OPTIMAL RECOVERY OF FUNCTIONS AND THEIR DERIVATIVES FROM INACCURATE INFORMATION ABOUT THE SPECTRUM AND INEQUALITIES FOR DERIVATIVES

G. G. MAGARIL-IL'YAEV AND K. YU. OSIPENKO

Abstract. We study optimal recovery problems for functions and their derivatives in the L_2 metric on the line from information about the Fourier transform of the function in question known approximately on a finite interval or on the entire line. Exact values of optimal recovery errors and closed-form expressions for optimal recovery methods are obtained. We also prove a sharp inequality for derivatives (closely related to these recovery problems), which estimates the kth derivative of a function in the L_2-norm on the line via the L_2-norm of the nth derivative and the L_p-norm of the Fourier transform of the function.

1. Statement of the problem

We start from the general statement of the optimal recovery problem. Let C be a set (class) in a linear space X. For every element $x \in C$, we know information $I(x)$, where I is a mapping (which is called an *information mapping*) of C into another linear space Y. The information about elements of C can be inaccurate, and therefore, I is a multivalued mapping in general. Next, let Z be a normed space and $\Lambda\colon X \to Z$ a linear operator. The problem is to recover Λ on C in the metric of Z in the best possible way from the given information I. Namely, a mapping $\varphi\colon Y \to Z$ will be called a *recovery method* (of Λ on C from I). The number

$$e(\Lambda, C, I, \varphi) = \sup_{x \in C,\ y \in I(x)} \|\Lambda x - \varphi(y)\|_Z$$

is called the *error* of the method. The number

$$E(\Lambda, C, I) = \inf_{\varphi\colon Y \to Z} e(\Lambda, C, I, \varphi), \tag{1}$$

where the infimum is taken over all mappings $\varphi\colon Y \to Z$, is called the *optimal recovery error*, and any method for which the infimum is attained is called an *optimal recovery method*.

This research was financially supported by the Russian Foundation for Basic Research (grants Nos. 00-15-96109 and 02-01-00386) and the Program "Universities of Russia" (grant No. UR.04.03.013) as well as by US CRDF–R.F. Ministry of Education Award VZ-0100-0.

The optimal recovery problem was originally stated by S. A. Smolyak [1] for the case in which $Z = \mathbb{R}$, I is a linear operator, and $\dim Y < \infty$. More recently, this range of problems has been intensively developed in various directions (see [2]–[5]). An approach to recovery problems on the basis of general principles of extremum theory was developed in [6]–[8]. We use this approach in the present paper.

Let us proceed to the statement of recovery problems considered in this paper. Let S be the Schwartz space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}$, S' the dual space of distributions, and $F\colon S' \to S'$ the Fourier transform. Let $1 \le p \le \infty$ and $n \in \mathbb{N}$. We take the space

$$X_p^n = \{\, x \in S' : Fx(\cdot) \in L_p(\mathbb{R}),\ x^{(n)}(\cdot) \in L_2(\mathbb{R})\,\}$$

as the space X in problem (1) and the set

$$C_p^n = \{\, x(\cdot) \in X_p^n : \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})} \le 1\,\}$$

as the class C.

Let us describe information mappings to be studied here. Let $1 \le p < \infty$, $0 < \sigma \le \infty$, $\Delta_\sigma = (-\sigma, \sigma)$, and $\delta > 0$. Suppose that the information available about the element $x(\cdot) \in C_p^n$ is as follows: we know a function $y(\cdot) \in L_p(\Delta_\sigma)$ such that $\|Fx(\cdot) - y(\cdot)\|_{L_p(\Delta_\sigma)} \le \delta$. Thus the information mapping $I = I_p^{\delta,\sigma}\colon C_p^n \to L_p(\Delta_\sigma)$ has the form $I_p^{\delta,\sigma}x(\cdot) = Fx(\cdot)\big|_{\Delta_\sigma} + \delta BL_p(\Delta_\sigma)$, where $BL_p(\Delta_\sigma)$ is the unit ball in $L_p(\Delta_\sigma)$.

If $p = \infty$, then we consider a more general situation. Let $\delta(\cdot)$ be a nonnegative function in $L_\infty(\Delta_\sigma)$. We assume that the information about $x(\cdot)$ is a function $y(\cdot) \in L_\infty(\Delta_\sigma)$ such that $|Fx(t) - y(t)| \le \delta(t)$ for almost all $t \in \Delta_\sigma$. If we set

$$B(\delta(\cdot)) = \{\, y(\cdot) \in L_\infty(\Delta_\sigma) : |y(t)| \le \delta(t) \text{ a. e.}\,\},$$

then the information mapping $I = I_\infty^{\delta(\cdot),\sigma}\colon C_\infty^n \to L_\infty(\Delta_\sigma)$ has the form $I_\infty^{\delta(\cdot),\sigma}x(\cdot) = Fx(\cdot)\big|_{\Delta_\sigma} + B(\delta(\cdot))$.

We state the problem on the optimal recovery of the linear operator Λ, $\Lambda x(\cdot) = x^{(k)}(\cdot)$, $0 \le k \le n-1$, on the class C_p^n in the $L_2(\mathbb{R})$-metric from the information mappings defined above. (In the following, we show that $\Lambda\colon X_p^n \to L_2(\mathbb{R})$ for $2 \le p \le \infty$.)

Thus in our case problem (1) for $p < \infty$ has the form

(2)
$$E(x^{(k)}(\cdot), C_p^n, I_p^{\delta,\sigma}) = \inf_{\varphi} \sup_{\substack{x(\cdot)\in C_p^n,\ y(\cdot)\in L_p(\Delta_\sigma)\\ \|Fx(\cdot)-y(\cdot)\|_{L_p(\Delta_\sigma)}\le\delta}} \|x^{(k)}(\cdot) - \varphi(y)(\cdot)\|_{L_2(\mathbb{R})},$$

where the infimum is taken over all $\varphi\colon L_p(\Delta_\sigma)\to L_2(\mathbb{R})$. If $p=\infty$, then it has the form

(3)
$$E(x^{(k)}(\cdot),C^n_\infty,I^{\delta(\cdot),\sigma}_\infty)=\inf_{\varphi}\sup_{\substack{x(\cdot)\in C^n_\infty,\ y(\cdot)\in L_\infty(\Delta_\sigma)\\ |Fx(t)-y(t)|\le\delta(t)\ \text{a. e.}}}\|x^{(k)}(\cdot)-\varphi(y)(\cdot)\|_{L_2(\mathbb{R})},$$

where the infimum is taken over all $\varphi\colon L_\infty(\Delta_\sigma)\to L_2(\mathbb{R})$.

In problem (2) with $p=2$ and problem (3), we find the exact values of optimal recovery errors and closed-form expressions for optimal recovery methods. In both cases, the following phenomenon occurs: for a given error in problem (3), there exists a finite $\widehat{\sigma}>0$ such that the knowledge of the Fourier transform on an interval larger than $(-\widehat{\sigma},\widehat{\sigma})$ does not result in a decrease in the optimal recovery error. This conclusion is apparently important in practical applications of the results obtained here.

In problem (2) with $p=2$ and $\sigma=\infty$, the information is equivalent (in view of the Plancherel theorem) to the knowledge of the function itself with accuracy δ in the $L_2(\mathbb{R})$-metric. In this setting, the problem was solved in [9]. Periodic analogs of problem (2) whith $p=2$ and problem (3), as well as their extensions to larger function classes, were studied in the paper [10].

For $2<p<\infty$, we obtain a lower bound for the number (2). Moreover, we prove a sharp estimate of the k-th derivative of a function in the L_2-norm via the L_2-norm of the n-th derivative and the L_p-norm of the Fourier transform.

2. Statement of the main results

Theorem 1. *Let $n\in\mathbb{N}$, $0\le k\le n-1$, $0<\sigma\le\infty$, $\Delta_\sigma=(-\sigma,\sigma)$, $\delta(\cdot)\in L_\infty(\Delta_\sigma)$, $\delta(\cdot)\ge 0$, and*

$$\sigma_0=\sup\left\{a:0<a<\sigma,\ \frac{1}{2\pi}\int_{-a}^{a}t^{2n}\delta^2(t)\,dt\le 1\right\}.$$

If $\sigma_0<\infty$, then

(4)
$$E(x^{(k)}(\cdot),C^n_\infty,I^{\delta(\cdot),\sigma}_\infty)=\sqrt{\sigma_0^{-2(n-k)}+\frac{1}{2\pi}\int_{-\sigma_0}^{\sigma_0}(t^{2k}-\sigma_0^{-2(n-k)}t^{2n})\delta^2(t)\,dt}$$

and the method

$$\widehat{\varphi}(y)(t)=\frac{1}{2\pi}\int_{-\sigma_0}^{\sigma_0}(i\tau)^k\left(1-\left(\frac{\tau}{\sigma_0}\right)^{2(n-k)}\right)y(\tau)e^{i\tau t}\,d\tau \tag{5}$$

is optimal.

If $\sigma_0=\infty$, then

$$E(x^{(k)}(\cdot),C_\infty^n,I_\infty^{\delta(\cdot),\sigma})=\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}t^{2k}\delta^2(t)\,dt} \tag{6}$$

and the method

$$\widehat{\varphi}(y)(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\tau)^k y(\tau)e^{i\tau t}\,d\tau \tag{7}$$

is optimal.

Corollary 1. *Let $\delta(t)\equiv\delta>0$ and*

$$\widehat{\sigma}=(\pi(2n+1))^{\frac{1}{2n+1}}\delta^{-\frac{2}{2n+1}}.$$

Then

$$E(x^{(k)}(\cdot),C_\infty^n,I_\infty^{\delta,\sigma})=\begin{cases}\sqrt{\sigma^{-2(n-k)}+\dfrac{2\delta^2(n-k)}{\pi(2k+1)(2n+1)}\sigma^{2k+1}}, & \sigma<\widehat{\sigma},\\[2ex] \sqrt{\dfrac{2n+1}{2k+1}}\left(\dfrac{1}{\pi(2n+1)}\right)^{\frac{n-k}{2n+1}}\delta^{\frac{2(n-k)}{2n+1}}, & \sigma\ge\widehat{\sigma},\end{cases}$$

and the method (5) with $\sigma_0=\min(\sigma,\widehat{\sigma})$ is optimal.

It follows from this corollary that for a given δ, starting from $\widehat{\sigma}$, further extension of the interval on which the Fourier transform of a function in C_∞^n is given with error δ in the uniform metric does not result in a decrease in the recovery error. In other words, if the relation

$$\delta^2\sigma^{2n+1}\le\pi(2n+1) \tag{8}$$

between the input data and the size of the interval on which the data is measured is violated, then the available information turns out to be redundant.

Theorem 2. *Let $n\in\mathbb{N}$, $0<k\le n-1$, $0<\sigma\le\infty$, $\delta>0$, and*

$$\widehat{\sigma}=\left(\frac{n}{k}\right)^{\frac{1}{2(n-k)}}\left(\frac{2\pi}{\delta^2}\right)^{\frac{1}{2n}}.$$

Then

$$E(x^{(k)}(\cdot),C_2^n,I_2^{\delta,\sigma})=\begin{cases}\sigma^k\sqrt{\dfrac{n-k}{2\pi n}\left(\dfrac{k}{n}\right)^{\frac{k}{n-k}}\delta^2+\dfrac{1}{\sigma^{2n}}}, & \sigma<\widehat{\sigma},\\[2ex] \left(\dfrac{\delta^2}{2\pi}\right)^{\frac{n-k}{2n}}, & \sigma\ge\widehat{\sigma}\end{cases}$$

and the method

$$\widehat{\varphi}(y)(t)=\frac{1}{2\pi}\int_{-\sigma}^{\sigma}(i\tau)^k\left(1+\frac{n}{n-k}\left(\frac{n}{k}\right)^{\frac{k}{n-k}}\left(\frac{\tau}{\sigma_0}\right)^{2n}\right)^{-1}y(\tau)e^{i\tau t}\,d\tau,$$

where $\sigma_0 = \min(\sigma, \widehat{\sigma})$, *is optimal.*

If $k = 0$ *and* $0 < \sigma < \infty$, *then*

$$E(x(\cdot), C_2^n, I_2^{\delta,\sigma}) = \sqrt{\frac{\delta^2}{2\pi} + \frac{1}{\sigma^{2n}}}$$

and the method

$$\widehat{\varphi}(y)(t) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \left(1 + \left(\frac{\tau}{\sigma}\right)^{2n}\right)^{-1} y(\tau) e^{i\tau t} \, d\tau \tag{9}$$

is optimal.

It follows from Theorem 2 that the "saturation" effect is also observed in problem (2) with $p = 2$. Here the analogue of relation (8) is given by the inequality

$$\delta^2 \sigma^{2n} \le 2\pi \left(\frac{n}{k}\right)^{\frac{n}{n-k}};$$

the available information is redundant whenever this inequality is violated.

Note that the method

$$\widetilde{\varphi}(y)(t) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} (i\tau)^k y(\tau) e^{i\tau t} \, d\tau,$$

which takes each function $y(\cdot)$ to the k-th derivative of its inverse Fourier transform, is also optimal for $k = 0$ (along with the method (9)) in the recovery problem in Theorem 2. This can readily be verified by a straightforward estimate. However, it can be shown that this seemingly natural method is not optimal for $k > 0$ and, moreover, its error is equal to ∞.

Theorem 3. *Let* $n \in \mathbb{N}$, $0 \le k \le n-1$, *and* $2 \le p \le \infty$. *The following sharp inequality holds:*

$$\|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \le K(k,n,p) \|Fx(\cdot)\|_{L_p(\mathbb{R})}^{\frac{n-k}{n+1/2-1/p}} \|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{k+1/2-1/p}{n+1/2-1/p}}, \tag{10}$$

where

$$K(k,n,p) = \sqrt{\frac{n+1/2-1/p}{k+1/2-1/p}} \left(\frac{\sqrt{k+1/2-1/p}\, B^{1/2-1/p}}{(2\pi)^{1/p}(n-k)^{1-1/p}}\right)^{\frac{n-k}{n+1/2-1/p}},$$

$$B = B\left(\frac{k+1/2-1/p}{(n-k)(1-2/p)}, 2\frac{1-1/p}{1-2/p}\right) \tag{11}$$

for $2 < p < \infty$, *and* $B(\cdot,\cdot)$ *is the Eiler beta function; moreover*

(12)

$$K(k,n,\infty) = \sqrt{\frac{2n+1}{2k+1}} \left(\frac{1}{\pi(2n+1)}\right)^{\frac{n-k}{2n+1}}, \quad K(k,n,2) = \left(\frac{1}{2\pi}\right)^{\frac{n-k}{2n}}.$$

For $p = 2$, inequality (10) coincides with the well-known Hardy–Littlewood–Pólya inequality by the Plancherel theorem. Inequalities of the form (10) with various metrics and with the function itself occurring instead of its Fourier transform are usually called Kolmogorov type inequalities for derivatives. They play an important role in various problems of analysis and approximation theory. There is a wide literature dealing with them (e.g., see [11, 12]).

3. Proofs

We start from a simple statement concerning a lower bound for the optimal recovery error.

Lemma 1. *Suppose that the set*

$$\operatorname{gr} I = \{\,(x,y) \in X \times Y : x \in C,\ y \in I(x)\,\}$$

is centrally symmetric and the set

$$I^{-1}(0) = \{x \in C : 0 \in I(x)\}$$

is nonempty in problem (1)*. Then*

$$E(\Lambda, C, I) \geq \sup_{x\in C,\ x\in I^{-1}(0)} \|\Lambda x\|_Z.$$

Proof. Let φ be an arbitrary method. Then

$$2\|\Lambda x\|_Z \leq \|\Lambda x - \varphi(0)\|_Z + \|\Lambda(-x) - \varphi(0)\|_Z \leq 2e(\Lambda, C, I, \varphi)$$

for all $x \in C$ such that $x \in I^{-1}(0)$. Consequently,

$$e(\Lambda, C, I, \varphi) \geq \sup_{x\in C,\ x\in I^{-1}(0)} \|\Lambda x\|_Z$$

for each method φ, which readily implies the desired estimate. $\square$

The proofs of Theorems 1 and 2 are carried out by a common scheme. For this reason, first we prove a general result (containing some reasoning based on convex optimization principles) and then use it to obtain the cited theorems. The optimal recovery problem for which this result will be stated is a refinement of the general setting mentioned in the beginning of the paper.

Suppose that T is a finite set with the discrete measure or an interval (finite or infinite) on the line with the Lebesgue measure, X and Y_t, $t \in T$, are linear spaces with semi-inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_{Y_t}$ and the corresponding seminorms $\|\cdot\|_X$ and $\|\cdot\|_{Y_t}$, Z is a normed space, and $\delta(\cdot)$ is a nonnegative measurable function on T. Let Y be a subspace of functions $y(\cdot)$ on T ranging in $\cup_{t\in T} Y_t$ such that $y(t) \in Y_t$ and $t \to \|y(t)\|_{Y_t}$ is a measurable function on T. We consider the problem of optimal recovery of an operator $\Lambda\colon X \to Z$ on the class $BX = \{x \in X : \|x\|_X \leq 1\}$ from information about a linear operator

$I\colon X\to Y$ given with an error $\delta(\cdot)$. More precisely, for every $x\in BX$ we know a function $y(\cdot)\in Y$ such that

$$\|Ix(t)-y(t)\|_{Y_t}\le\delta(t)$$

for almost all $t\in T$.

In this case, the optimal recovery error has the form

$$E(\Lambda,BX,I,\delta(\cdot))=\inf_{\varphi\colon Y\to Z}\ \sup_{\substack{x\in BX,\ y(\cdot)\in Y\\ \|Ix(t)-y(t)\|_{Y_t}\le\delta(t)\text{ a. e.}}}\|\Lambda x-\varphi(y)\|_Z. \tag{13}$$

We wish to compute this number and find an optimal method.

It follows from Lemma 1 that

$$E(\Lambda,BX,I,\delta(\cdot))\ge\sup_{\substack{x\in BX\\ \|Ix(t)\|_{Y_t}\le\delta(t)\text{ a. e.}}}\|\Lambda x\|_Z. \tag{14}$$

Consider the following extremal problem (whose value coincides with the square of the right-hand side of (14)):

(15)
$$\|\Lambda x\|_Z^2\to\max,\quad \|Ix(t)\|_{Y_t}^2\le\delta^2(t)\text{ for almost all }t\in T,\quad \|x\|_X^2\le1.$$

Set

$$\mathcal{L}(x,\lambda_1(\cdot),\lambda_2)=-\|\Lambda x\|_Z^2+\int_T\lambda_1(t)\|Ix(t)\|_{Y_t}^2\,d\mu+\lambda_2\|x\|_X^2,$$

where $\lambda_1(\cdot)$ is a measurable nonnegative function, $\lambda_2\ge0$, and $d\mu$ is either the Lebesgue measure (if T is an interval of the line) or the discrete measure (if T is a finite set).

Theorem 4. *Suppose that there exists a measurable nonnegative function $\widehat{\lambda}_1(\cdot)$ and a number $\widehat{\lambda}_2\ge0$ such that the function $\widehat{\lambda}_1(\cdot)\delta^2(\cdot)$ and all functions $t\to\widehat{\lambda}_1(t)(y^1(t),y^2(t))_{Y_t}$, $y^1(\cdot),y^2(\cdot)\in Y$, are integrable on T and*

(a) $\mathcal{L}(x,\widehat{\lambda}_1(\cdot),\widehat{\lambda}_2)\ge0\quad\forall x\in X.$

Furthermore, suppose that there exists a sequence $\{x_m\}$ of admissible elements in (15) such that the following conditions hold:

(b) $\displaystyle\lim_{m\to\infty}\mathcal{L}(x_m,\widehat{\lambda}_1(\cdot),\widehat{\lambda}_2)=0,$

(c) $\displaystyle\lim_{m\to\infty}\left(\int_T\widehat{\lambda}_1(t)\left(\|Ix_m(t)\|_{Y_t}^2-\delta^2(t)\right)d\mu+\widehat{\lambda}_2\left(\|x_m\|_X^2-1\right)\right)=0.$

If x_y is a solution of the extremal problem

$$\int_T\widehat{\lambda}_1(t)\|Ix(t)-y(t)\|_{Y_t}^2\,d\mu+\widehat{\lambda}_2\|x\|_X^2\to\min,\quad x\in X, \tag{16}$$

then the recovery method

$$\varphi(y)=\Lambda x_y \tag{17}$$

is optimal and

(18)
$$E(\Lambda, BX, I, \delta(\cdot)) = \sup_{\substack{x\in BX \\ \|Ix(t)\|_{Y_t}\le\delta(t)\ a.\ e.}} \|\Lambda x\|_Z = \sqrt{\int_T \widehat{\lambda}_1(t)\delta^2(t)\,d\mu + \widehat{\lambda}_2}.$$

Proof. 1. Let us show that the values of problem (15) and the problem

$$\|\Lambda x\|_Z^2 \to \max, \quad \int_T \widehat{\lambda}_1(t)\|Ix(t)\|_{Y_t}^2\,d\mu + \widehat{\lambda}_2\|x\|_X^2 \le S, \tag{19}$$

where
$$S = \int_T \widehat{\lambda}_1(t)\delta^2(t)\,d\mu + \widehat{\lambda}_2,$$

coincide and are equal to S. Indeed, with regard to (a),

$$-\|\Lambda x\|_Z^2 \ge -\|\Lambda x\|_Z^2 + \int_T \widehat{\lambda}_1(t)\left(\|Ix(t)\|_{Y_t}^2 - \delta^2(t)\right)\,d\mu + \widehat{\lambda}_2\left(\|x\|_X^2 - 1\right) \ge -S$$

for each element $x \in X$ admissible in (15). On the other hand, using (c) and (b), we obtain

$$-\lim_{m\to\infty}\|\Lambda x_m\|^2 = \lim_{m\to\infty}\Bigg(-\|\Lambda x_m\|_Z^2 + \int_T \widehat{\lambda}_1(t)\left(\|Ix_m(t)\|_{Y_t}^2 - \delta^2(t)\right)\,d\mu + \widehat{\lambda}_2\left(\|x_m\|_X^2 - 1\right)\Bigg) = -S;$$

that is, S is the value of problem (15). But the same arguments obviously prove that S is the value of problem (19).

2. An upper bound. Consider the linear space $H = X \times Y$ with the semi-inner product

$$((x^1, y^1(\cdot)), (x^2, y^2(\cdot)))_H = \int_T \widehat{\lambda}_1(t)(y^1(t), y^2(t))_{Y_t}\,d\mu + \widehat{\lambda}_2(x^1, x^2)_X.$$

Then the extremal problem (16) can be rewritten in the form

$$\|(x, Ix(\cdot)) - (0, y(\cdot))\|_H^2 \to \min, \quad x \in X.$$

If x_y is a solution of this problem, then it is easily seen that

$$((x_y, Ix_y(\cdot)) - (0, y(\cdot)), (x, Ix(\cdot)))_H = 0$$

for all $x \in X$. It follows that

$$\|(x, Ix(\cdot)) - (0, y(\cdot))\|_H^2 = \|(x, Ix(\cdot)) - (x_y, Ix_y(\cdot))\|_H^2 + \|(x_y, Ix_y(\cdot)) - (0, y(\cdot))\|_H^2.$$

If $\|x\|_X \le 1$ and $\|Ix(t) - y(t)\|_{Y_t} \le \delta(t)$, $t \in T$, then it follows from the last inequality that

$$\|(x, Ix(\cdot)) - (x_y, Ix_y(\cdot))\|_H^2 \le \|(x, Ix(\cdot)) - (0, y(\cdot))\|_H^2$$
$$= \int_T \widehat{\lambda}_1(t)\|Ix(t) - y(t)\|_{Y_t}^2\, d\mu + \widehat{\lambda}_2\|x\|_X^2 \le S.$$

By setting $z = x - x_y$, we arrive at the inequality

$$\int_T \widehat{\lambda}_1(t)\|Iz(t)\|_{Y_t}^2\, d\mu + \widehat{\lambda}_2\|z\|_X^2 \le S$$

and therefore,

$$\|\Lambda x - \Lambda x_y\|_Z = \|\Lambda z\|_Z \le \sup_{\int_T \widehat{\lambda}_1(t)\|Ix_t\|_{Y_t}^2\, d\mu + \widehat{\lambda}_2\|x\|_X^2 \le S} \|\Lambda x\|_Z = \sqrt{S}.$$

Taking account of (14), we obtain the equality (18) and prove the optimality of the method (17). $\square$

Proof of Theorem 1. First, let $\sigma_0 < \infty$. Let us rewrite this problem in terms of Theorem 4. Here $T = \Delta_\sigma$, $X = X_\infty^n$ is a linear space with the semi-inner product

$$(x_1(\cdot), x_2(\cdot))_{X_\infty^n} = \int_{\mathbb{R}} x_1^{(n)}(t)\overline{x_2^{(n)}(t)}\, dt,$$

$Y_t = \mathbb{C}$ for all $t \in \Delta_\sigma$, $Z = L_2(\mathbb{R})$, $Y = L_\infty(\Delta_\sigma)$, $\Lambda x(\cdot) = x^{(k)}(\cdot)$, $BX = C_\infty^n$, and the operator $I\colon X_\infty^n \to L_\infty(\Delta_\sigma)$ is defined by the formula $Ix(\cdot) = Fx(\cdot)$. With this notation, problem (3) coincides with problem (13). In this case, the function $\mathcal{L}(x(\cdot), \lambda_1(\cdot), \lambda_2)$ in Theorem 4 has the form

$$\mathcal{L}(x(\cdot), \lambda_1(\cdot), \lambda_2) = -\|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})}^2 + \int_{\Delta_\sigma} \lambda_1(t)|Fx(t)|^2\, dt$$
$$+ \lambda_2\|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}^2.$$

Passing to Fourier transforms and writing $(2\pi)^{-1}|Fx(\cdot)|^2 = u(\cdot)$, we have

$$\mathcal{L}(x(\cdot), \lambda_1(\cdot), \lambda_2) = \int_{\mathbb{R}} \left(-t^{2k} + \lambda_2 t^{2n}\right) u(t)\, dt + 2\pi \int_{\Delta_\sigma} \lambda_1(t)u(t)\, dt$$

by the Plancherel theorem. Let $\widehat{\lambda}_2 = \sigma_0^{-2(n-k)}$ and

$$\widehat{\lambda}_1(t) = \begin{cases} (2\pi)^{-1}\left(t^{2k} - \widehat{\lambda}_2 t^{2n}\right), & |t| < \sigma_0, \\ 0, & |t| \ge \sigma_0. \end{cases}$$

Then

$$\mathcal{L}(x(\cdot), \widehat{\lambda}_1(\cdot), \widehat{\lambda}_2) = \int_{|t| \ge \sigma_0} \left(-t^{2k} + \sigma_0 t^{2n}\right) u(t)\, dt \ge 0$$

for all $x(\cdot) \in X_\infty^n$.

Set

$$\gamma = 1 - \frac{1}{2\pi}\int_{-\sigma_0}^{\sigma_0} t^{2n}\delta^2(t)\,dt.$$

If $\gamma = 0$, then, denoting the inverse Fourier transform of the function coinciding with $\delta(\cdot)$ on the interval $(-\sigma_0, \sigma_0)$ and vanishing outside it by $\widehat{x}(\cdot)$, we can readily verify that conditions (b) and (c) in Theorem 4 are satisfied for the constant sequence $x_m(\cdot) = \widehat{x}(\cdot)$.

If $\gamma > 0$ (in this case, it is obvious that $\sigma_0 = \sigma$), then we set

$$u_m(t) = \begin{cases} \delta^2(t)/(2\pi), & |t| < \sigma, \\ \sigma^{-2n}(m\gamma - \sqrt{m})/2, & \sigma \le |t| \le \sigma + 1/m, \\ 0, & |t| > \sigma + 1/m, \end{cases}$$

for $m > \gamma^{-2}$. (We denote the inverse Fourier transform of $\sqrt{2\pi u_m(\cdot)}$ by $x_m(\cdot)$.) It is easily seen that

$$\lim_{m\to\infty} \mathcal{L}(x_m(\cdot), \widehat{\lambda}_1(\cdot), \widehat{\lambda}_2) = 0.$$

Moreover,

$$\int_{-\sigma}^{\sigma} \widehat{\lambda}_1(t)(2\pi u_m(t) - \delta^2(t))\,dt = 0$$

and

$$\begin{aligned} \|x_m^{(n)}(\cdot)\|^2_{L_2(\mathbb{R})} &= \int_{\mathbb{R}} t^{2n}u_m(t)\,dt \\ &= \frac{1}{2\pi}\int_{-\sigma}^{\sigma} t^{2n}\delta^2(t)\,dt + \frac{m\gamma - \sqrt{m}}{\sigma^{2n}}\int_{\sigma}^{\sigma+1/m} t^{2n}\,dt \\ &= 1 - \gamma + \frac{(m\gamma - \sqrt{m})\left((\sigma+1/m)^{2n+1} - \sigma^{2n+1}\right)}{\sigma^{2n}(2n+1)} \overset{m\to\infty}{\to} 1. \end{aligned}$$

It also follows from the last equations that

$$\|x_m^{(n)}(\cdot)\|^2_{L_2(\mathbb{R})} < 1 - \frac{1}{2\sqrt{m}}$$

for sufficiently large m; that is, the functions $x_m(\cdot)$ are admissible in problem (15).

Problem (16) with $y(\cdot) \in L_\infty(\Delta_\sigma)$ has the form

$$\int_{\Delta_\sigma} \widehat{\lambda}_1(t)|Fx(t) - y(t)|^2\,dt + \widehat{\lambda}_2\|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})} \to \min, \quad x(\cdot) \in X^n_\infty.$$

With regard to the Plancherel theorem, it can be rewritten as

$$\int_{\Delta_\sigma} \widehat{\lambda}_1(t)|Fx(t) - y(t)|^2\,dt + \frac{\widehat{\lambda}_2}{2\pi}\int_{\mathbb{R}} t^{2n}|Fx(t)|^2\,dt \to \min, \quad x(\cdot) \in X^n_\infty.$$

One can readily verify that its solution is the function $\widehat{x}_y(\cdot)$ such that

$$F\widehat{x}_y(t) = \begin{cases} \dfrac{2\pi\widehat{\lambda}_1(t)}{2\pi\widehat{\lambda}_1(t) + \widehat{\lambda}_2 t^{2n}} y(t), & |t| < \sigma_0, \\ 0, & |t| \geq \sigma_0, \end{cases}$$

that is,

$$F\widehat{x}_y(t) = \begin{cases} \left(1 - \sigma_0^{-2(n-k)} t^{2(n-k)}\right) y(t), & |t| < \sigma_0, \\ 0, & |t| \geq \sigma_0. \end{cases}$$

It follows from Theorem 4 that the optimal recovery method has the form (5) and its error is given by (4).

If $\sigma_0 = \infty$ (in this case, obviously, $\sigma = \infty$), then it follows from Lemma 1 that

$$E(x^{(k)}(\cdot), C_\infty^n, I^{\delta(\cdot),\infty}) \geq \sup_{\substack{x(\cdot)\in C_\infty^n \\ |Fx(t)|\leq\delta(t) \text{ п. в.}}} \|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})}$$
$$\geq \|\widehat{x}^{(k)}(\cdot)\|_{L_2(\mathbb{R})} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{2k}\delta^2(t)\,dt},$$

where $\widehat{x}(\cdot)$ is the inverse Fourier transform of $\delta(\cdot)$. On the other hand,

$$e(x^{(k)}(\cdot), C_\infty^n, I^{\delta(\cdot),\infty}, \widehat{\varphi}) = \sup_{\substack{x(\cdot)\in C_\infty^n,\ y(\cdot)\in L_\infty(\mathbb{R}) \\ |Fx(t)-y(t)|\leq\delta(t) \text{ a. e.}}} \|x^{(k)}(\cdot) - \widehat{\varphi}(y)(\cdot)\|_{L_2(\mathbb{R})}$$
$$= \sup_{\substack{x(\cdot)\in C_\infty^n,\ y(\cdot)\in L_\infty(\mathbb{R}) \\ |Fx(t)-y(t)|\leq\delta(t) \text{ a. e.}}} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{2k}|Fx(t)-y(t)|^2\,dt\right)^{1/2}$$
$$\leq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{2k}\delta^2(t)\,dt}$$

for the method (7). □

In particular, it follows from Theorem 1 that if $\delta(t) \equiv \delta > 0$ and $\sigma = \infty$ (see Corollary 1), then

$$\sup_{\substack{x(\cdot)\in C_\infty^n \\ |Fx(t)|\leq\delta(t) \text{ a. e.}}} \|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} = K(k,n,\infty)\delta^{\frac{2(n-k)}{2n+1}},$$

where the constant $K(k,n,\infty)$ is defined in (12). Hence we arrive at the sharp inequality

$$\|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \leq K(k,n,\infty)\|Fx(\cdot)\|_{L_\infty(\mathbb{R})}^{\frac{2(n-k)}{2n+1}}\|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{2k+1}{2n+1}}.$$

Proof of Theorem 2. We again rewrite the problem in terms of Theorem 4. Here T consists of one point, say, $t=1$. Further, $X=X_2^n$ is a linear space with the semi-inner product

$$(x_1(\cdot),x_2(\cdot))_{X_2^n}=\int_{\mathbb{R}} x_1^{(n)}(t)\overline{x_2^{(n)}(t)}\,dt,$$

$Y_1=Y=L_2(\Delta_\sigma)$, $Z=L_2(\mathbb{R})$, $\Lambda x(\cdot)=x^{(k)}(\cdot)$, $BX=C_2^n$, the information operator $I\colon X_2^n\to L_2(\Delta_\sigma)$ is given by the formula $Ix(\cdot)=Fx(\cdot)$, and $\delta(1)=\delta>0$. Here the function $\mathcal{L}(x(\cdot),\lambda_1,\lambda_2)$ in Theorem 4 has the form

$$\mathcal{L}(x(\cdot),\lambda_1,\lambda_2)=-\|x^{(k)}(\cdot)\|^2_{L_2(\mathbb{R})}+\lambda_1\int_{\Delta_\sigma}|Fx(t)|^2\,dt+\lambda_2\|x^{(n)}(\cdot)\|^2_{L_2(\mathbb{R})}.$$

Passing to Fourier transforms, we have

$$\mathcal{L}(x(\cdot),\lambda_1,\lambda_2)=\int_{\mathbb{R}}\left(-t^{2k}+2\pi\lambda_1\chi_\sigma(t)+\lambda_2t^{2n}\right)u(t)\,dt,$$

where $(2\pi)^{-1}|Fx(\cdot)|^2=u(\cdot)$ and $\chi_\sigma(\cdot)$ is the characteristic function of the interval Δ_σ.

Set

$$\widehat\lambda_1=\begin{cases}\dfrac{1}{2\pi}\left(\dfrac{k}{n}\right)^{\frac{k}{n-k}}\dfrac{n-k}{n}\sigma_0^{2k}, & k>0,\\[2mm] \dfrac{1}{2\pi}, & k=0,\end{cases}\qquad \widehat\lambda_2=\begin{cases}\sigma_0^{-2(n-k)}, & k>0,\\ \sigma^{-2n}, & k=0,\end{cases}$$

$$\widehat t=\begin{cases}\left(\dfrac{k}{n}\right)^{\frac{1}{2(n-k)}}\sigma_0, & k>0,\\[2mm] 0, & k=0.\end{cases}$$

One can readily verify that

$$\mathcal{L}(x(\cdot),\widehat\lambda_1,\widehat\lambda_2)\ge 0$$

for all $x(\cdot)\in X_2^n$.

First, suppose that $k>0$ and $\sigma<\widehat\sigma$ (in this case, $\sigma_0=\sigma$). For sufficiently large m, consider the sequence of functions

$$u_m(t)=\begin{cases}m\dfrac{\delta^2}{4\pi}, & \widehat t\le|t|\le\widehat t+1/m,\\[2mm] \dfrac{m-\sqrt{m}}{2\sigma^{2n}}\left(1-\dfrac{\delta^2}{2\pi}\widehat t^{2n}\right), & \sigma\le|t|\le\sigma+1/m,\\[2mm] 0, & \text{in all other cases.}\end{cases}$$

Then

$$2\pi\int_{\Delta_\sigma}u_m(t)\,dt=\delta^2,$$

$$\int_{\mathbb{R}} t^{2n} u_m(t)\,dt = \frac{m\delta^2}{2\pi}\frac{(\widehat{t}+1/m)^{2n+1}-\widehat{t}^{2n+1}}{2n+1} + \frac{m-\sqrt{m}}{2\sigma^{2n}}\left(1-\frac{\delta^2}{2\pi}\widehat{t}^{2n}\right)\frac{(\sigma+1/m)^{2n+1}-\sigma^{2n+1}}{2n+1} \overset{m\to\infty}{\to} 1.$$

Moreover,

$$\int_{\mathbb{R}} t^{2n} u_m(t)\,dt < 1 - \left(1-\frac{\delta^2}{2\pi}\widehat{t}^{2n}\right)\frac{1}{2\sqrt{m}}$$

for sufficiently large m. It follows that condition (c) in Theorem 4 is satisfied for the inverse Fourier transform $x_m(\cdot)$ of the functions $\sqrt{2\pi u_m(\cdot)}$. A straightforward verification readily shows that condition (b) in the same theorem is also satisfied.

If $k>0$ and $\sigma \geq \widehat{\sigma}$ or $k=0$, then one should consider the functions

$$u_m(t) = \begin{cases} (m-\sqrt{m})\delta^2/(4\pi), & \widehat{t} \leq |t| \leq \widehat{t}+1/m, \\ 0, & \text{otherwise.} \end{cases}$$

Problem (16) with $y(\cdot)\in L_2(\mathbb{R})$ has the form

$$\widehat{\lambda}_1\|Fx(\cdot)-y(\cdot)\|^2_{L_2(\Delta_\sigma)} + \widehat{\lambda}_2\|x^{(n)}(\cdot)\|_{L_2(\mathbb{R})} \to \min, \quad x(\cdot)\in X_2^n.$$

It can be rewritten in the form

$$\widehat{\lambda}_1\int_{\Delta_\sigma}|Fx(t)-y(t)|^2\,dt + \frac{\widehat{\lambda}_2}{2\pi}\int_{\mathbb{R}} t^{2n}|Fx(t)|^2\,dt \to \min, \quad x(\cdot)\in X_2^n$$

with regard to the Plancherel theorem. The solution of this problem is the function $\widehat{x}_y(\cdot)$ such that

$$F\widehat{x}_y(t) = \begin{cases} \left(1+\dfrac{n}{n-k}\left(\dfrac{n}{k}\right)^{\frac{k}{n-k}}\left(\dfrac{t}{\sigma_0}\right)^{2n}\right)^{-1} y(t), & |t|<\sigma, \\ 0, & |t|\geq\sigma, \end{cases}$$

for $k>0$ and

$$F\widehat{x}_y(t) = \begin{cases} \left(1+\left(\dfrac{t}{\sigma}\right)^{2n}\right)^{-1} y(t), & |t|<\sigma, \\ 0, & |t|\geq\sigma, \end{cases}$$

for $k=0$. Now the desired assertion follows from Theorem 4. $\square$

It follows from Theorem 2 that

$$\sup_{\substack{x(\cdot)\in C_2^n \\ \|Fx(\cdot)\|_{L_2(\mathbb{R})}\leq\delta}} \|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} = \left(\frac{\delta^2}{2\pi}\right)^{\frac{n-k}{2n}},$$

whence we obtain the sharp inequality

$$\|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \leq \left(\frac{1}{2\pi}\right)^{\frac{n-k}{2n}} \|Fx(\cdot)\|^{\frac{n-k}{n}}_{L_2(\mathbb{R})}\|x^{(n)}(\cdot)\|^{\frac{k}{n}}_{L_2(\mathbb{R})}.$$

Proof of Theorem 3. The cases $p=\infty,2$ follow from Theorems 1 and 2. Suppose that $2<p<\infty$. Consider the extremal problem

$$\|x^{(k)}(\cdot)\|^2_{L_2(\mathbb{R})}\to\max,\quad \|Fx(\cdot)\|^2_{L_p(\mathbb{R})}\le\delta^2,\quad \|x^{(n)}(\cdot)\|^2_{L_2(\mathbb{R})}\le 1.$$

This problem can be rewritten in terms of the Fourier transforms as

$$\int_{\mathbb{R}} t^{2k}u(t)\,dt\to\max,\quad \int_{\mathbb{R}} u^{p/2}(t)\,dt\le\frac{\delta^p}{2\pi}, \quad \int_{\mathbb{R}} t^{2n}u(t)\,dt\le 1,\ u(t)\ge 0, \tag{20}$$

where $u(\cdot)=(2\pi)^{-1}|Fx(\cdot)|^2$. To problem (20) we assign the Lagrange function

$$\mathcal{L}(u(\cdot),\lambda_1,\lambda_2)=\int_{\mathbb{R}}(-t^{2k}u(t)+\lambda_1u^{p/2}(t)+\lambda_2t^{2n}u(t))\,dt.$$

If we find a function $\widehat{u}(\cdot)$ admissible in (20) and Lagrange multipliers $\widehat{\lambda}_1,\widehat{\lambda}_2\ge 0$ such that

$$(a)\quad \min_{u(t)\ge 0}\mathcal{L}(u(\cdot),\widehat{\lambda}_1,\widehat{\lambda}_2)=\mathcal{L}(\widehat{u}(\cdot),\widehat{\lambda}_1,\widehat{\lambda}_2),$$

$$(b)\quad \widehat{\lambda}_1\left(\int_{\mathbb{R}}u(t)^{p/2}\,dt-\frac{\delta^p}{2\pi}\right)=0,$$

$$(c)\quad \widehat{\lambda}_2\left(\int_{\mathbb{R}}t^{2n}u(t)\,dt-1\right)=0,$$

then $\widehat{u}(\cdot)$ will be a solution of the problem (20). Indeed,

$$-\int_{\mathbb{R}}t^{2k}u(t)\,dt\ge-\int_{\mathbb{R}}t^{2k}u(t)\,dt+\widehat{\lambda}_1\left(\int_{\mathbb{R}}u(t)^{p/2}\,dt-\frac{\delta^p}{2\pi}\right)$$
$$+\widehat{\lambda}_2\left(\int_{\mathbb{R}}t^{2n}u(t)\,dt-1\right)\ge-\int_{\mathbb{R}}t^{2k}\widehat{u}(t)\,dt+\widehat{\lambda}_1\left(\int_{\mathbb{R}}\widehat{u}(t)^{p/2}\,dt-\frac{\delta^p}{2\pi}\right)$$
$$+\widehat{\lambda}_2\left(\int_{\mathbb{R}}t^{2n}\widehat{u}(t)\,dt-1\right)=-\int_{\mathbb{R}}t^{2k}\widehat{u}(t)\,dt$$

for any admissible function $u(\cdot)$.

Set $\widehat{\lambda}_2=\sigma^{-2(n-k)}$, where the parameter $\sigma>0$ will be defined later. Then

$$-t^{2k}u(t)+\lambda_1u^{p/2}(t)+\frac{t^{2n}}{\sigma^{2(n-k)}}u(t)\ge-t^{2k}\widehat{u}(t)+\lambda_1\widehat{u}^{p/2}(t)+\frac{t^{2n}}{\sigma^{2(n-k)}}\widehat{u}(t)$$

for all $u(t)\ge 0$ and any $\widehat{\lambda}_1>0$, where

$$\widehat{u}(t)=\begin{cases}\left(\dfrac{2}{p\widehat{\lambda}_1}\left(t^{2k}-\dfrac{t^{2n}}{\sigma^{2(n-k)}}\right)\right)^{\frac{1}{p/2-1}}, & |t|\le\sigma,\\ 0, & |t|>\sigma.\end{cases}$$

Thus condition (a) is satisfied. We take σ and $\widehat{\lambda}_1$ such that conditions (b) and (c) are satisfied:

$$\left(\frac{2}{p\widehat{\lambda}_1}\right)^{\frac{p/2}{p/2-1}} \int_{-\sigma}^{\sigma} \left(t^{2k} - \frac{t^{2n}}{\sigma^{2(n-k)}}\right)^{\frac{p/2}{p/2-1}} dt = \frac{\delta^p}{2\pi},$$

$$\left(\frac{2}{p\widehat{\lambda}_1}\right)^{\frac{1}{p/2-1}} \int_{-\sigma}^{\sigma} t^{2n} \left(t^{2k} - \frac{t^{2n}}{\sigma^{2(n-k)}}\right)^{\frac{1}{p/2-1}} dt = 1.$$

Making the change of variable $t = \sigma y$ and expressing the resulting integrals via the value of beta function B defined in (11), we obtain

$$\sigma^{\frac{pk}{p/2-1}+1} \left(\frac{2}{p\widehat{\lambda}_1}\right)^{\frac{p/2}{p/2-1}} \frac{B}{n-k} = \frac{\delta^p}{2\pi},$$

$$\sigma^{\frac{2k}{p/2-1}+2n+1} \left(\frac{2}{p\widehat{\lambda}_1}\right)^{\frac{1}{p/2-1}} \frac{(k+1/2-1/p)B}{(n-k)^2} = 1.$$

Hence

$$\left(\frac{2}{p\widehat{\lambda}_1}\right)^{\frac{1}{p/2-1}} = \frac{(n-k)^2}{(k+1/2-1/p)B}\sigma^{-\frac{2k}{p/2-1}-2n-1} \tag{21}$$

and

$$\sigma = \left(\frac{(2\pi)^{1/p}(n-k)^{1-1/p}}{\delta(k+1/2-1/p)^{1/2}B^{1/2-1/p}}\right)^{\frac{1}{n+1/2-1/p}}. \tag{22}$$

Taking account of (21), we have

$$\int_{\mathbb{R}} t^{2k}\widehat{u}(t)\,dt = \frac{n+1/2-1/p}{k+1/2-1/p}\sigma^{-2(n-k)}.$$

Substituting there the value σ given by (22), we obtain

$$\sup_{\substack{x(\cdot)\in F_p^n \\ \|Fx(\cdot)\|_{L_p(\mathbb{R})}\le\delta}} \|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} = K\delta^{\frac{n-k}{n+1/2-1/p}}$$

for all $2 < p < \infty$. It readily follows that the desired sharp inequality for derivatives is valid. $\square$

References

[1] S. A. Smolyak, On Optimal Recovery of Functions and Functionals of Them [Russian], Ph.D. thesis, Moscow State University, Moscow, 1965. (Russian)

[2] C. A. Micchelli and T. J. Rivlin, A survey of optimal recovery. In: Optimal Estimation in Approximation Theory (C. A. Micchelli and T. J. Rivlin, eds.), Plenum Press, New York, 1977, pp. 1–54.

[3] J. F. Traub and H. Woźniakowski, A General Theory of Optimal Algorithms, Academic Press, New York, 1980.

[4] C. A. Micchelli and T. J. Rivlin, Lectures on Optimal Recovery, Lect. Notes in Math., Vol. 1129, Springer–Verlag, Berlin, 1985, pp. 21–93.

[5] K. Yu. Osipenko, Optimal Recovery of Analytic Functions, Nova Science Publ., Inc., Huntington, New York, 2000.

[6] G. G. Magaril-Il'yaev and V. M. Tikhomirov, "Kolmogorov type inequalities for derivatives," Mat. Sb., **188**. No. 12. 73–106 (1997); English transl. in Sb. Math., **188** (1997).

[7] G. G. Magaril-Il'yaev and V. M. Tikhomirov, Convex Analysis and Its Applications, Editorial URSS, Moscow, 2000.

[8] G. G. Magaril-Il'yaev, K. Yu. Osipenko, and V. M. Tikhomirov, "Optimal recovery and the theory of extremum," Dokl. Ross. Akad. Nauk, **379**, No. 2. 161–164 (2001); English transl. in Dokl. Math., **64** (2001).

[9] A. A. Melkman and C. A. Micchelli, "Optimal estimation of linear operators in Hilbert spaces from inaccurate data," SIAM J. Numer. Anal., **16** (1979), 87–105.

[10] G. G. Magaril-Il'yaev and K. Yu. Osipenko, "Optimal recovery of functions and their derivatives from Fourier coefficients prescribed with an error," Mat. Sb., **193**. No. 3. 79–100 (2002); English transl. in Sb. Math., **193** (2002).

[11] V. M. Tikhomirov and G. G. Magaril-Il'yaev, Inequalities for derivatives. In: Selected works of A. N. Kolmogorov. Mathematics and mechanics, Nauka, Moscow, 1985, pp. 387–390; English transl., Kluwer, Dordrecht, 1991.

[12] V. V. Arestov, "Approximation of unbounded operators by bounded operators and related extremal problems," Uspekhi Mat. Nauk, **51**. No. 6. 89–124 (1996); English transl. in Russian Math. Surveys, **51** (1996).

Moscow State Institute of Radio Engineering, Electronics and Automation (Technological University)

MATI — Russian State Technological University

Translated by K. Yu. Osipenko