Optimal Recovery of the Derivative of Periodic Analytic Functions from Hardy Classes

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Let $S_\beta := \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$. For 2π-periodic functions which are analytic in S_β with p-integrable boundary values, we construct an optimal method of recovery of $f'(\xi)$, $\xi \in S_\beta$, using information about the values $f(z_1), \dots, f(z_n)$, $z_j \in [0, 2\pi)$.

Introduction

Let X and Y be linear spaces, L a linear functional on X, and $I\colon X \to Y$ a linear operator (which is usually called an information operator). Suppose that $W \subset X$. Consider the problem of the optimal recovery of Lx, $x \in W$, on the basis of the information Ix. The value

$$e(L, W, I) := \inf_F \sup_{x \in W} |Lx - F(Ix)|, \tag{1}$$

where $F\colon Y \to \mathbb{C}$ are any functionals (not necessarily linear or continuous) is called the intrinsic error. A functional F_0 for which

$$\sup_{x \in W} |Lx - F_0(Ix)| = e(L, W, I)$$

is said to be an optimal algorithm or optimal method.

General settings of recovery problems can be found in [3, 4, 13, 2].

Denote by $\mathcal{H}_{p,\beta}$, $1 \le p \le \infty$, the space of all 2π-periodic functions f, which are analytic in $S_\beta := \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$ and satisfy

$$\|f\|_{\mathcal{H}_{p,\beta}} := \sup_{0 \le \eta < \beta} \left(\frac{1}{4\pi} \int_0^{2\pi} (|f(t + i\eta)|^p + |f(t - i\eta)|^p)\, dt \right)^{1/p} < \infty, \quad 1 \le p < \infty,$$

$$\|f\|_{\mathcal{H}_{\infty,\beta}} := \sup_{z \in S_\beta} |f(z)| < \infty.$$

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Set

$$H_{p,\beta} := \{ f \in \mathcal{H}_{p,\beta} : \|f\|_{\mathcal{H}_{p,\beta}} \le 1 \}.$$

We will consider the problem (1) for $W = H_{p,\beta}$, $Lf = f'(\xi)$, $\xi \in S_\beta$, and

$$If = (f(x_1), \ldots, f(x_n)),$$

where x_j are distinct points from $\mathbb{T} := [0, 2\pi)$. In this case we denote the intrinsic error (1) by $e'(\xi, H_{p,\beta}, I)$. From the well-known Smolyak's formula (the complex version of this result was proved in Osipenko [5]) it follows that

$$e'(\xi, H_{p,\beta}, I) = \sup_{\substack{f \in H_{p,\beta} \\ If=0}} |f'(\xi)|. \tag{2}$$

For the unit ball H_p of the Hardy space of nonperiodic functions analytic in the unit disk the analogous problem of optimal recovery was solved in Micchelli, Rivlin [3, 4] ($p = \infty$) and Osipenko, Stessin [8] ($1 \le p < \infty$). The problem of recovery of $f^{(k)}(\xi)$ in H_p was considered in Osipenko [6]. An interesting extremal problem concerning minimization of the intrinsic error by choosing points $x_1, \ldots, x_n$ was studied by Rivlin, Ruscheweyh, Shaffer, Wirths [12]. Several results relating to optimal recovery of $f'(\xi)$, $f \in H_p$, from inaccurate values of f can be found in Osipenko, Stessin [9, 10]. An optimal method of recovery of $f(\xi)$, $f \in H_{p,\beta}$, was recently obtained by Osipenko, Wilderotter [11].

In Section 1 we construct an optimal method of recovery of $f'(\xi)$, $f \in H_{p,\beta}$ and calculate the appropriate intrinsic error. In Section 2 we examine the intrinsic error of optimal recovery for the classes $H_{\infty,\beta}$ and $H_{2,\beta}$ in the case where the values of functions are known at equidistant nodes.

1. Optimal method of recovery

Extremal problems for periodic analytic functions are often solved in terms of elliptic functions (see, for example, [7, 11]). We shall recall some notions from this theory. The Jacobi elliptic function $w = \operatorname{sn}(z, k)$ is defined by the equation

$$z = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

We shall also deal with the elliptic functions

$$\operatorname{cn}(z,k) := \sqrt{1 - \operatorname{sn}^2(z,k)}, \quad \operatorname{dn}(z,k) := \sqrt{1 - k^2 \operatorname{sn}^2(z,k)}$$

($\operatorname{cn}(0,k) = \operatorname{dn}(0,k) = 1$), and complete elliptic integrals of the first kind with moduli k and $k' := \sqrt{1-k^2}$:

$$K := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad K' := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}.$$

We always assume that the modulus k is defined from the equation

$$\frac{\pi K'}{2K} = \beta.$$

It can be shown (see, for example, Akhiezer [1]) that

$$k = 4e^{-\beta}\left(\frac{\sum_{m=0}^{\infty} e^{-2\beta m(m+1)}}{1+2\sum_{m=1}^{\infty} e^{-2\beta m^2}}\right)^2.$$

Henceforth we shall not note the dependence of the Jacobi elliptic functions on the modulus k.

In what follows all expressions with p for $p=\infty$ are considered in their limits as $p \to \infty$.

Set

$$W(z) := k^{n/2}\prod_{j=1}^{n} \operatorname{sn}\frac{K}{\pi}(z-x_j), \quad \omega_j(z) := \prod_{\substack{s=1\\ s\neq j}}^{n} \operatorname{sn}\frac{K}{\pi}(z-x_s).$$

Assume that $\xi \notin \{x_1, \ldots, x_n\}$ and consider the equation

$$\operatorname{sn}\gamma\left(\frac{1}{\operatorname{sn}(\gamma+K)} + k^2\frac{p-2}{p}\operatorname{sn}(\gamma+K)\right) = \frac{\pi}{K}\frac{W'(\xi)}{W(\xi)}. \tag{3}$$

Denote the function in the left hand side of (3) by $s(\gamma)$. Since $s(\gamma)$ is a continuous function in $(-K, K)$ and $s(\gamma) \to \pm\infty$ as $\gamma \to \pm K$ there exists a solution of (3) $\gamma_0 \in (-K, K)$. For $\xi \in \{x_1, \ldots, x_n\}$ put $\gamma_0 = K$.

Set $x_0 := \xi - \pi\gamma_0/K$,

$$w(z) := k \operatorname{sn}\frac{K}{\pi}(z-\xi)\operatorname{sn}\frac{K}{\pi}(z-\xi+\pi),$$

$$T_1 := \left\{\zeta \in \mathbb{T} : \frac{\pi}{2kK}|W'(\zeta)| < \frac{p-1}{p}|W(\zeta)|\right\}, \qquad T_0 := \mathbb{T}\setminus T_1,$$

$$b := \begin{cases} \dfrac{p}{p-1}\,\dfrac{\pi}{2kK}\dfrac{W'(\xi)}{W(\xi)}, & \xi \in T_1,\ n=2m, \\[2ex] \dfrac{W(\xi)\operatorname{sign} W'(\xi)}{\dfrac{\pi}{2kK}|W'(\xi)| + \sqrt{\left(\dfrac{\pi}{2kK}W'(\xi)\right)^2 - \dfrac{p-2}{p}W^2(\xi)}}, & \xi \in T_0,\ n=2m, \\[2ex] -w(x_0), & n = 2m-1, \end{cases}$$

$$u_\xi(z) := \begin{cases} 1, & \xi \in T_1,\ n=2m, \\[1ex] \dfrac{w(z)+b}{1+bw(z)}, & \xi \in T_0,\ n=2m, \\[2ex] \dfrac{w(z)+b}{1+bw(z)}\left(\sqrt{k}\operatorname{sn}\dfrac{K}{\pi}(z-x_0)\right)^{-1}, & n=2m-1. \end{cases}$$

Theorem 1. *For all $1 \le p \le \infty$ the method*

$$f'(\xi) \approx \sum_{j=1}^{n} c_j(\xi) f(x_j),$$

where for $\xi \ne x_j$

$$c_j(\xi) = -\frac{\pi\,\alpha(\xi)}{2k^{n/2+1}K}\,\frac{u_\xi(x_j)(1+bw(x_j))^{\frac{2(p-1)}{p}}\,\mathrm{dn}^{\frac{2(p-1)}{p}}\frac{K}{\pi}(\xi - x_j)}{\omega_j(x_j)\,\mathrm{sn}^2\frac{K}{\pi}(\xi - x_j)},$$

$$c_j(x_j) = \frac{\omega_j'(x_j)}{\omega_j(x_j)},$$

and

$$\alpha(\xi) = \begin{cases} \dfrac{2kK^2}{\pi^2}\dfrac{W(\xi)}{u_\xi(\xi)}, & \xi \notin \{x_1, \dots, x_n\}, \\ \dfrac{2k^{\left[\frac{n+1}{2}\right]}K^2}{\pi^2}\omega_j(x_j), & \xi = x_j,\ j = 1, \dots, n, \end{cases}$$

is an optimal method of recovery on the class $H_{p,\beta}$. Moreover, the following equality holds

$$e'(\xi, H_{p,\beta}, I) = \begin{cases} \dfrac{k}{2}\left(\dfrac{2K}{\pi}\right)^{\frac{p+1}{p}}\dfrac{|W(\xi)|}{|u_\xi(\xi)|}(1+b^2)^{\frac{p-1}{p}}, & \xi \notin \{x_1, \dots, x_n\}, \\ \dfrac{k^{\left[\frac{n+1}{2}\right]}}{2}\left(\dfrac{2K}{\pi}\right)^{\frac{p+1}{p}}|\omega_j(x_j)|, & \xi = x_j. \end{cases}$$

Proof. The function

$$v(z) := \sqrt{k}\,\mathrm{sn}\,\frac{K}{\pi}z$$

is analytic in S_β. Moreover, $v(z+2\pi) = -v(z)$ and $|v(x+i\beta)| \equiv 1$ for all $x \in \mathbb{R}$. Thus $\overline{W(z)} = W^{-1}(z)$ for $z \in \partial S_\beta$. Using the definition of b, it can be shown that $b \in [-1,1]$. Consider the function

$$g(z) := \frac{w(z)+b}{1+bw(z)}\frac{W(z)}{u_\xi(z)}(1+bw(z))^{2/p}\,\mathrm{dn}^{2/p}\frac{K}{\pi}(z-\xi).$$

Since $\mathrm{dn}\,\frac{K}{\pi}z$ and $w(z)$ are 2π-periodic, $|w(z)| < 1$, $z \in S_\beta$, and $\mathrm{dn}\,\frac{K}{\pi}z$ does not vanish in S_β, $g \in \mathcal{H}_{p,\beta}$.

For $f \in H_{p,\beta}$ and $1 \le p < \infty$ set

$$Jf := \frac{\alpha(\xi)}{4\pi}\int_0^{2\pi}\Big(\overline{g(x+i\beta)}|g(x+i\beta)|^{p-2}f(x+i\beta) + \overline{g(x-i\beta)}|g(x-i\beta)|^{p-2}f(x-i\beta)\Big)\,dx. \quad (4)$$

Using the properties of elliptic functions, we have for all $x \in \mathbb{R}$

$$\overline{\operatorname{dn} \frac{K}{\pi}(x \pm i\beta)} = \pm i \frac{\operatorname{cn} \frac{K}{\pi}(x \pm i\beta)}{\operatorname{sn} \frac{K}{\pi}(x \pm i\beta)}. \tag{5}$$

The element of integration in Jf is a 2π-periodic function. Consequently, we can rewrite Jf in the form

$$\begin{aligned} Jf &:= \frac{\alpha(\xi)}{4\pi i} \int_{\Gamma_\varepsilon} \frac{u_\xi(z)(1+bw(z))^{\frac{2(p-1)}{p}}}{W(z)w(z)} \operatorname{dn}^{\frac{p-2}{p}} \frac{K}{\pi}(z-\xi) \frac{\operatorname{cn} \frac{K}{\pi}(z-\xi)}{\operatorname{sn} \frac{K}{\pi}(z-\xi)} f(z)\, dz \\ &= \frac{\alpha(\xi)}{4\pi i} \int_{\Gamma_\varepsilon} \frac{u_\xi(z)(1+bw(z))^{\frac{2(p-1)}{p}} \operatorname{dn}^{\frac{2(p-1)}{p}} \frac{K}{\pi}(z-\xi)}{kW(z) \operatorname{sn}^2 \frac{K}{\pi}(z-\xi)} f(z)\, dz, \end{aligned} \tag{6}$$

where Γ_ε is the boundary of rectangle $-\varepsilon < \operatorname{Re} z < 2\pi - \varepsilon$, $|\operatorname{Im} z| < \beta$, and ε is such that $\xi, x_1, \ldots, x_{2n}$ lie inside this rectangle. Assume that $\xi \notin \{x_1, \ldots, x_{2n}\}$. By the residue theorem

$$Jf = f'(\xi) + Cf(\xi) - \sum_{j=1}^{2n} c_j(\xi) f(x_j),$$

where

$$\begin{aligned} C &= \frac{\alpha(\xi)}{2k} \lim_{z \to \xi} \left(\frac{(z-\xi)^2 u_\xi(z)(1+bw(z))^{\frac{2(p-1)}{p}} \operatorname{dn}^{\frac{2(p-1)}{p}} \frac{K}{\pi}(z-\xi)}{W(z) \operatorname{sn}^2 \frac{K}{\pi}(z-\xi)} \right)' \\ &= \frac{W(\xi)}{u_\xi(\xi)} \left(\frac{u_\xi(z)(1+bw(z))^{\frac{2(p-1)}{p}}}{W(z)} \right)' \Bigg|_{z=\xi}. \end{aligned}$$

It is not hard to check that b is defined from the condition $C = 0$. Thus we have

$$Jf = f'(\xi) - \sum_{j=1}^{n} c_j(\xi) f(x_j). \tag{7}$$

From (4) and Hölder's inequality

$$|Jf| \le |\alpha(\xi)| \|g\|_{\mathcal{H}_{p,\beta}}^{p-1}.$$

Hence

$$e'(\xi, H_{p,\beta}, I) \le |\alpha(\xi)| \|g\|_{\mathcal{H}_{p,\beta}}^{p-1}.$$

On the other hand for $g_0 := g/\|g\|_{\mathcal{H}_{p,\beta}}$ we have $Jg_0 = g_0'(\xi)$. Using equality (2), we obtain

$$e'(\xi, H_{p,\beta}, I) \geq |g_0'(\xi)| = |Jg_0| = |\alpha(\xi)| \|g\|^{p-1}_{\mathcal{H}_{p,\beta}}.$$

Thus

$$e'(\xi, H_{p,\beta}, I) = |\alpha(\xi)| \|g\|^{p-1}_{\mathcal{H}_{p,\beta}}.$$

To calculate $\|g\|_{\mathcal{H}_{p,\beta}}$ substitute $f(z) = g(z)$ in (6)

$$\alpha(\xi)\|g\|_{\mathcal{H}_{p,\beta}} = \frac{\alpha(\xi)}{4\pi i}\int_{\Gamma_\varepsilon} \frac{(w(z)+b)(1+bw(z))\operatorname{dn}^2\dfrac{K}{\pi}(z-\xi)}{k\operatorname{sn}^2\dfrac{K}{\pi}(z-\xi)}\,dz = \alpha(\xi)\frac{\pi}{2K}(1+b^2) \tag{8}$$

(we omit here some technical details concerned with the application of the residue theorem). Consequently,

$$\|g\|_{\mathcal{H}_{p,\beta}} = \left(\frac{\pi}{2K}(1+b^2)\right)^{1/p}$$

and

$$e'(\xi, H_{p,\beta}, I) = \frac{k}{2}\left(\frac{2K}{\pi}\right)^{\frac{p+1}{p}} \frac{|W(\xi)|}{|u_\xi(\xi)|}(1+b^2)^{\frac{p-1}{p}}.$$

If $\xi = x_j$, then $b = 0$ and $g(z) = w(z)W(z)u_\xi^{-1}(z)\operatorname{dn}^{2/p}\dfrac{K}{\pi}(z - x_j)$. In this case the assertion of the theorem can be obtained by the same scheme.

For $p = \infty$ consider the integral

$$Jf := \frac{\alpha(\xi)}{4\pi}\int_0^{2\pi}\Big(\overline{g(x+i\beta)}\varphi(x+i\beta)f(x+i\beta) + \overline{g(x-i\beta)}\varphi(x-i\beta)f(x-i\beta)\Big)\,dx,$$

where

$$\varphi(z) = \left|(1+bw(z))\operatorname{dn}\frac{K}{\pi}(z-\xi)\right|^2.$$

The representation (7) follows from (5) and the residue theorem. We have

$$|Jf| \leq |\alpha(\xi)|\|\varphi\|_{\mathcal{H}_{1,\beta}}.$$

Taking in account that $\varphi(z) \geq 0$, we obtain

$$|g'(\xi)| = |Jg| = |\alpha(\xi)|\|\varphi\|_{\mathcal{H}_{1,\beta}}.$$

Hence

$$e'(\xi, H_{\infty,\beta}, I) = |\alpha(\xi)|\|\varphi\|_{\mathcal{H}_{1,\beta}}.$$

Using the residue theorem, by analogy with (8) we obtain

$$\|\varphi\|_{\mathcal{H}_{1,\beta}} = \frac{\pi}{2K}(1+b^2). \quad \blacksquare$$

Let us consider our problem in the case when $\xi = 0$ and

$$If = I_h f := (f(-h), f(h)), \quad h \in (0, \pi).$$

In other words we wish to construct an optimal formula of numerical differentiation at the point $\xi = 0$, using the information about values of function at the points $\pm h$.

In this particular case we have

$$W(z) = k \operatorname{sn} \frac{K}{\pi}(z+h) \operatorname{sn} \frac{K}{\pi}(z-h), \quad W(0) = -k \operatorname{sn}^2 \frac{K}{\pi} h, \quad W'(0) = 0.$$

Moreover, $0 \in T_1$ and $b = 0$. Thus we obtain that an optimal method has the form

$$f'(0) \approx \frac{K}{\pi} \frac{f(h) - f(-h)}{\operatorname{sn} \dfrac{2K}{\pi} h} \operatorname{dn}^{\frac{2(p-1)}{p}} \frac{K}{\pi} h$$

and

$$e'(0, H_{p,\beta}, I_h) = \frac{k^2}{2} \left(\frac{2K}{\pi}\right)^{\frac{p+1}{p}} \operatorname{sn}^2 \frac{K}{\pi} h = k^2 2^{1/p} \left(\frac{K}{\pi}\right)^{\frac{3p+1}{p}} h^2 + O(h^4).$$

2. Optimal recovery using an equidistant system of points

For optimal recovery of periodic functions the most natural system of points is an equidistant system. We will estimate the error of optimal recovery of the derivative from the information

$$If = I^{(2n)} f := (f(t_1^0), \ldots, f(t_{2n}^0)),$$

where

$$t_j^0 = (j-1)\frac{\pi}{n}, \quad j = 1, \ldots, 2n.$$

Set

$$e'_{2n}(H_{p,\beta}) := \sup_{\xi \in \mathbb{T}} e'(\xi, H_{p,\beta}, I^{(2n)}).$$

Theorem 2. *For all $\beta > 0$*

$$e'_{2n}(H_{\infty,\beta}) = \sqrt{\lambda} \frac{2n\Lambda}{\pi} = 2n e^{-\beta n} + O(n e^{-5\beta n}),$$

$$e'_{2n}(H_{2,\beta}) = \sqrt{\frac{2K\lambda}{\pi}} \frac{2n\Lambda}{\pi} = \sqrt{\frac{2K}{\pi}} 2n e^{-\beta n} + O(n e^{-5\beta n}),$$

where

$$\lambda = 4 e^{-2\beta n} \left(\frac{\sum_{m=0}^{\infty} e^{-4\beta n m(m+1)}}{1 + 2\sum_{m=1}^{\infty} e^{-4\beta n m^2}} \right)^2$$

and Λ is the complete elliptic integral of the first kind for modulus λ.

Proof. Using the first principal transform of elliptic functions of degree $2n$ (see [1]), we find

$$\begin{aligned} W\left(z-\frac{\pi}{2n}\right) &= k^n \prod_{j=1}^{2n} \operatorname{sn}\left(\frac{K}{\pi}z-\frac{2j-1}{2n}K\right) \\ &= (-1)^n k^n \prod_{j=1}^{n} \operatorname{sn}\left(\frac{K}{\pi}z-\frac{2j-1}{2n}K\right) \operatorname{sn}\left(\frac{K}{\pi}z+\frac{2j-1}{2n}K\right) \\ &= k^n \prod_{j=1}^{n} \frac{\operatorname{sn}^2 \frac{2j-1}{2n}K - \operatorname{sn}^2 \frac{K}{\pi}z}{1-k^2\operatorname{sn}^2 \frac{2j-1}{2n}K \operatorname{sn}^2 \frac{K}{\pi}z} = \sqrt{\lambda}\operatorname{sn}\left(\frac{2n\Lambda}{\pi}z+\Lambda,\lambda\right). \end{aligned}$$

Hence

$$W(z) = -\sqrt{\lambda}\operatorname{sn}\left(\frac{2n\Lambda}{\pi}z,\lambda\right).$$

In view of the equalities

$$\frac{d}{dt}\operatorname{sn}(t,\lambda) = \operatorname{cn}(t,\lambda)\operatorname{dn}(t,\lambda) = \sqrt{(1-\operatorname{sn}^2(t,\lambda))(1-\lambda^2\operatorname{sn}^2(t,\lambda))},$$

from Theorem 1 we obtain

$$e'_{2n}(H_{p,\beta}) = \sup_{s\in[0,1]} \frac{k}{2}\left(\frac{2K}{\pi}\right)^{\frac{1+p}{p}} \sqrt{\lambda}\Phi_p(s),$$

where

$$\Phi_p(s) = \begin{cases} s\left(1+\left(\dfrac{pa}{p-1}\right)^2 \dfrac{(1-s^2)(1-\lambda^2 s^2)}{s^2}\right)^{\frac{p-1}{p}}, & s\in S_p \\ \gamma(s)\left(1+\dfrac{s^2}{\gamma^2(s)}\right)^{\frac{p-1}{p}}, & s\in[0,1]\setminus S_p \end{cases}$$

$$a = \frac{n\Lambda}{kK}, \quad S_p = \left\{s\in[0,1] : a^2(1-s^2)(1-\lambda^2 s^2) < \left(\frac{p-1}{p}\right)^2 s^2\right\},$$

$$\gamma(s) = a\sqrt{(1-s^2)(1-\lambda^2 s^2)} + \sqrt{a^2(1-s^2)(1-\lambda^2 s^2) - \frac{p-2}{p}s^2}.$$

Let us begin with the case $p = 2$. It is easy to check that

$$\Phi_2(s) = \sqrt{s^2+4a^2(1-s^2)(1-\lambda^2 s^2)}.$$

From properties of the first principal transformations of elliptic functions of degree $2n$ it follows that

$$\frac{2n\Lambda}{K} = \prod_{j=1}^{n} \frac{\operatorname{sn}^2\left(\frac{j}{n}K\right)}{\operatorname{sn}^2\left(\frac{2j-1}{2n}K\right)} > 1. \tag{9}$$

Hence $2a > 1$ and

$$\Phi_2^2(s) \le s^2 + 4a^2(1-s^2) \le 4a^2.$$

This estimate is attained for $s = 0$. Thus

$$e'_{2n}(H_{p,\beta}) = \sqrt{\frac{2K}{\pi}\sqrt{\lambda}\frac{2n\Lambda}{\pi}}.$$

The asymptotic equality follows from the equations

$$\sqrt{\lambda} = 2e^{-\beta n} + O\left(e^{-5\beta n}\right),$$
$$\Lambda = \frac{\pi}{2} + O\left(e^{-4\beta n}\right).$$

Let $p = \infty$. It can be easily shown that $S_\infty = (s^*, 1]$ where s^* is the unique solution of the equation

$$a^2(1-s^2)(1-\lambda^2 s^2) = s^2.$$

We have

$$\Phi_\infty(s) = \begin{cases} 2a\sqrt{(1-s^2)(1-\lambda^2 s^2)}, & s \in [0, s^*] \\ s + a^2\dfrac{(1-s^2)(1-\lambda^2 s^2)}{s}, & s \in (s^*, 1] \end{cases}$$

Since the function

$$F(s) := s + a^2\frac{(1-s^2)(1-\lambda^2 s^2)}{s}$$

is convex for $s \in (0,1)$ we obtain

$$\max_{s\in[s^*,1]} F(s) = \max\{F(s^*), F(1)\} = \max\{\Phi_\infty(s^*), 1\}.$$

The function $\Phi_\infty(s)$ decreases while $s \in [0, s^*]$. Consequently

$$\max_{s\in[0,1]} \Phi_\infty(s) = \max\{\Phi_\infty(0), 1\} = 2a. \quad \blacksquare$$

References

[1] N. I. Akhiezer, "Elements of the Theory of Elliptic Functions," Nauka, Moscow, 1970.

[2] G. G. Magaril-Il'yaev and K. Yu. Osipenko, Optimal recovery of functionals based on inaccurate data, *Mat. Zametki* **50** (1991), 85–93; English translation *Math. Notes* **50** (1991), 1274–1279.

[3] C. A. Micchelli and T. J. Rivlin, A survey of optimal recovery, *in* "Optimal Estimation in Approximation Theory" (C. A. Micchelli and T. J. Rivlin, Eds.), pp. 1–54, Plenum Press, New York, 1977.

[4] C. A. Micchelli and T. J. Rivlin, "Lectures on Optimal Recovery," Lecture Notes in Math., Vol. 1129, pp. 21–93, Springer–Verlag, New York/Berlin, 1985.

[5] K. Yu. Osipenko, Best approximation of analytic functions from information about their values at a finite number of points, *Mat. Zametki* **19**, (1976), 29–40; English translation *Math. Notes* **19**, (1976), 17–23.

[6] K. Yu. Osipenko, Caratheodory–Fejer problem and optimal recovery of derivatives in Hardy spaces, *Mat. Sb.* **185** (1994), 27–42; English translation *Russian Acad. Sci. Sb. Math.* **81**, (1995), 21–33.

[7] K. Yu. Osipenko, Exact n-widths of Hardy-Sobolev classes, *Constr. Approx.* **13** (1997), 17–27.

[8] K. Yu. Osipenko and M.I. Stessin, On problems of recovery in Hardy and Bergman spaces, *Mat. Zametki* **49** (1991), 95–104; English translation *Math. Notes* **49** (1991), 395–401.

[9] K. Yu. Osipenko and M.I. Stessin, On some problem of optimal recovery of analytic and harmonic functions from inaccurate data, *J. Approx. Theory* **70** (1992), 206–228.

[10] K. Yu. Osipenko and M.I. Stessin, Optimal recovery of derivatives of bounded analytic and harmonic functions from inaccurate data, *Mat. Zametki* **53** (1993), 87–97; English translation *Math. Notes* **53** (1993), 513–520.

[11] K. Yu. Osipenko and K. Wilderotter, Optimal information for approximating periodic analytic functions, *Math. Comput.* **66** (1997), 1579–1592.

[12] T. J. Rivlin, St. Ruscheweyh, D. Shaffer and K. J. Wirths, Optimal recovery of the derivative of bounded analytic functions, *IMA J. of Numer. Anal.* **3** (1983), 327–332.

[13] J. F. Traub and H. Woźniakowski, "A General Theory of Optimal Algorithms," Academic Press, New York, 1980.