OPTIMAL RECOVERY OF VALUES OF FUNCTIONS AND THEIR DERIVATIVES ON THE LINE BY INACCURATE DATA ABOUT THE FOURIER TRANSFORM

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Abstract. The problems of the optimal recovery of derivatives of functions from the information about the Fourier transform of these functions given inaccurately on the finite interval or on the whole line are considered. The problem of S. B. Stechkin about approximation of derivatives by bounded linear functionals which is closely connected with these problems are also studied. The exact Kolmogorov-type inequalities for derivatives corresponding to these settings are obtained.

1. Statement of the problems

We begin with the formulation of concrete problems which are studied in this paper and then give the general statement of the optimal recovery problem for functionals combining these problems. Let S be the Schwartz space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}$, S' the corresponding space of distributions, $F\colon S'\to S'$ the Fourier transform, $n\in\mathbb{N}$, and $1\le p\le\infty$. Set

$$X_p^n=\{\,x\in S'\mid Fx(\cdot)\in L_p(\mathbb{R}),\ x^{(n)}(\cdot)\in L_2(\mathbb{R})\,\}$$

and

$$C_p^n=\{\,x(\cdot)\in X_p^n\mid \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}\le 1\,\}.$$

The optimal recovery problem of $x^{(k)}(\tau)$ where $0\le k<n$, $\tau\in\mathbb{R}$, on the class C_p^n by the information about the Fourier transform $Fx(\cdot)$ given on the interval $\Delta_\sigma=(-\sigma,\sigma)$, $0<\sigma\le\infty$, with the error $\delta>0$ in the metric $L_p(\Delta_\sigma)$ is to find the value

$$E_p(n,k,\sigma,\delta)=\inf_{\varphi}\sup_{\substack{x(\cdot)\in C_p^n,\ y(\cdot)\in L_p(\Delta_\sigma)\\ \|Fx(\cdot)-y(\cdot)\|_{L_p(\Delta_\sigma)}\le\delta}}|x^{(k)}(\tau)-\varphi(y(\cdot))| \tag{1}$$

(where the infimum is taken over all functions $\varphi\colon L_p(\Delta_\sigma)\to\mathbb{C}$), which is called the *error of optimal recovery*, and a function $\widehat{\varphi}$, delivering the lower bound in (1), which is called an *optimal method of recovery.*

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In this paper we also study the problem of best approximation of $x^{(k)}(\tau)$, $0 \le k < n$, $\tau \in \mathbb{R}$, on the class C_p^n by the information about the Fourier transform $Fx(\cdot)$ given on the interval $\Delta_\sigma = (-\sigma, \sigma)$ by linear continuous functionals on $L_p(\Delta_\sigma)$ with the norm not greater than some fixed positive number N. It is in finding the value

$$e_p(n,k,\sigma,N) = \inf_{y^*} \sup_{x(\cdot)\in C_p^n} |x^{(k)}(\tau) - \langle y^*, Fx(\cdot)\rangle| \tag{2}$$

(where the lower bound is taken over all linear functionals y^* on $L_p(\Delta_\sigma)$ such that $\|y^*\| \le N$), and also a functional $\widehat{y}^*$ delivering the lower bound in (2) which is called *extremal.*

If we put $x(\cdot)$ in (2) instead of $Fx(\cdot)$ then we obtain the classical problem of S. B. Stechkin, so (2) is its generalization which we also call the problem of Stechkin.

Now we give the general setting of the optimal recovery problem of a linear functional on a class of elements by some information about elements themselves. Let X be a real or complex linear space and C a nonempty subset (a class of elements) of X. For every element $x \in C$ we have available the information $I(x)$ where I is an *information* map from C into another real or complex linear space Y. In the case when the information is given inaccurately I is a multivalued map. Let x' be a given linear functional on X and Φ a set of functions $\varphi\colon Y \to \mathbb{R}(\mathbb{C})$. The problem of the optimal recovery of the functional x' on the class C by the information I using functions (*methods of recovery*) from Φ is in finding the value

$$E(x', C, I) = \inf_{\varphi\in\Phi} \sup_{\substack{x\in C,\\ y\in I(x)}} |\langle x', x\rangle - \varphi(y)|, \tag{3}$$

which is called the *error of optimal recovery* (of the functional x' on C by the information I) and a method delivering the lower bound in (3) which is called an *optimal recovery method.*

The problems (1) and (2) are contained in this general scheme. In the first case $X = X_p^n$, $C = C_p^n$, $Y = L_p(\Delta_\sigma)$, $I\colon X_p^n \to L_p(\Delta_\sigma)$, $Ix(\cdot) = Fx(\cdot)|_{\Delta_\sigma} + \delta BL_p(\Delta_\sigma)$ ($BL_p(\Delta_\sigma)$ is the unit ball of $L_p(\Delta_\sigma)$), $\langle x', x(\cdot)\rangle = x^{(k)}(\tau)$, and Φ is the set of all functions on $L_p(\Delta_\sigma)$.

For the second problem $X = X_p^n$, $C = C_p^n$, $Y = L_p(\Delta_\sigma)$, $I\colon X_p^n \to L_p(\Delta_\sigma)$, $Ix(\cdot) = Fx(\cdot)|_{\Delta_\sigma}$, $\langle x', x(\cdot)\rangle = x^{(k)}(\tau)$, and $\Phi = NBY^*$, where BY^* is the unit ball of the dual space of Y.

The problem of optimal recovery of a linear functional on a class of elements for the case when I is a linear map, $\dim Y < \infty$, and Φ is the set of all functions from Y to $\mathbb{R}$ was stated by S. A. Smolyak [1]. In this situation he proved that if C is a convex centrally-symmetric set, then there exists a linear method among optimal methods. Further this problem were generalized and developed in several directions (see [2]–[7]).

The problems of optimal recovery of functions and their derivatives in the L_2 metric (that is, the problem of optimal recovery of an operator and not a functional) by inaccurate Fourier coefficients (for periodic functions) and by inaccurate Fourier transform (for functions defined on the line) were studied in [8], [9]. The range of problems connected with Stechkin's problem was elucidated in the survey paper [10].

In the periodic case for $p=\infty$ an analogue of the problem (1) was considered in [11].

2. Statement of the main results

In view of the translation invariance of the classes under consideration throughout what follows we assume that $\tau=0$. We start with the case when $p=\infty$.

Theorem 1. *Let* $\delta>0$, $k,n\in\mathbb{Z}$, $0\le k<n$, $0<\sigma\le\infty$,

$$\widehat{\sigma}=\left(\frac{\pi(2n+1)(2n-2k-1)}{2\delta^2(2n-k)}\right)^{\frac{1}{2n+1}},$$

and $\sigma_0=\min(\sigma,\widehat{\sigma})$. *Then*

$$E_\infty(n,k,\sigma,\delta)=\frac{\sigma_0^{k+1}}{\pi}\left(\frac{\delta}{k+1}+\sqrt{\frac{1}{2n-2k-1}\left(\frac{\pi}{\sigma_0^{2n+1}}-\frac{\delta^2}{2n+1}\right)}\right)$$

and the method

$$\widehat{\varphi}(y(\cdot))=\frac{1}{2\pi}\int_{|t|<\sigma_0}(it)^k\left(1-\delta\lambda|t|^{2n-k}\right)y(t)\,dt, \tag{4}$$

where

$$\lambda=\frac{\sigma_0^{-2n+k}}{\sqrt{2n-2k-1}}\left(\frac{\pi}{\sigma_0^{2n+1}}-\frac{\delta^2}{2n+1}\right)^{-1/2}, \tag{5}$$

is optimal.

It follows by Theorem 1 that for $\sigma\ge\widehat{\sigma}$

$$E_\infty(n,k,\sigma,\delta)=K\delta^{\frac{2n-2k-1}{2n+1}}, \tag{6}$$

where

$$K=\frac{(n+1/2)^{\frac{k+1}{2n+1}}}{k+1}\left(\frac{2n-k}{\pi(2n-2k-1)}\right)^{\frac{2n-k}{2n+1}}. \tag{7}$$

Thus in the problem under consideration the "saturation" effect of the optimal recovery error is occured which is in the fact that for a fixed $\delta>0$ the knowledge of the Fourier transform of a function from C_∞^n given with the error δ in the uniform metric on the intervals larger than $(-\widehat{\sigma},\widehat{\sigma})$ does not result in a decrease in the optimal recovery error. Thus the violation of the relation

$$\delta^2\sigma^{2n+1}\le\frac{\pi(2n+1)(2n-2k-1)}{2(2n-k)} \tag{8}$$

leads to the fact that the available information turns out to be redundant. This fact is apparently important in practical applications when we have to take into account that obtaining the additional information requires some expense.

In view of the translation invariance of the space X^n_∞ it follows from (6) the following result.

Corollary 1. *Let $k, n \in \mathbb{Z}$ and $0 \le k < n$. Then we have the sharp inequality*

$$\|x^{(k)}(\cdot)\|_{L_\infty(\mathbb{R})} \le K \|Fx(\cdot)\|_{L_\infty(\mathbb{R})}^{\frac{2n-2k-1}{2n+1}} \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2k+2}{2n+1}},$$

where the constant K is defined by the equation (7).

We proceed now to the problem (1) for $p = 1$. If $k > 0$ and $\sigma < \infty$, we set

$$\Phi(\varepsilon) = \frac{4\pi}{\sigma^{2n-1}} \frac{\varepsilon^{2(2n-k-1)} \left(\displaystyle\int_1^\varepsilon (x^k - 1) x^{-2n}\, dx \right)^2}{\varepsilon^{2n-2k-1} \displaystyle\int_1^\varepsilon (x^k - 1)^2 x^{-2n}\, dx + (2n - 2k - 1)^{-1}}$$

(here and throughout what follows for brevity we do not give the expressions for the integrals which can be explicitly calculated). Obviously, the function $\Phi(\cdot)$ is continuous on $(1, +\infty)$. It is not difficult to verify that $\Phi(\varepsilon) \to 0$ as $\varepsilon \to 1$ and $\Phi(\varepsilon) \to +\infty$ as $\varepsilon \to +\infty$. Thus, for any $\delta > 0$ the equation

$$\Phi(\varepsilon) = \delta^2 \tag{9}$$

has a solution from the interval $(1, +\infty)$.

Theorem 2. *Let $\delta > 0$, $k, n \in \mathbb{N}$, $0 < k < n$, $0 < \sigma \le \infty$. Set*

$$a = \begin{cases} \sigma/\varepsilon_\delta, & 0 < \sigma < \infty, \\ \left(\dfrac{2\pi(2n - 2k - 1)}{\delta^2 (2n - 1)(2n - k - 1)} \right)^{\frac{1}{2n-1}}, & \sigma = \infty, \end{cases}$$

$$\lambda = \begin{cases} \dfrac{2}{\delta} \left(\dfrac{\varepsilon_\delta}{\sigma} \right)^{2n-k-1} \displaystyle\int_1^{\varepsilon_\delta} (x^k - 1) x^{-2n}\, dx, & 0 < \sigma < \infty, \\ \dfrac{k \delta^{\frac{2n-2k-1}{2n-1}}}{\pi(2n - 2k - 1)} \left(\dfrac{2\pi(2n - 2k - 1)}{(2n - 1)(2n - k - 1)} \right)^{\frac{k}{2n-1}}, & \sigma = \infty, \end{cases}$$

where ε_δ is a solution of (9). *Then*

$$E_1(n, k, \sigma, \delta) = \lambda + \frac{\delta}{2\pi} a^k$$

and the method

$$\widehat{\varphi}(y(\cdot)) = \frac{1}{2\pi} \int_{|t| < \sigma} \mu_\delta(t) y(t)\, dt, \tag{10}$$

where

$$\mu_\delta(t)=\begin{cases}(it)^k, & |t|\le a,\\ (ia)^k\operatorname{sign}t^k, & a<|t|<\sigma,\end{cases}\tag{11}$$

is optimal. For $k=0$

$$E_1(n,0,\sigma,\delta)=\frac{\delta}{2\pi}+\frac{1}{\sigma^{n-1/2}\sqrt{\pi(2n-1)}}$$

and the method

$$\widehat{\varphi}(y(\cdot))=\frac{1}{2\pi}\int_{|t|<\sigma}y(t)\,dt\tag{12}$$

is optimal.

For $\sigma=\infty$ it follows from Theorem 2

Corollary 2. *Let* $k,n\in\mathbb{Z}$ *and* $0\le k<n$*. Then the sharp inequality*

$$\|x^{(k)}(\cdot)\|_{L_\infty(\mathbb{R})}\le K_1\|Fx(\cdot)\|_{L_1(\mathbb{R})}^{\frac{2n-2k-1}{2n-1}}\|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2k}{2n-1}}$$

holds, where

$$K_1=\frac{1}{(2n-k-1)^{\frac{k}{2n-1}}}\left(\frac{2n-1}{2\pi(2n-2k-1)}\right)^{\frac{2n-k-1}{2n-1}}.$$

We proceed now to the case $p=2$. If $\sigma<\infty$, then put

$$\Psi(h)=\frac{2\pi\sigma^{2n-2k-1}h^{4n-2k-1}\displaystyle\int_0^{\sigma h}x^{2k}(1+x^{2n})^{-2}\,dx}{(\sigma h)^{2n-2k-1}\displaystyle\int_0^{\sigma h}x^{2(n+k)}(1+x^{2n})^{-2}\,dx+(2n-2k-1)^{-1}}.$$

It is easy to verify that the function $\Psi(\cdot)$ is continuous on $(0,+\infty)$, $\Psi(h)\to0$ as $h\to0$ and $\Phi(h)\to+\infty$ as $h\to+\infty$. Thus, for any $\delta>0$ the equation

$$\Psi(h)=\delta^2\tag{13}$$

has a solution from the interval $(0,+\infty)$.

Theorem 3. *Let* $\delta>0$*,* $k,n\in\mathbb{Z}$*,* $0\le k<n$*,* $0<\sigma\le\infty$*. If* $0<\sigma<\infty$*, then denote by* h_δ *a solution of the equation* (13)*, and if* $\sigma=\infty$*, then put*

$$h_\delta=\left(\frac{(2k+1)\delta^2}{2\pi(2n-2k-1)}\right)^{\frac{1}{2n}}.$$

Then

$$E_2(n,k,\sigma,\delta)=\frac{\delta^2+2\pi h_\delta^{2n}}{2\pi\delta h_\delta^{k+1/2}}\left(2\int_0^{\sigma h_\delta}x^{2k}(1+x^{2n})^{-2}\,dx\right)^{1/2}$$

and the method

$$\widehat{\varphi}(y(\cdot)) = \frac{1}{2\pi}\int_{|t|<\sigma}\frac{(it)^k}{1+(h_\delta t)^{2n}}y(t)\,dt$$

is optimal.

For $\sigma=\infty$ we have

$$E_2(n,k,\infty,\delta) = K_2\delta^{\frac{2n-2k-1}{2n}},$$

where

$$K_2 = \left((2k+1)\sin\pi\frac{2k+1}{2n}\right)^{-1/2}\left(\frac{2k+1}{2\pi(2n-2k-1)}\right)^{\frac{2n-2k-1}{4n}}.$$

Hence the sharp inequality

$$\|x^{(k)}(\cdot)\|_{L_\infty(\mathbb{R})} \le K_2\|Fx(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2n-2k-1}{2n}}\|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2k+1}{2n}}$$

holds. In view of Parseval's equality

$$\|Fx(\cdot)\|_{L_2(\mathbb{R})} = \sqrt{2\pi}\|x(\cdot)\|_{L_2(\mathbb{R})}$$

it can be written in the form

$$\|x^{(k)}(\cdot)\|_{L_\infty(\mathbb{R})} \le (2\pi)^{\frac{2n-2k-1}{4n}}K_2\|x(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2n-2k-1}{2n}}\|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2k+1}{2n}}.$$

This inequality was proved by L. V. Taikov [12].

We proceed now to the Stechkin problem of approximation of the k-th derivative of a function from the class C_p^n by the information about its Fourier transform on the interval Δ_σ by using linear functionals which have the norm not greater than a fixed positive number N.

Theorem 4. *Let $n,k\in\mathbb{Z}$, $0\le k<n$, $0<\sigma\le\infty$, $N>0$,*

$$\widehat{\sigma}_N = \left(\frac{\pi N(k+1)(2n+1)}{2n-k}\right)^{\frac{1}{k+1}},$$

and $\sigma_N=\min(\sigma,\widehat{\sigma}_N)$. Then

$$e_\infty(n,k,\sigma,N) = \frac{\sigma_N^{-n+k+1/2}}{\sqrt{\pi}}\sqrt{\frac{1}{2n-2k-1}+\frac{\gamma^2(\sigma,N)}{2n+1}}, \tag{14}$$

where

$$\gamma(\sigma,N) = \max\left\{0,(2n+1)\left(\frac{1}{k+1}-\frac{\pi N}{\sigma_N^{k+1}}\right)\right\},$$

and the functional

$$\langle\widehat{y}^*,Fx(\cdot)\rangle = \frac{1}{2\pi}\int_{|t|<\sigma_N}(it)^k\left(1-\gamma(\sigma,N)\left(\frac{|t|}{\sigma_N}\right)^{2n-k}\right)Fx(t)\,dt \tag{15}$$

is extremal.

It follows from Theorem 4 that for a fixed N for $\sigma \geq \widehat{\sigma}_N$ we have

$$e_\infty(n,k,\sigma,N) = \sqrt{\frac{2k+2}{2n-2k-1}}\left(\frac{2n-k}{\pi(k+1)(2n+1)}\right)^{\frac{2n-k}{2k+2}} N^{-\frac{2n-2k-1}{2k+2}}.$$

It means that analogously to the recovery problem in Stechkin's problem under consideration the "saturation" effect which is in the fact that for a fixed N the knowledge of the Fourier transform on an interval larger than $(-\widehat{\sigma}_N, \widehat{\sigma}_N)$ does not result in a decrease in the error $e_\infty(n,k,\sigma,N)$ is also observed. Here the analogue of the relation (8) is the inequality

$$\frac{\sigma^{k+1}}{N} \leq \frac{\pi(k+1)(2n+1)}{2n-k},$$

which in the case of violation leads to the redundancy of the obtained information about the Fourier transform.

Theorem 5. *Let $n,k \in \mathbb{N}$, $0<k<n$, and $N>0$. Then for $\sigma < \infty$*

$$e_1(n,k,\sigma,N) = \frac{1}{\sqrt{\pi}\sigma^{n-k-1/2}} \times \sqrt{\frac{2k^2\varepsilon^{2n-2k-1}}{(2n-2k-1)(2n-k-1)(2n-1)} + \frac{2\varepsilon^{-k}}{2n-k-1} - \frac{\varepsilon^{-2k}}{2n-1}},$$

where

$$\varepsilon = \frac{\sigma}{(2\pi N_0)^{1/k}}, \qquad N_0 = \min\left\{N, \frac{\sigma^k}{2\pi}\right\},$$

and the functional

$$(16)\quad \langle \widehat{y}^*, Fx(\cdot)\rangle = \frac{1}{2\pi}\int_{|t|<(2\pi N_0)^{1/k}} (it)^k Fx(t)\,dt + N_0 i^k \int_{(2\pi N_0)^{1/k}\leq |t|<\sigma} \operatorname{sign} t^k Fx(t)\,dt$$

is extremal. For $\sigma = \infty$

$$e_1(n,k,\infty,N) = \frac{\sqrt{2}k(2\pi N)^{-\frac{2n-2k-1}{2k}}}{\sqrt{\pi(2n-1)(2n-k-1)(2n-2k-1)}}$$

and the functional

$$\langle \widehat{y}^*, Fx(\cdot)\rangle = \frac{1}{2\pi}\int_{|t|<(2\pi N)^{1/k}} (it)^k Fx(t)\,dt + N\int_{|t|\geq(2\pi N)^{1/k}} i^k \operatorname{sign} t^k Fx(t)\,dt$$

is extremal. If $k=0$, then

$$e_1(n,0,\sigma,N)=\begin{cases}\infty, & 0<N<\dfrac{1}{2\pi},\\ \dfrac{1}{\sigma^{n-1/2}\sqrt{\pi(2n-1)}}, & N\ge\dfrac{1}{2\pi},\ \sigma<\infty,\\ 0 & N\ge\dfrac{1}{2\pi},\ \sigma=\infty,\end{cases}$$

moreover, the functional

$$\langle\widehat{y}^*,Fx(\cdot)\rangle=\frac{1}{2\pi}\int_{|t|<\sigma}Fx(t)\,dt$$

is extremal.

Consider now the Stechkin problem for the case when $p=2$. Set

$$\Omega(h)=\frac{1}{2\pi^2h^{2k+1}}\int_0^{\sigma h}x^{2k}(1+x^{2n})^{-2}\,dx.$$

The function $\Omega(h)$ is continuous for $h\in(0,+\infty)$. Moreover,

$$\lim_{h\to0}\Omega(h)=\frac{\sigma^{2k+1}}{2\pi^2(2k+1)},$$

and $\lim_{h\to\infty}\Omega(h)=0$. Therefore, for all

$$0<N<\frac{\sigma^{k+1/2}}{\pi\sqrt{2(2k+1)}}$$

the equation

$$\Omega(h)=N^2 \tag{17}$$

has a solution.

Theorem 6. *Let $n,k\in\mathbb{Z}$, $0\le k<n$, and $N>0$. Then for all $0<\sigma<\infty$*

$$e_2(n,k,\sigma,N)=\left(\frac{\widehat{h}_N^{2n-2k-1}}{\pi}\int_0^{\sigma\widehat{h}_N}\frac{x^{2(k+n)}}{(1+x^{2n})^2}\,dx+\frac{\sigma^{-(2n-2k-1)}}{\pi(2n-2k-1)}\right)^{1/2},$$

where

$$\widehat{h}_N=\begin{cases}h_N, & 0<N<\widehat{N},\\ 0, & N\ge\widehat{N},\end{cases}\qquad \widehat{N}=\frac{\sigma^{k+1/2}}{\pi\sqrt{2(2k+1)}},$$

and h_N is a solution of (17)*. Moreover, the functional*

$$\langle\widehat{y}^*,Fx(\cdot)\rangle=\frac{1}{2\pi}\int_{|t|<\sigma}\frac{(it)^k}{1+(\widehat{h}_Nt)^{2n}}y(t)\,dt \tag{18}$$

is extremal. For $\sigma=\infty$

$$e_2(n,k,\infty,N)=\frac{\sqrt{2k+1}}{\left(4n^2\sin\pi\dfrac{2k+1}{2n}\right)^{\frac{n}{2k+1}}}\left(\frac{2n-2k-1}{2\pi N^2}\right)^{\frac{2n-2k-1}{2(2k+1)}}$$

and the functional (18) *in which*

$$\widehat{h}_N = \left(\frac{2n-2k-1}{8\pi n^2 N^2 \sin \pi \dfrac{2k+1}{2n}} \right)^{\frac{1}{2k+1}}$$

is extremal.

Note that in the case $\sigma = \infty$ the result stated in this theorem can be obtained from the paper [12].

3. Proofs

The arguments connected with general principals of extremum theory underlie proofs of the stated theorems. The main point is in the fact that the problems under consideration here are reduced to some convex problems for which necessary and sufficient conditions for an admissible point to be a solution of the problem are vanishing of the derivative (or belonging of zero to the subdifferential) of the Lagrange function in this point. This condition is some identity. On the other hand, the recovery problem itself is dual to the mentioned convex problems, and therefore solving them (that is, obtaining the required identity) we also solve in general the dual problem (see [13]–[15] for details about such approach to the solution of various extremal problems). The following theorem is a summarizing result of the mentioned arguments.

Theorem 7. *Let $n, k \in \mathbb{Z}$, $0 \le k < n$, $0 < \sigma \le \infty$, $\delta > 0$, $1 \le p \le \infty$, and for all $x(\cdot) \in X_p^n$ the equlity*

$$x^{(k)}(0) = \langle \widehat{y}^*, Fx(\cdot) \rangle + \lambda \int_{\mathbb{R}} x^{(n)}(t) \overline{\widehat{x}^{(n)}(t)}\, dt \tag{19}$$

holds, where $\widehat{y}^*$ is some linear continuous functional on $L_p(\Delta_\sigma)$, $\lambda \in \mathbb{R}_+$, and $\widehat{x}(\cdot) \in X_p^n$ satisfies the following conditions

(*i*) $\|F\widehat{x}(\cdot)\|_{L_p(\Delta_\sigma)} = \delta$,
(*ii*) $\|\widehat{x}^{(n)}(\cdot)\|_{L_2(\mathbb{R})} = 1$,
(*iii*) $\langle \widehat{y}^*, F\widehat{x}(\cdot) \rangle = \delta \|\widehat{y}^*\|$.

Then

$$E_p(n, k, \sigma, \delta) = \lambda + \delta \|\widehat{y}^*\| \tag{20}$$

and $\widehat{y}^*$ is an optimal method of recovery. Moreover, for Stechkin's problem for $N = \|\widehat{y}^*\|$

$$e_p(n, k, \sigma, N) = \lambda$$

and $\widehat{y}^*$ is an extremal functional.

Proof. Taking (19) into account we have

$$(21)\quad E_p(n,k,\sigma,\delta)\le \sup_{\substack{x(\cdot)\in C_p^n,\ y(\cdot)\in L_p(\Delta_\sigma)\\ \|Fx(\cdot)-y(\cdot)\|_{L_p(\Delta_\sigma)}\le\delta}} |x^{(k)}(0)-\langle \widehat{y}^*,y(\cdot)\rangle|$$
$$\le \sup_{x(\cdot)\in C_p^n} |x^{(k)}(0)-\langle \widehat{y}^*,Fx(\cdot)\rangle|+\delta\|\widehat{y}^*\|=\lambda+\delta\|\widehat{y}^*\|.$$

On the other hand, in view of (i) for any recovery methof $\varphi(y(\cdot))$ we have

$$2|\widehat{x}^{(k)}(0)|\le|\widehat{x}^{(k)}(0)-\varphi(0)|+|-\widehat{x}^{(k)}(0)-\varphi(0)|$$
$$\le 2\sup_{\substack{x(\cdot)\in C_p^n,\ y(\cdot)\in L_p(\Delta_\sigma)\\ \|Fx(\cdot)-y(\cdot)\|_{L_p(\Delta_\sigma)}\le\delta}} |x^{(k)}(0)-\varphi(y(\cdot))|.$$

Hence $E_p(n,k,\sigma,\delta)\ge|\widehat{x}^{(k)}(0)|$. Taking (19), (ii), and (iii) into account we obtain

$$E_p(n,k,\sigma,\delta)\ge|\widehat{x}^{(k)}(0)|=\left|\langle\widehat{y}^*,F\widehat{x}(\cdot)\rangle+\lambda\|\widehat{x}^{(n)}(\cdot)\|_{L_2(\mathbb{R})}\right|=\lambda+\delta\|\widehat{y}^*\|.$$

It follows from this inequality and (21) inequality (20) and the optimality of the method $\widehat{y}^*$.

We now proceed to the Stechkin problem. As was proved in the optimal recovery problem among all optimal methods there exists a method defined by a linear continuous functional, therefore

$$E_p(n,k,\sigma,\delta)=\inf_{N>0}\inf_{\|y^*\|\le N}\sup_{\substack{x(\cdot)\in C_p^n,\ y(\cdot)\in L_p(\Delta_\sigma)\\ \|Fx(\cdot)-y(\cdot)\|_{L_p(\Delta_\sigma)}\le\delta}} |x^{(k)}(0)-\langle y^*,y(\cdot)\rangle|$$
$$\le\inf_{\|y^*\|\le N}\sup_{x(\cdot)\in C_p^n}|x^{(k)}(0)-\langle y^*,Fx(\cdot)\rangle|+\delta N=e_p(n,k,\sigma,N)+\delta N.$$

Consequently, for all $N>0$

$$(22)\qquad e_p(n,k,\sigma,N)\ge E_p(n,k,\sigma,\delta)-\delta N.$$

Hence from (20) for $N=\|\widehat{y}^*\|$ we obtain

$$e_p(n,k,\sigma,N)\ge\lambda.$$

On the other hand, in view of (19) we have

$$e_p(n,k,\sigma,N)\le\sup_{x(\cdot)\in C_p^n}|x^{(k)}(0)-\langle\widehat{y}^*,Fx(\cdot)\rangle|=\lambda.$$

□

Proof of Theorem 1. Let us prove that for all $x(\cdot)\in X_\infty^n$ the equality

$$(23)\quad x^{(k)}(0)=\frac{1}{2\pi}\int_{|t|<\sigma_0}(it)^k\left(1-\delta\lambda|t|^{2n-k}\right)Fx(t)\,dt$$
$$+\lambda\int_{\mathbb{R}}x^{(n)}(t)\overline{\widehat{x}^{(n)}(t)}\,dt$$

holds, where the function $\widehat{x}(\cdot) \in X_\infty^n$ is such that

$$F\widehat{x}(t) = \begin{cases} (-i)^k \delta \operatorname{sign} t^k, & |t| < \sigma_0, \\ \dfrac{(-i)^k}{\lambda t^{2n-k}}, & |t| \ge \sigma_0. \end{cases}$$

By the Plancherel theorem we have

$$\int_{\mathbb{R}} x^{(n)}(t)\overline{\widehat{x}^{(n)}(t)}\, dt = \frac{1}{2\pi}\int_{\mathbb{R}} t^{2n} Fx(t)\overline{F\widehat{x}(t)}\, dt. \tag{24}$$

Therefore,

$$\begin{aligned} \frac{1}{2\pi}\int_{|t|<\sigma_0} &(it)^k \left(1 - \delta\lambda|t|^{2n-k}\right) Fx(t)\, dt + \lambda\int_{\mathbb{R}} x^{(n)}(t)\overline{\widehat{x}^{(n)}(t)}\, dt \\ &= \frac{1}{2\pi}\int_{|t|<\sigma_0} \left((it)^k \left(1 - \delta\lambda|t|^{2n-k}\right) + \lambda t^{2n} i^k \delta \operatorname{sign} t^k\right) Fx(t)\, dt \\ &\qquad + \frac{1}{2\pi}\int_{|t|\ge\sigma_0} (it)^k Fx(t)\, dt = \frac{1}{2\pi}\int_{\mathbb{R}} (it)^k Fx(t)\, dt = x^{(k)}(0). \end{aligned}$$

The equality $\|\widehat{x}^{(n)}(\cdot)\|_{L_2(\mathbb{R})} = 1$ is easily verified. Let us prove that $\|F\widehat{x}(\cdot)\|_{L_\infty(\Delta_\sigma)} = \delta$. For $\sigma_0 \ge \sigma$ it is immediately follows from the definition of $F\widehat{x}(\cdot)$. Let $\sigma_0 < \sigma$. Then $\sigma_0 = \widehat{\sigma}$ and it is not difficult to verify that $(\lambda\widehat{\sigma}^{2n-k})^{-1} = \delta$. Thus, $|F\widehat{x}(t)| \le \delta$ for $|t| \ge \widehat{\sigma}$. We now verify the fulfilment of the condition (iii) of Theorem 7. We have

$$\langle \widehat{y}^*, F\widehat{x}(\cdot)\rangle = \frac{\delta}{2\pi}\int_{|t|<\sigma_0} |t|^k \left(1 - \delta\lambda|t|^{2n-k}\right)\, dt. \tag{25}$$

Let us prove that $1 - \delta\lambda|t|^{2n-k} > 0$ for $|t| < \sigma_0$. In view of the definition of σ_0 we have

$$\delta^2\sigma_0^{2n+1} 2(2n-k) \le \delta^2\widehat{\sigma}^{2n+1} 2(2n-k) = \pi(2n+1)(2n-2k-1).$$

Hence

$$\begin{aligned} \delta^2\sigma_0^{2n+1}(2n+1) &\le (2n-2k-1)(\pi(2n+1) - \delta^2\sigma_0^{2n+1}) \\ &= \sigma_0^{-2n+2k+1}(2n+1)\lambda^{-2}, \end{aligned}$$

that is, $\delta\lambda\sigma_0^{2n-k} \le 1$. Thus, for $|t| < \sigma_0$, $1 - \delta\lambda|t|^{2n-k} > 1 - \delta\lambda\sigma_0^{2n-k} \ge 0$. Consequently, the right-hand side of (25) is equal to $\delta\|\widehat{y}^*\|$. To complete the proof it remains to apply Theorem 7. □

Proof of Theorem 2. First of all, we consider the case $0 < k < n$. We prove that for all $x(\cdot) \in X_1^n$ the equality

$$x^{(k)}(0) = \frac{1}{2\pi}\int_{\mathbb{R}} \mu_\delta(t) Fx(t)\, dt + \lambda\int_{\mathbb{R}} x^{(n)}(t)\overline{\widehat{x}^{(n)}(t)}\, dt \tag{26}$$

holds, where the function $\widehat{x}(\cdot) \in X_1^n$ is such that

$$F\widehat{x}(t) = \begin{cases} 0, & |t| \le a, \\ (-i)^k \dfrac{|t|^k - a^k}{\lambda t^{2n}} \operatorname{sign} t^k, & a < |t| < \sigma, \\ \dfrac{(it)^k}{\lambda t^{2n}}, & |t| \ge \sigma. \end{cases}$$

Indeed, taking (24) into account we have

$$\begin{aligned} \frac{1}{2\pi} \int_{|t|<\sigma} \mu_\delta(t) Fx(t)\, dt + \lambda \int_{\mathbb{R}} x^{(n)}(t) \overline{\widehat{x}^{(n)}(t)}\, dt = \frac{1}{2\pi} \int_{|t|\le a} (it)^k Fx(t)\, dt \\ + \frac{1}{2\pi} \int_{a<|t|<\sigma} \left((ia)^k \operatorname{sign} t^k + i^k(|t|^k - a^k) \operatorname{sign} t^k \right) Fx(t)\, dt \\ + \frac{1}{2\pi} \int_{|t|\ge\sigma} (it)^k Fx(t)\, dt = \frac{1}{2\pi} \int_{\mathbb{R}} (it)^k Fx(t)\, dt = x^{(k)}(0). \end{aligned}$$

By direct calculations one can verify that the equations

$$\|F\widehat{x}(\cdot)\|_{L_1(\Delta_\sigma)} = \delta, \quad \|\widehat{x}^{(n)}(\cdot)\|_{L_2(\mathbb{R})} = 1 \tag{27}$$

hold. It remains to apply Theorem 7.

Assume now that $k = 0$. If $\sigma = \infty$, then in view of the equality

$$x(0) = \frac{1}{2\pi} \int_{\mathbb{R}} Fx(t)\, dt$$

considering any function $\widehat{x}(\cdot)$ satisfying the conditions $\|F\widehat{x}(\cdot)\|_{L_1(\mathbb{R})} = \delta$, $\|\widehat{x}^{(n)}(\cdot)\|_{L_2(\mathbb{R})} = 1$, and applying Theorem 7, we obtain the assertion of the theorem for $k = 0$ and $\sigma = \infty$.

The case $k = 0$, $\sigma < \infty$ requires an individual consideration. For sufficiently small $\varepsilon > 0$ define the function $\widehat{x}_\varepsilon(\cdot) \in X_1^n$ so that

$$F\widehat{x}_\varepsilon(t) = \begin{cases} 0, & |t| \le \varepsilon, \\ \dfrac{c_1}{t^{2n}}, & \varepsilon < |t| < \sigma, \\ \dfrac{c_2}{t^{2n}}, & |t| \ge \sigma. \end{cases}$$

Putting

$$c_1 = \frac{(2n-1)\delta}{2} \left(\frac{1}{\varepsilon^{2n-1}} - \frac{1}{\sigma^{2n-1}} \right)^{-1},$$

$$c_2 = \sqrt{2n-1}\,\sigma^{n-1/2} \left(\pi - \frac{(2n-1)\delta^2}{4} \left(\frac{1}{\varepsilon^{2n-1}} - \frac{1}{\sigma^{2n-1}} \right)^{-1} \right)^{1/2}$$

it is not difficult to verify that for the function $\widehat{x}_\varepsilon(\cdot)$ conditions (27) hold. Similarly to the proof of Theorem 7 it can be shown that

$$E_1(n,0,\sigma,\delta) \ge |\widehat{x}_\varepsilon(0)| = \frac{1}{2\pi}\int_{\mathbb{R}} F\widehat{x}_\varepsilon(t)\,dt$$
$$= \frac{1}{2\pi}\int_{|t|<\sigma} F\widehat{x}_\varepsilon(t)\,dt + \frac{1}{\pi}\int_\sigma^\infty \frac{c_2}{t^{2n}}\,dt = \frac{\delta}{2\pi} + \frac{c_2}{\sigma^{2n-1}\pi(2n-1)}.$$

By tending ε to zero we obtain the estimate

$$E_1(n,0,\sigma,\delta) \ge \frac{\delta}{2\pi} + \frac{1}{\sigma^{n-1/2}\sqrt{\pi(2n-1)}}.$$

On the other hand, for the method defined by (12) we have

$$E_1(n,0,\sigma,\delta) \le \sup_{\substack{x(\cdot)\in C_1^n,\ y(\cdot)\in L_1(\Delta_\sigma)\\ \|Fx(\cdot)-y(\cdot)\|_{L_1(\Delta_\sigma)}\le\delta}} \left| x(0) - \frac{1}{2\pi}\int_{|t|<\sigma} y(t)\,dt\right|$$
$$\le \sup_{x(\cdot)\in C_1^n} \left| x(0) - \frac{1}{2\pi}\int_{|t|<\sigma} Fx(t)\,dt\right| + \frac{\delta}{2\pi}$$
$$= \frac{\delta}{2\pi} + \sup_{x(\cdot)\in C_1^n} \frac{1}{\pi}\int_\sigma^\infty |Fx(t)|\,dt \le \frac{\delta}{2\pi} + \frac{1}{\pi}\sqrt{\int_\sigma^\infty t^{2n}|Fx(t)|^2\,dt}\sqrt{\int_\sigma^\infty \frac{dt}{t^{2n}}}$$
$$\le \frac{\delta}{2\pi} + \frac{1}{\sigma^{n-1/2}\sqrt{\pi(2n-1)}}.$$

□

The proof of Theorem 3 is carried out by the same scheme as in Theorems 1 and 2 using the identity

$$x^{(k)}(0) = \frac{1}{2\pi}\int_{|t|<\sigma} \frac{(it)^k}{1+(h_\delta t)^{2n}} Fx(t)\,dt + \lambda\int_{\mathbb{R}} x^{(n)}(t)\overline{\widehat{x}^{(n)}(t)}\,dt,$$

in which

$$\lambda = \frac{h_\delta^{2n-k-1/2}}{\delta}\left(2\int_0^{\sigma h_\delta} x^{2k}(1+x^{2n})^{-2}\,dx\right)^{1/2},$$

and the function $\widehat{x}(\cdot)\in X_2^n$ is such that

$$F\widehat{x}(t) = \begin{cases} \dfrac{h_\delta^{2n}}{\delta}\dfrac{(it)^k}{1+(h_\delta t)^{2n}}, & |t|<\sigma,\\[2mm] \dfrac{(-i)^k}{\lambda} t^{k-2n}, & |t|\ge\sigma. \end{cases}$$

Proof of Theorem 4. Denote by $N(\delta)$ the norm of the linear functional (4) (as a functional on $L_\infty(\Delta_\sigma)$). We have

$$N(\delta) = \frac{1}{\pi}\left(\frac{\sigma_0^{k+1}}{k+1} - \delta\lambda\frac{\sigma_0^{2n+1}}{2n+1}\right),$$

where

$$\sigma_0 = \begin{cases} \sigma, & 0 < \delta \le \delta_0, \\ \left(\dfrac{\pi(2n+1)(2n-2k-1)}{2\delta^2(2n-k)}\right)^{\frac{1}{2n+1}}, & \delta > \delta_0, \end{cases}$$

$$\delta_0 = \sigma^{-n-1/2}\sqrt{\frac{\pi(2n+1)(2n-2k-1)}{2(2n-k)}}.$$

Thus, taking (5) into account for $0 < \delta \le \delta_0$

$$N(\delta) = \frac{\sigma^{k+1}}{\pi}\left(\frac{1}{k+1} - \frac{\delta\left(\dfrac{\pi}{\sigma^{2n+1}} - \dfrac{\delta^2}{2n+1}\right)^{-1/2}}{(2n+1)\sqrt{2n-2k-1}}\right).$$

It is not difficult to verify that for $0 < \delta \le \delta_0$ the function $N(\delta)$ monotonically decreases from N_2 to N_1, where

$$N_2 = \frac{\sigma^{k+1}}{\pi(k+1)}, \quad N_1 = \frac{\sigma^{k+1}(2n-k)}{\pi(k+1)(2n+1)}.$$

Consequently, for $N_1 \le N < N_2$ the equation $N(\delta) = N$ has the unique solution

$$\delta_N = \sigma^{-n-1/2}\sqrt{\frac{\pi(2n+1)(2n-2k-1)}{(2n+1)\gamma^{-2}(\sigma, N) + 2n - 2k - 1}}.$$

Moreover, taking into account that for $N_1 \le N < N_2$, $\sigma_N = \sigma$, it follows by Theorem 7 that an extremal functional has the form (15).

If $\delta \ge \delta_0$, then

$$N(\delta) = \left(\frac{\pi(2n+1)(2n-2k-1)}{2\delta^2(2n-k)}\right)^{\frac{k+1}{2n+1}} \frac{2n-k}{\pi(k+1)(2n+1)}.$$

For $\delta \ge \delta_0$ the function $N(\delta)$ monotonically decreases from N_1 to 0. Hence for all $0 < N \le N_1$ there exists the unique solution of the equation $N(\delta) = N$ given by the equality

$$\delta_N = \sqrt{n-k-1/2}\left(\frac{2n-k}{\pi(2n+1)}\right)^{\frac{2n-k}{2k+2}}\left(\frac{1}{(k+1)N}\right)^{\frac{2n+1}{2k+2}}.$$

For $0 < N \le N_1$, $\sigma_N = \widehat{\sigma}_N$ and an extremal functional has again the form (15). The expression for $e_\infty(n,k,\sigma,N)$ for $0 < N < N_2$ is obtained accordingly Theorem 7 by substitution in (5) $\delta = \delta_N$.

Assume now that $N \ge N_2$. Then it follows from (22) that for all $\delta > 0$

$$e_\infty(n,k,\sigma,N) \ge E_\infty(n,k,\sigma,\delta) - \delta N.$$

Tending δ to zero we obtain

$$e_\infty(n,k,\sigma,N) \geq \frac{1}{\sigma^{n-k-1/2}\sqrt{\pi(2n-2k-1)}}.$$

The immediate estimate of the functional (15) which for $N \geq N_2$ has the form

$$\langle \widehat{y}^*, Fx(\cdot)\rangle = \frac{1}{2\pi}\int_{|t|<\sigma}(it)^k Fx(t)\,dt \tag{28}$$

(its norm is equal to N_2) gives us

$$\begin{aligned}
|x^{(k)}(0) - \langle \widehat{y}^*, Fx(\cdot)\rangle &\leq \frac{1}{2\pi}\int_{|t|\geq\sigma}|t|^k|Fx(t)|\,dt \\
&= \frac{1}{2\pi}\int_{|t|\geq\sigma}|t|^n|Fx(t)||t|^{k-n}\,dt \\
&\leq \frac{1}{2\pi}\sqrt{\int_{|t|\geq\sigma}|t|^{2n}|Fx(t)|^2\,dt}\sqrt{\int_{|t|\geq\sigma}|t|^{2k-2n}\,dt} \\
&\leq \frac{1}{\sigma^{n-k-1/2}\sqrt{\pi(2n-2k-1)}}.
\end{aligned} \tag{29}$$

□

Proof of Theorem 5. Let $k>0$, $0<\sigma<\infty$, $\varepsilon \in (1,+\infty)$, and δ is defined by (9). Denote by $N(\varepsilon)$ the norm of the linear functional (10) (as a functional on $L_1(\Delta_\sigma)$). We have

$$N(\varepsilon) = \frac{\sigma^k}{2\pi\varepsilon^k}.$$

Therefore for $0<N<\sigma^k/(2\pi)$ taking (26) into account the statement of the theorem follows immediately from Theorem 7. If $N \geq \sigma^k/(2\pi)$, then it follows from (22) that for all $\delta > 0$

$$e_1(n,k,\sigma,N) \geq E_1(n,k,\sigma,\delta) - \delta N.$$

Tending ε to one (in this case $\delta \to 0$) we obtain

$$e_1(n,k,\sigma,N) \geq \frac{1}{\sigma^{n-k-1/2}\sqrt{\pi(2n-2k-1)}}.$$

For $N \geq \sigma^k/(2\pi)$ the functional (16) has the form (28). Taking into account the estimate (29) we have

$$e_1(n,k,\sigma,N) \leq \frac{1}{\sigma^{n-k-1/2}\sqrt{\pi(2n-2k-1)}}.$$

For $k>0$ and $\sigma=\infty$ the norm of the linear functional (10) is equal to

$$N(\delta) = \frac{1}{2\pi}\left(\frac{2\pi(2n-2k-1)}{\delta^2(2n-1)(2n-k-1)}\right)^{\frac{k}{2n-1}}.$$

Therefore for all $N > 0$ there exists a δ_N for which $N(\delta_N) = N$ and taking into account the identity (26) the statement of the theorem follows from Theorem 7.

Assume now that $k = 0$. Then for all $\delta > 0$

$$e_1(n,0,\sigma,N) \geq E_1(n,0,\sigma,\delta) - \delta N = \left(\frac{1}{2\pi} - N\right)\delta + \frac{1}{\sigma^{n-1/2}\sqrt{\pi(2n-1)}}.$$

Tending δ to infinity we obtain

$$e_1(n,0,\sigma,N) = \infty.$$

If $N \geq 1/(2\pi)$, then tending δ to zero by the same inequality we have

$$e_1(n,0,\sigma,N) \geq \frac{1}{\sigma^{n-1/2}\sqrt{\pi(2n-1)}}.$$

The inverse inequality follows from (29) for $k = 0$. $\square$

The proof of Theorem 6 is carried out by the same scheme which was used in the proof of Theorems 4 and 5.

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