\alpha^0)} \frac{|\varphi(\xi)|^{\alpha^0}}{\sum_{s=1}^{l} \widehat{\lambda}_{j_s} |\varphi(\xi)|^{\alpha^{j_s}}} \le e^{-S(\alpha^0)}.$$

□

Note that it follows from the proof of Theorem 1 that

$$\sup_{\substack{x(\cdot)\in W_p^{\mathcal{A}}(\mathbb{R}^d,\overline{\delta}) \\ \|\Lambda_j x(\cdot)\|_{L_p(\mathbb{R}^d)} \le \delta_j,\ j=1,\dots,m}} \|\Lambda_0 x(\cdot)\|_{L_p(\mathbb{R}^d)} = e^{-S(\alpha^0)} = \prod_{s=1}^{l} \delta_{j_s}^{\theta_{j_s}} \tag{15}$$

$$= \min\left\{ \prod_{j=1}^{N} \delta_j^{\theta_j} : \theta_j \ge 0,\ j = 1, \dots, N,\ \sum_{j=1}^{N} \theta_j = 1,\ \alpha^0 = \sum_{j=1}^{N} \theta_j \alpha^j \right\}.$$

## 3. Exact Carlson type inequalities

The Carlson inequality [8]

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \le \sqrt{\pi} \|x(t)\|_{L_2(\mathbb{R}_+)}^{1/2} \|tx(t)\|_{L_2(\mathbb{R}_+)}^{1/2}, \quad \mathbb{R}_+ = [0, +\infty),$$

was generalized by many authors (see [9–15]). In [14] we found exact constants in inequalities of the form

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \le K \|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu))}^{\gamma} \|w_1(\cdot)x(\cdot)\|_{L_r(T,\mu)}^{1-\gamma}, \tag{16}$$

where $T$ is a cone in a linear space, $w(\cdot)$, $w_0(\cdot)$, and $w_1(\cdot)$ are homogenous functions, $\mu$ is a homogenous measure, and $1 \le q < p, r < \infty$ (for $T = \mathbb{R}^d$ the exact inequality was obtained in [12]). Recall that a constant $K$ is called exact if it cannot be replaced by a smaller value. The inequality in this case is called exact.

Weighted inequalities and inequalities for derivatives do not always have a multiplicative form. For example, for analytic and bounded in the strip $S_\beta = \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$ functions $x(t) \not\equiv 0$ the inequality

$$\|x'(\cdot)\|_{C(\mathbb{R})} \le \frac{1}{2\beta} \|x(\cdot)\|_{C(\mathbb{R})} \|x(\cdot)\|_{C(S_\beta)}^2 \times \int_0^{\pi/2} \frac{dt}{\sqrt{\|x(\cdot)\|_{C(S_\beta)}^4 \cos^2 t + \|x(\cdot)\|_{C(\mathbb{R})}^4 \sin^2 t}}$$

holds (see [2, c. 177]).

In this regard, it becomes necessary to use a more general definition of exact inequalities. Let $X$ be a linear space, $Y_0, Y_1, \dots, Y_N$ be normed linear spaces, and $\Lambda_j\colon X \to Y_j$, $j = 0, 1, \dots, N$, be linear operators. We say that the inequality

$$\|\Lambda_0 x\|_{Y_0} \le \kappa(\Lambda x), \quad \Lambda x = (\|\Lambda_1 x\|_{Y_1}, \dots, \|\Lambda_N x\|_{Y_N}), \quad \kappa\colon \mathbb{R}_+^N \to \mathbb{R}_+, \tag{17}$$

is exact, if it is fulfilled for all $x \in X$ and there is no such $x_0 \in X$ for which

$$\sup_{\substack{x\in X\\ \|\Lambda_j x\|_{Y_j} \le \|\Lambda_j x_0\|_{Y_j},\ j=1,\dots,N}} \|\Lambda_0 x\|_{Y_0} < \kappa(\Lambda x_0).$$

**Proposition 1.** *Let $\delta = (\delta_1, \dots, \delta_N) \in \mathbb{R}_+^N$. Set*

$$\kappa(\delta) = \sup_{\substack{x\in X\\ \|\Lambda_j x\|_{Y_j} \le \delta_j,\ j=1,\dots,N}} \|\Lambda_0 x\|_{Y_0}.$$

*Then inequality* (17) *is exact.*

*Proof.* For $x \in X$ we put

$$\delta = (\|\Lambda_1 x\|_{Y_1}, \dots, \|\Lambda_N x\|_{Y_N}).$$

From the definition of $\kappa(\cdot)$ inequality (17) holds. Assume that it is not exact. Then there is an element $x_0 \in X$ for which

$$\sup_{\substack{x\in X\\ \|\Lambda_j x\|_{Y_j} \le \delta_j^0,\ j=1,\dots,N}} \|\Lambda_0 x\|_{Y_0} < \kappa(\delta^0),$$

where

$$\delta^0 = (\delta_1^0, \dots, \delta_N^0) = (\|\Lambda_1 x_0\|_{Y_1}, \dots, \|\Lambda_N x_0\|_{Y_N}).$$

This contradicts the definition of $\kappa(\cdot)$. $\square$

The solution of extremal problem (15) allows us to obtain new exact inequalities of Carlson type.

Let $\varphi(\cdot) = (\varphi_1(\cdot), \dots, \varphi_k(\cdot))$. Assume that $\varphi_1(\cdot), \dots, \varphi_k(\cdot)$ are continuous on $\mathbb{R}^d$ and for any $a_1, \dots, a_k > 0$ there exists $\widehat{\xi} \in \mathbb{R}^d$ for which $|\varphi_j(\widehat{\xi})| = a_j$, $j = 1, \dots, k$. It follows from Proposition 1 and (15) that for $\alpha^0 \in \operatorname{co}\{\alpha^1, \dots, \alpha^N\}$ we have the exact inequality

$$\|\varphi^{\alpha^0}(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)} \le \min\Bigg\{\prod_{j=1}^N \|\varphi^{\alpha^j}(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)}^{\theta_j} : \theta_j \ge 0,\ j = 1, \dots, N,\ \sum_{j=1}^N \theta_j = 1,\ \alpha^0 = \sum_{j=1}^N \theta_j \alpha^j, \Bigg\}.$$

In particular, for $\varphi(\xi)=i\xi$ and $\alpha^0\in\operatorname{co}\{\alpha^1,\ldots,\alpha^N\}$ we obtain the exact inequality

$$\||\xi|^{\alpha^0}x(\xi)\|_{L_p(\mathbb{R}^d)}\le\min\left\{\prod_{j=1}^N\||\xi|^{\alpha^j}x(\xi)\|_{L_p(\mathbb{R}^d)}^{\theta_j}:\right.$$
$$\left.\theta_j\ge0,\ j=1,\ldots,N,\ \sum_{j=1}^N\theta_j=1,\ \alpha^0=\sum_{j=1}^N\theta_j\alpha^j\right\}.$$

Consider one more example. Let

$$\varphi(\xi)=\psi_\theta(\xi)=\left(|\xi_1|^\theta+\ldots+|\xi_d|^\theta\right)^{2/\theta},\quad\theta>0. \tag{18}$$

In this case $k=1$, $Q$ is a convex set on $\mathbb{R}^2$, and $S(\cdot)$ is a broken line. For $\alpha^0\in\operatorname{co}\{\alpha^1,\ldots,\alpha^N\}$ we have the exact inequality

$$\|\psi_\theta^{\alpha^0}(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)}\le\min\left\{\|\psi_\theta^{\alpha^{j_1}}(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)}^{\lambda}\|\psi_\theta^{\alpha^{j_2}}(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)}^{1-\lambda}:\right.$$
$$\left.0\le\lambda\le1,\quad\alpha^0=\lambda\alpha^{j_1}+(1-\lambda)\alpha^{j_2}\right\}.$$

## 4. Recovery of differential operators and exact inequalities

Using notation (2), we set

$$\mathcal{W}_F^{\mathcal{A}}(\mathbb{R}^d)=\left\{\,x(\cdot)\in L_2(\mathbb{R}^d):\varphi^{\alpha^j}(\cdot)Fx(\cdot)\in L_2(\mathbb{R}^d),\ j=1,\ldots,N\,\right\},$$

where $Fx(\cdot)$ is the Fourier trancform of $x(\cdot)$. Define operators $\Lambda_j\colon\mathcal{W}_F^{\mathcal{A}}(\mathbb{R}^d)\to L_2(\mathbb{R}^d)$, $j=0,1,\ldots,N$, as follows

$$\Lambda_jx(\cdot)=F^{-1}(\varphi^{\alpha^j}(\cdot)Fx(\cdot))(\cdot),\quad j=0,1,\ldots,N. \tag{19}$$

Consider problem (1) of optimal recovery of $\Lambda_0$ on the set

$$W_F^{\mathcal{A}}(\mathbb{R}^d)=\{\,x(\cdot)\in\mathcal{W}_F^{\mathcal{A}}(\mathbb{R}^d):\|\Lambda_jx(\cdot)\|_{L_2(\mathbb{R}^d)}\le\delta_j,\ \delta_j>0,$$
$$j=m+1,\ldots,N\,\}$$

by values of $\Lambda_1,\ldots,\Lambda_m$ given with errors.

Passing to Fourier transforms, we have

$$e(\Lambda_0, W_F^{\mathcal{A}}(\mathbb{R}^d), \delta, \Phi) = \frac{1}{(2\pi)^d} \sup_{\substack{x(\cdot)\in W_F^{\mathcal{A}}(\mathbb{R}^d),\ y=(y_1,\dots,y_m)\in(L_2(\mathbb{R}^d))^m \\ \frac{1}{(2\pi)^d}\|\varphi^{\alpha^j}(\cdot)Fx(\cdot)-Fy_j(\cdot)\|_{Y_j}\le\delta_j,\ j=1,\dots,m}} \|\varphi^{\alpha^0}(\cdot)Fx(\cdot) - F(\Phi(y))(\cdot)\|_{L_2(\mathbb{R}^d)}.$$

Putting

$$z(\cdot) = \frac{1}{(2\pi)^d}Fx(\cdot), \quad z_j(\cdot) = \frac{1}{(2\pi)^d}Fy_j(\cdot) \quad j = 1,\dots,m,$$

it is easy to verify that the problem under consideration is reduced to the one considered earlier in Theorem 1 for $p = 2$.

**Theorem 2.** *Let the conditions of Theorem 1 be satisfied with respect to the functions $\varphi_j(\cdot)$, $j = 1,\dots,k$, and $\alpha^0 \in \operatorname{co}\mathcal{A}$. Then*

$$E(\Lambda_0, W_F^{\mathcal{A}}(\mathbb{R}^d), \delta) = e^{-S(\alpha^0)}.$$

*If $M \neq \emptyset$, then all methods of the form*

$$\Phi(y(\cdot))(\cdot) = F^{-1}\Bigg(\sum_{j\in M} a_j(\cdot)Fy_j(\cdot)\Bigg),$$

*where functions $a_{j_s}(\cdot)$, $s = 1,\dots,l$, satisfy the conditions*

$$\sum_{s=1}^{l} \varphi^{\alpha^{j_s}}(\xi)a_{j_s}(\xi) = \varphi^{\alpha^0}(\xi),$$

$$\sum_{s=1}^{l} \frac{\delta_{j_s}^2|a_{j_s}(\xi)|^2}{\theta_{j_s}} \le e^{-2S(\alpha^0)}$$

*for almost all $\xi \in \mathbb{R}^d$, are optimal.*

*If $M = \emptyset$, then the method $\Phi(y(\cdot))(\cdot) = 0$ is optimal.*

*The exact inequality*

$$\|\Lambda_0(\cdot)x(\cdot)\|_{L_2(\mathbb{R}^d)} \le \prod_{s=1}^{l} \|\Lambda_{j_s}(\cdot)x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\theta_{j_s}}$$

*holds.*

For $\varphi(\xi) = i\xi$ the operators $\Lambda_j$ defined by (19) are the Weyl derivatives of orders $\alpha^j$ which are denoted by $D^{\alpha^j}$. Thus, it follows from Theorem 2 the following result.

**Corollary 1** ( [2], Theorem 5.19)**.** *Let* $\alpha^0\in\operatorname{co}\mathcal{A}$. *Then*

$$E(D^{\alpha^0},W_F^{\mathcal{A}}(\mathbb{R}^d),\delta)=e^{-S(\alpha^0)}.$$

*If* $M\ne\emptyset$, *then all methods of the form*

$$\Phi(y(\cdot))(\cdot)=F^{-1}\biggl(\sum_{j\in M}a_j(\cdot)Fy_j(\cdot)\biggr),$$

*where functions* $a_{j_s}(\cdot)$, $s=1,\dots,l$, *satisfy the conditions*

$$\sum_{s=1}^l(i\xi)^{\alpha^{j_s}}a_{j_s}(\xi)=(i\xi)^{\alpha^0},$$

$$\sum_{s=1}^l\frac{\delta_{j_s}^2|a_{j_s}(\xi)|^2}{\theta_{j_s}}\le e^{-2S(\alpha^0)}$$

*for almost all* $\xi\in\mathbb{R}^d$, *are optimal.*

*If* $M=\emptyset$, *then the method* $\Phi(y(\cdot))(\cdot)=0$ *is optimal.*

*The exact inequality*

$$\|D^{\alpha^0}x(\cdot)\|_{L_2(\mathbb{R}^d)}\le\prod_{s=1}^l\|D^{\alpha^{j_s}}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\theta_{j_s}}$$

*holds.*

Set

$$\Lambda_\theta^{\alpha^j}=F^{-1}(\psi_\theta^{\alpha^j}(\cdot)Fx(\cdot))(\cdot),\quad j=0,1,\dots,N,$$

where the function $\psi_\theta(\cdot)$ is defined by (18). Note that $\Lambda_2=-\Delta$, where $\Delta$ is the Laplace operator. In this case $Q$ is the set on $\mathbb{R}^2$ because $k=1$. Consequently, $1\le l\le2$. Assume that $\alpha_0\notin\mathcal{A}$ (otherwise the answer is written out in a trivial way). Then $l=2$.

**Corollary 2.** *Let* $\alpha^0\in\operatorname{co}\mathcal{A}$ *and* $\alpha_0\notin\mathcal{A}$. *Then*

$$E(\Lambda_\theta^{\alpha^0},W_F^{\mathcal{A}}(\mathbb{R}^d),\delta)=e^{-S(\alpha^0)}.$$

*If* $M\ne\emptyset$, *then all methods of the form*

$$\Phi(y(\cdot))(\cdot)=F^{-1}\biggl(\sum_{j\in M}a_j(\cdot)Fy_j(\cdot)\biggr),$$

*where functions* $a_{j_1}(\cdot)$, $a_{j_2}(\cdot)$ *satisfy the conditions*

$$\psi_\theta^{\alpha^{j1}}(\xi)a_{j_1}(\xi)+\psi_\theta^{\alpha^{j2}}(\xi)a_{j_2}(\xi)=\psi_\theta^{\alpha^0}(\xi),$$

$$\frac{\delta_{j_1}^2|a_{j_1}(\xi)|^2}{\theta_1}+\frac{\delta_{j_2}^2|a_{j_2}(\xi)|^2}{1-\theta_1}\le e^{-2S(\alpha^0)}$$

*for almost all* $\xi\in\mathbb{R}^d$, *are optimal.*

*If $M=\emptyset$, then the method $\Phi(y(\cdot))(\cdot)=0$ is optimal.*
*The exact inequality*

$$\|\Lambda_\theta^{\alpha^0}x(\cdot)\|_{L_2(\mathbb{R}^d)}\le\|\Lambda_\theta^{\alpha^{j_1}}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\theta_{j_1}}\|\Lambda_\theta^{\alpha^{j_2}}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\theta_{j_1}}$$

*holds.*

In particular, for $\theta=2$ we obtain the exact inequality

$$\|(-\Delta)^{\alpha^0}x(\cdot)\|_{L_2(\mathbb{R}^d)}\le\|(-\Delta)^{\alpha^{j_1}}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\theta_{j_1}}\|(-\Delta)^{\alpha^{j_2}}x(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\theta_{j_1}}.$$

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