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Optimal Recovery of Pipe Temperature from Inaccurate Measurements

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Abstract—The problem of optimal recovery of the solution of the heat equation on a manifold at a given instant of time from inaccurate measurements of this solution at other instants of time is considered, the manifold being the product of the real line and a circle. A family of optimal recovery methods is constructed.

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1. INTRODUCTION

This paper is devoted to the problem of optimal recovery of the solution of the heat equation on the manifold $\mathbb{R} \times \mathbb{T}$ ($\mathbb{T}$ is the unit circle) at a given instant of time from approximate measurements of this solution at other instants of time. The manifold $\mathbb{R} \times \mathbb{T}$ simulates a pipe, and in this sense the problem consists in the best recovery of the pipe temperature at a time instant τ from its approximate measurements at times t_i, $i = 1, \dots, n$. In this study, we find a family of linear optimal recovery methods for the pipe temperature, each method involving at most two measurements. Note that the results of the study can be extended to the case of manifolds of the form $\mathbb{R}^n \times \mathbb{T}^m$. However, in order not to overload the presentation, we restrict ourselves to the case $n = m = 1$.

The field of research related to optimal recovery of linear operators on classes of sets whose elements are known approximately emerged in the 1960s, and the studies in this field were initiated in [11–13]. Subsequently, the main attention has been paid to the problems of recovery of functions and their derivatives from inaccurate spectral data and to the problems of optimal recovery of solutions to equations of mathematical physics (see, for example, [1, 2, 4–9]). The present paper pertains to the second class of problems.

2. STATEMENT OF THE PROBLEM AND THE MAIN RESULT

We address the following problem. Suppose that we can measure the temperature of a pipe at time instants $0 \le t_1 < \ldots < t_n$ with known error. How can we recover the pipe temperature at a time instant $\tau \neq t_i$, $i = 1, \dots, n$, from this information?

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The precise statement is as follows. The heat propagation along the manifold $M = \mathbb{R} \times \mathbb{T}$ is described by the heat equation

$$\frac{\partial u(x,y,t)}{\partial t} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}, \tag{2.1}$$

where $x \in \mathbb{R}$, $y \in \mathbb{T}$, and $t \geq 0$, with an initial function $f(\,\cdot\,)$ of the variables (x,y).

We assume that $f(\,\cdot\,)$ belongs to $L_2(M)$. In this case, for every $t > 0$, there exists a unique solution $u(\,\cdot\,,t)$ of equation (2.1) that belongs to $L_2(M)$ and converges to $f(\,\cdot\,)$ in the metric of $L_2(M)$ as $t \to 0$.

Suppose we are given functions $g_i(\,\cdot\,) \in L_2(M)$ and numbers $\delta_i > 0$, $i = 1,\dots,n$, such that

$$\|u(\,\cdot\,,t_i) - g_i(\,\cdot\,)\|_{L_2(M)} \leq \delta_i, \qquad i = 1,\dots,n.$$

By the problem of optimal recovery of the function $u(\,\cdot\,,\tau)$ from this information we mean the following. Any recovery method is a map $\varphi\colon (L_2(M))^n \to L_2(M)$. The error of a method φ is defined as

$$e_\tau(\overline{g}(\,\cdot\,),\overline{\delta},\varphi) = \sup_{\substack{f(\,\cdot\,)\in L_2(M),\ g_i(\,\cdot\,)\in L_2(M),\ i=1,\dots,n \\ \|u(\,\cdot\,,t_i)-g_i(\,\cdot\,)\|_{L_2(M)}\leq\delta_i,\ i=1,\dots,n}} \|u(\,\cdot\,,\tau) - \varphi(\overline{g}(\,\cdot\,))\|_{L_2(M)},$$

where $\overline{g} = (g_1(\,\cdot\,),\dots,g_n(\,\cdot\,))$ and $\overline{\delta} = (\delta_1,\dots,\delta_n)$.

We are interested in the quantity

$$E_\tau(\overline{g}(\,\cdot\,),\overline{\delta}) = \inf_{\varphi} e_\tau(\overline{g}(\,\cdot\,),\overline{\delta},\varphi),$$

where the infimum is taken over all maps $\varphi\colon (L_2(M))^n \to L_2(M)$, and in the methods $\widehat{\varphi}$ on which the lower bound is attained, i.e., methods such that

$$E_\tau(\overline{g}(\,\cdot\,),\overline{\delta}) = e_\tau(\overline{g}(\,\cdot\,),\overline{\delta},\widehat{\varphi}). \tag{2.2}$$

The quantity $E_\tau(\overline{g}(\,\cdot\,),\overline{\delta})$ is called the *error of optimal recovery*, and the methods $\widehat{\varphi}$ for which equality (2.2) holds are called *optimal recovery methods*.

A similar problem on the real line was considered in [5], and the general scheme of the proof is largely similar to that used here. However, the presence of a periodic component significantly changes some steps of reasoning.

To formulate a theorem, we need some definitions. Recall that the Fourier transform F in $L_2(M)$ acts from $L_2(M)$ to $L_2(\widehat{M})$, where $\widehat{M} = \mathbb{R} \times \mathbb{Z}$, and is an isometric isomorphism of these spaces. The squared norm of a function $a(\,\cdot\,) \in L_2(\widehat{M})$ is defined by

$$\frac{1}{(2\pi)^2}\int_{\widehat{M}} |a(\xi,k)|^2\,d\xi\,dk = \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} |a(\xi,k)|^2\,d\xi.$$

If $g(\,\cdot\,) \in L_2(M)$, then $F[g](\,\cdot\,)$ denotes the Fourier transform of $g(\,\cdot\,)$.

Next, on the two-dimensional plane (t,x), consider the convex set

$$A = \operatorname{conv}\left\{\left(t_i, \ln\frac{1}{\delta_i}\right)\colon\ 1 \leq i \leq n\right\} + \{(t,0)\colon\ t \geq 0\},$$

which is the algebraic sum of the convex hull of the finite set of points $(t_i, \ln(1/\delta_i))$, $i = 1,\dots,n$, and the positive half-axis.

Define a function $\theta(\,\cdot\,)$ on $[0,\infty)$ by the equality $\theta(t) = \max\{x\colon\ (t,x) \in A\}$, with $\theta(t) = -\infty$ if the set of such x is empty. It is clear that $\theta(\,\cdot\,)$ is a nondecreasing concave polygonal line on $[t_1,\infty)$. Denote by $t_{s_1} < \ldots < t_{s_k}$ its break points (assuming that t_1 is also a break point, i.e., $t_{s_1} = t_1$).

Clearly, if $k>1$, then the function $\theta(\cdot)$ is increasing on $[t_{s_1},t_{s_k}]$ and is a constant equal to $\ln(1/\delta_{s_k})$ on $[t_{s_k},\infty)$.

If $\tau\in(t_{s_i},t_{s_{i+1}})$, $1\le i\le k-1$, then we set

$$\begin{aligned}\widehat{\lambda}_{s_i}&=\widehat{\lambda}_{s_i}(\tau)=\frac{t_{s_{i+1}}-\tau}{t_{s_{i+1}}-t_{s_i}}\left(\frac{\delta_{s_i}}{\delta_{s_{i+1}}}\right)^{-2(\tau-t_{s_i})/(t_{s_{i+1}}-t_{s_i})},\\ \widehat{\lambda}_{s_{i+1}}&=\widehat{\lambda}_{s_{i+1}}(\tau)=\frac{\tau-t_{s_i}}{t_{s_{i+1}}-t_{s_i}}\left(\frac{\delta_{s_i}}{\delta_{s_{i+1}}}\right)^{2(t_{s_{i+1}}-\tau)/(t_{s_{i+1}}-t_{s_i})}.\end{aligned}\tag{2.3}$$

It is easy to see that these are positive numbers such that $\widehat{\lambda}_{s_i}<1$.

Given a τ, define another positive number

$$r_i=r_i(\tau)=-\frac{\ln\widehat{\lambda}_{s_i}}{\tau-y_{s_i}}$$

and put $B(r_i)=\{(\xi,k)\in\mathbb{R}\times\mathbb{Z}\colon\ \xi^2+k^2\le r_i\}$.

Theorem 1. *For any $\tau>0$, we have the equality*

$$E_\tau(\overline{g}(\cdot),\overline{\delta})=e^{-\theta(\tau)}.$$

If $\tau\in(t_{s_i},t_{s_{i+1}})$, $1\le i\le k-1$, then the set of functions $a(\cdot)$ on $\mathbb{R}\times\mathbb{Z}$ that are measurable for every $k\in\mathbb{Z}$, satisfy the inequality

$$\frac{|b(\xi,k)|^2}{\widehat{\lambda}_{s_i}}+\frac{|a(\xi,k)|^2}{\widehat{\lambda}_{s_{i+1}}}\le 1\tag{2.4}$$

with

$$b(\xi,k)=e^{-(\xi^2+k^2)(\tau-y_{s_i})}-a(\xi,k)e^{-(\xi^2+k^2)(t_{s_{i+1}}-t_{s_i})}$$

for $(\xi,k)\in B(r_i)$, and vanish outside $B(r_i)$ is nonempty. For every such function $a(\cdot)$, the method $\widehat{\varphi}_a$ defined by

$$\widehat{\varphi}_a(g_1(\cdot),\dots,g_n(\cdot))=(K_b*g_{s_i})(\cdot)+(K_a*g_{s_{i+1}})(\cdot),\tag{2.5}$$

where $F[K_b](\xi,k)=b(\xi,k)$ and $F[K_a](\xi,k)=a(\xi,k)$, is optimal.

If $\tau>t_{s_k}$, then the method $\widehat{\varphi}$ defined by the formula $\widehat{\varphi}(g_1(\cdot),\dots,g_n(\cdot))=(K*g_{s_k})(\cdot)$, where $F[K](\xi)=e^{-(\xi^2+k^2)t_{s_k}}$, is optimal.

Note that the optimal methods are well defined. Indeed, the function $a(\cdot)$ is bounded (as follows from (2.4)) and vanishes outside some ball. Thus, it belongs to $L_2(\widehat{M})$ and, hence, is the image of some function $K_a\in L_2(M)$. Similarly we find that $K_b\in L_2(M)$, and so the convolution makes sense.

Note that if $t_1>0$ and $0<\tau<t_1$, then by definition $\theta(\tau)=-\infty$, and thus $E_\tau(\overline{g}(\cdot),\overline{\delta})=+\infty$; i.e., "the past cannot be recovered from the inaccurate present."

Note in addition that optimal methods are linear, "smooth out" observations, and use information on at most two measurements.

3. PROOF OF THEOREM 1

The general scheme of further reasoning is as follows. We obtain a lower bound on $E_\tau(\overline{g}(\cdot),\overline{\delta})$ for any $\tau>0$ and then show that the error of any of the methods (2.5) does not exceed this bound. This obviously implies that these methods are optimal. Then we demonstrate that the set of such methods is nonempty.

As already mentioned, for every $t>0$, the solution of equation (2.1) is uniquely defined by the initial function $f(\cdot)$; therefore, below it is convenient to write $u(\cdot,t,f)$ instead of $u(\cdot,t)$.

Lower bound for $E_\tau(\overline{g}(\,\cdot\,),\overline{\delta})$. The error of optimal recovery satisfies the following estimate:

$$E_\tau(\overline{g}(\,\cdot\,),\overline{\delta}) \ge \sup_{\substack{f(\,\cdot\,)\in L_2(M)\\ \|u(\,\cdot\,,t_i,f)\|_{L_2(M)}\le\delta_i,\ i=1,\dots,n}} \|u(\,\cdot\,,\tau,f)\|_{L_2(M)}. \tag{3.1}$$

This is a general fact, which is valid in a much more general situation than the one considered here. For its proof, see, for example, [5].

We are interested in the exact value of the quantity on the right-hand side of (3.1). This is the value of the maximum problem

$$\|u(\,\cdot\,,\tau,f)\|_{L_2(M)} \to \max, \qquad \|u(\,\cdot\,,t_i,f)\|_{L_2(M)} \le \delta_i, \quad i=1,\dots,n, \qquad f(\,\cdot\,)\in L_2(M), \tag{3.2}$$

i.e., the upper bound of the functional to be maximized under given constraints.

Seeking a solution of equation (2.1) by the Fourier method (see, for example, [3]), one can easily verify that

$$F[u(\,\cdot\,,t,f)](\xi,k) = e^{-(\xi^2+k^2)t}F[f](\xi,k)$$

for any $t>0$, a.e. $\xi\in\mathbb{R}$, and all $k\in\mathbb{Z}$.

Then, by the Plancherel theorem,

$$\int_M |u(x,y,t,f)|^2\,dx\,dy = \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} e^{-2(\xi^2+k^2)t}|F[f](\xi,k)|^2\,d\xi.$$

Therefore, in terms of Fourier transforms, the squared value of problem (3.2) is equal to the value of the following problem:

$$\begin{gathered}\frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} e^{-2(\xi^2+k^2)\tau}|F[f](\xi,k)|^2\,d\xi \to \max,\\ \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} e^{-2(\xi^2+k^2)t_i}|F[f](\xi,k)|^2\,d\xi \le \delta_i^2, \qquad i=1,\dots,n,\\ f(\,\cdot\,)\in L_2(M).\end{gathered} \tag{3.3}$$

For every $k\in\mathbb{Z}$, the integrals can be viewed as integration with respect to the positive measures $d\mu_k(\xi) = (2\pi)^{-2}|F[f](\xi,k)|^2\,d\xi$ on $\mathbb{R}$, which are generated by the functions $f(\,\cdot\,)\in L_2(M)$. To find the value of this problem, it is convenient to consider a more general statement, namely, the following problem on the set of all positive Borel measures on $\mathbb{R}$:

$$\begin{gathered}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} e^{-2(\xi^2+k^2)\tau}\,d\mu_k(\xi) \to \max,\\ \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} e^{-2(\xi^2+k^2)t_i}\,d\mu_k(\xi) \le \delta_i^2, \qquad i=1,\dots,n,\\ d\mu_k \ge 0, \qquad k\in\mathbb{Z}.\end{gathered} \tag{3.4}$$

We will find the value of this problem, which is obviously not less than the value of problem (3.3), and then show that the values of these problems actually coincide.

Problem (3.4) with respect to the variable $\{d\mu_k\}_{k\in\mathbb{Z}}$ is the problem of maximizing a linear functional on a convex set. To find its solution, we apply standard techniques.

With this problem we associate the Lagrange function

$$\mathcal{L}(\{d\mu_k\}_{k\in\mathbb{Z}},\lambda)=-\sum_{k\in\mathbb{Z}}\int\limits_{\mathbb{R}}e^{-2(\xi^2+k^2)\tau}\,d\mu_k(\xi)+\sum_{i=1}^{n}\lambda_i\left(\sum_{k\in\mathbb{Z}}\int\limits_{\mathbb{R}}e^{-2(\xi^2+k^2)t_i}\,d\mu_k(\xi)-\delta_i^2\right),$$

which is defined on all sequences $\{d\mu_k\}_{k\in\mathbb{Z}}$, where $d\mu_k$ are $\mathbb{R}$-valued Borel measures on $\mathbb{R}$, and on all n-tuples $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{R}^n$, called Lagrange multipliers.

According to the sufficiency conditions in the Karush–Kuhn–Tucker theorem (see, for example, [10]), an admissible sequence $\{d\widehat{\mu}_k\}_{k\in\mathbb{Z}}$ is a solution of problem (3.4) if there exists an n-tuple of Lagrange multipliers $\widehat{\lambda}=(\widehat{\lambda}_1,\widehat{\lambda}_2,\dots,\widehat{\lambda}_n)$ such that the following conditions hold:

(a) $\min_{\{d\mu_k\geq 0\}_{k\in\mathbb{Z}}}\mathcal{L}(\{d\mu_k\}_{k\in\mathbb{Z}},\widehat{\lambda})=\mathcal{L}(\{d\widehat{\mu}_k\}_{k\in\mathbb{Z}},\widehat{\lambda})$;

(b) $\widehat{\lambda}_i\geq 0$ for $1\leq i\leq n$;

(c) $\widehat{\lambda}_i\big(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}e^{-2(\xi^2+k^2)t_i}\,d\widehat{\mu}_k(\xi)-\delta_i^2\big)=0$ for $i=1,\dots,n$.

Analyzing conditions (a)–(c), one can derive formulas for $\{d\widehat{\mu}_k\}_{k\in\mathbb{Z}}$ and $\widehat{\lambda}$. However, we will not perform such an analysis here and just present an admissible sequence $\{d\widehat{\mu}_k\}_{k\in\mathbb{Z}}$ and Lagrange multipliers $\widehat{\lambda}=(\widehat{\lambda}_1,\widehat{\lambda}_2,\dots,\widehat{\lambda}_n)$ that satisfy conditions (a)–(c).

Let $\tau>0$ and $\tau\neq t_i$, $i=1,\dots,n$. Then either $\tau\in(t_{s_i},t_{s_{i+1}})$ for some $1\leq i\leq k-1$, or $\tau>t_{s_k}$, or $\tau<t_1$, if $t_1>0$. Consider the first case.

Let $\tau\in(t_{s_i},t_{s_{i+1}})$, $1\leq i\leq k-1$. Define a vector $\widehat{\lambda}=(\widehat{\lambda}_1,\dots,\widehat{\lambda}_n)$ such that $\widehat{\lambda}_{s_i}$ and $\widehat{\lambda}_{s_{i+1}}$ are positive numbers specified in (2.3) and $\widehat{\lambda}_j=0$ if $j\neq s_i,s_{i+1}$.

Define the sequence of measures $\{d\widehat{\mu}_k\}_{k\in\mathbb{Z}}$ as follows: $d\widehat{\mu}_k=0$ if $k\neq 0$, and $d\widehat{\mu}_0=A\delta_{\xi_0}$, where δ_{ξ_0} is the Dirac measure at the point ξ_0 such that

$$\xi_0^2=\frac{\ln(1/\delta_{s_{i+1}})-\ln(1/\delta_{s_i})}{t_{s_{i+1}}-t_{s_i}}\tag{3.5}$$

and

$$A=\delta_{s_i}^{2t_{s_{i+1}}/(t_{s_{i+1}}-t_{s_i})}\,\delta_{s_{i+1}}^{-2t_{s_i}/(t_{s_{i+1}}-t_{s_i})}.\tag{3.6}$$

Since the function $\theta(\cdot)$ defined above is increasing on $[t_{s_1},t_{s_k}]$, the numerator in (3.5) is positive.

One can easily verify that such $\{d\widehat{\mu}_k\}_{k\in\mathbb{Z}}$ and $\widehat{\lambda}$ satisfy condition (c), which in this case has the form

$$\int\limits_{\mathbb{R}}e^{-2\xi^2t_j}\,d\widehat{\mu}_0(\xi)=Ae^{-2\xi_0^2t_j}=\delta_j^2,\qquad j=s_i,s_{i+1}.\tag{3.7}$$

Let us verify that the sequence $\{d\widehat{\mu}_k\}_{k\in\mathbb{Z}}$ is admissible in problem (3.4). Indeed, the points $(t_i,\ln(1/\delta_i))$, $i=1,\dots,n$, lie on or below the graph of the function $\theta(\cdot)$; since this function is concave, its graph lies on or below the line

$$p(t)=\frac{\ln(1/\delta_{s_{i+1}})-\ln(1/\delta_{s_i})}{t_{s_{i+1}}-t_{s_i}}(t-t_{s_i})+\ln\frac{1}{\delta_{s_i}}=\ln\delta_{s_i}^{-(t_{s_{i+1}}-t)/(t_{s_{i+1}}-t_{s_i})}\delta_{s_{i+1}}^{-(t-t_{s_i})/(t_{s_{i+1}}-t_{s_i})},$$

which connects the points $(t_{s_i},\ln(1/\delta_{s_i}))$ and $(t_{s_{i+1}},\ln(1/\delta_{s_{i+1}}))$. Then, taking into account the expressions for ξ_0^2 and A, we have

$$\begin{aligned}\int\limits_{\mathbb{R}}e^{-2\xi^2t_i}\,d\widehat{\mu}_0(\xi)&=Ae^{-2\xi_0^2t_i}=\delta_{s_i}^{2(t_{s_{i+1}}-t_i)/(t_{s_{i+1}}-t_{s_i})}\delta_{s_{i+1}}^{2(t_i-t_{s_i})/(t_{s_{i+1}}-t_{s_i})}\\&=e^{-2p(t_i)}\leq e^{-2\ln(1/\delta_i)}=\delta_i^2,\qquad i=1,\dots,n;\end{aligned}$$

i.e., the sequence $\{d\widehat{\mu}_k\}_{k\in\mathbb{Z}}$ is admissible in problem (3.4).

It remains to verify condition (a), i.e., to verify that

$$\begin{aligned}\mathcal{L}(\{d\mu_k\}_{k\in\mathbb{Z}},\widehat{\lambda}) &= \sum_{k\in\mathbb{Z}}\int\limits_{\mathbb{R}} e^{-2(\xi^2+k^2)\tau}\big(-1+\widehat{\lambda}_{s_i}e^{-2(\xi^2+k^2)(t_{s_i}-\tau)}+\widehat{\lambda}_{s_{i+1}}e^{-2(\xi^2+k^2)(t_{s_{i+1}}-\tau)}\big)\,d\mu_k(\xi)\\ &\qquad -\widehat{\lambda}_{s_i}\delta_{s_i}^2-\widehat{\lambda}_{s_{i+1}}\delta_{s_{i+1}}^2\\ &\ge \mathcal{L}(\{d\widehat{\mu}_k\}_{k\in\mathbb{Z}},\widehat{\lambda}) = \int\limits_{\mathbb{R}} e^{-2\xi^2\tau}\big(-1+\widehat{\lambda}_{s_i}e^{-2\xi^2(t_{s_i}-\tau)}+\widehat{\lambda}_{s_{i+1}}e^{-2\xi^2(t_{s_{i+1}}-\tau)}\big)\,d\mu_0(\xi)\\ &\qquad -\widehat{\lambda}_{s_i}\delta_{s_i}^2-\widehat{\lambda}_{s_{i+1}}\delta_{s_{i+1}}^2\end{aligned}\tag{3.8}$$

for all sequences of positive measures $\{d\mu_k\}_{k\in\mathbb{Z}}$.

The Lagrange multipliers $\widehat{\lambda}_{s_i}$ and $\widehat{\lambda}_{s_{i+1}}$ have been chosen in such a way that (as can easily be verified) the convex function on the real line given by $\omega\mapsto -1+\widehat{\lambda}_{s_i}e^{-2\omega(t_{s_i}-\tau)}+\widehat{\lambda}_{s_{i+1}}e^{-2\omega(t_{s_{i+1}}-\tau)}$ vanishes together with its derivative at the point $\omega_0=\xi_0^2$. This means that this function is nonnegative everywhere, and thus $\mathcal{L}(\{d\mu_k\}_{k\in\mathbb{Z}},\widehat{\lambda})\ge -\widehat{\lambda}_{s_i}\delta_{s_i}^2-\widehat{\lambda}_{s_{i+1}}\delta_{s_{i+1}}^2$. Since $d\mu_0$ is the Dirac measure at ξ_0, the integral on the right-hand side of (3.8) vanishes, and so $\mathcal{L}(\{d\widehat{\mu}_k\}_{k\in\mathbb{Z}},\widehat{\lambda})=-\widehat{\lambda}_{s_i}\delta_{s_i}^2-\widehat{\lambda}_{s_{i+1}}\delta_{s_{i+1}}^2$. This proves condition (a).

Thus, $\{d\widehat{\mu}_k\}_{k\in\mathbb{Z}}$ is a solution of problem (3.4). Substituting $\{d\widehat{\mu}_k\}_{k\in\mathbb{Z}}$ into the functional to be maximized, we obtain the value of this problem:

$$\int\limits_{\mathbb{R}} e^{-2\xi^2\tau}\,d\widehat{\mu}_0(\xi)=Ae^{-2\xi_0^2\tau}=\delta_{s_i}^{2(t_{s_{i+1}}-\tau)/(t_{s_{i+1}}-t_{s_i})}\delta_{s_{i+1}}^{2(\tau-t_{s_i})/(t_{s_{i+1}}-t_{s_i})}=e^{-2p(\tau)}=e^{-2\theta(\tau)}.$$

On the other hand, it is clear that $\mathcal{L}(\{d\widehat{\mu}_k\}_{k\in\mathbb{Z}},\widehat{\lambda})$ is equal to $-\int_{\mathbb{R}}e^{-2\xi^2\tau}\,d\widehat{\mu}_0(\xi)$ (by condition (c)), and therefore

$$\int\limits_{\mathbb{R}} e^{-2\xi^2\tau}\,d\widehat{\mu}_0(\xi)=e^{-2\theta(\tau)}=\widehat{\lambda}_{s_i}\delta_{s_i}^2+\widehat{\lambda}_{s_{i+1}}\delta_{s_{i+1}}^2.\tag{3.9}$$

Now, consider a sequence of functions $\varphi_m(\,\cdot\,)\in L_2(M)$, $m\in\mathbb{N}$, whose Fourier transforms have the form

$$F[\varphi_m](\xi,k)=\begin{cases}\sqrt{2\pi m}\,\delta_{s_i}^{t_{s_{i+1}}/(t_{s_{i+1}}-t_{s_i})}\delta_{s_{i+1}}^{-t_{s_i}/(t_{s_{i+1}}-t_{s_i})}, & \xi\in[\xi_0,\xi_0+1/m],\ k=0,\\ 0 & \text{otherwise.}\end{cases}$$

Direct calculation shows that these functions are admissible in problem (3.3) and the functional to be maximized converges on them to $e^{-2\theta(\tau)}$ as $m\to\infty$.

Thus, the values of problems (3.3) and (3.4) coincide. Then, formula (3.9) and estimate (3.1) imply

$$E_\tau(\overline{g}(\,\cdot\,),\overline{\delta})\ge e^{-\theta(\tau)}=\sqrt{\widehat{\lambda}_{s_i}\delta_{s_i}^2+\widehat{\lambda}_{s_{i+1}}\delta_{s_{i+1}}^2}.\tag{3.10}$$

Upper bound for $E_\tau(\overline{g}(\,\cdot\,),\overline{\delta})$ and optimal methods. Let $\widehat{\varphi}_a$ be a method of the form (2.5). Let us estimate its error, which by definition is the value of the following problem:

$$\begin{gathered}\big\|u(\,\cdot\,,\tau,f)-(K_b*g_{s_i})(\,\cdot\,)-(K_a*g_{s_{i+1}})(\,\cdot\,)\big\|_{L_2(M)}\to\max,\\ \|u(\,\cdot\,,t_{s_j},f)-g_{s_j}(\,\cdot\,)\|_{L_2(M)}\le\delta_{s_j},\qquad z_{s_j}(\,\cdot\,)\in L_2(M),\qquad j=i,i+1,\\ f(\,\cdot\,)\in L_2(M).\end{gathered}\tag{3.11}$$

By the Plancherel theorem, the squared functional in this problem is equal to

$$\frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left|e^{-(\xi^2+k^2)\tau}F[f](\xi,k)-b(\xi,k)F[g_{s_i}](\xi,k)-a(\xi,k)F[g_{s_{i+1}}](\xi,k)\right|^2 d\xi$$
$$=\frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left|b(\xi,k)z_i(\xi,k)+a(\xi,k)z_{i+1}(\xi,k)\right|^2 d\xi, \quad (3.12)$$

where $z_j(\xi,k)=e^{-(\xi^2+k^2)t_{s_j}}F[f](\xi,k)-F[g_{s_j}](\xi,k),\ j=i,i+1$.

We estimate the integrand on the right-hand side of (3.12) by the Cauchy–Bunyakovsky–Schwarz inequality ($\widehat{\lambda}_{s_i}$ and $\widehat{\lambda}_{s_{i+1}}$ are the Lagrange multipliers defined above):

$$\left|\frac{b(\xi,k)}{\sqrt{\widehat{\lambda}_{s_i}}}\sqrt{\widehat{\lambda}_{s_i}}\,z_i(\xi,k)+\frac{a(\xi,k)}{\sqrt{\widehat{\lambda}_{s_{i+1}}}}\sqrt{\widehat{\lambda}_{s_{i+1}}}\,z_{i+1}(\xi,k)\right|^2$$
$$\le\left(\frac{|b(\xi,k)|^2}{\widehat{\lambda}_{s_i}}+\frac{|a(\xi,k)|^2}{\widehat{\lambda}_{s_{i+1}}}\right)\left(\widehat{\lambda}_{s_i}|z_i(\xi,k)|^2+\widehat{\lambda}_{s_{i+1}}|z_{i+1}(\xi,k)|^2\right).$$

The first factor on the right-hand side does not exceed 1 for a.e. $\xi\in\mathbb{R}$ and all $k\in\mathbb{Z}$. Indeed, if $(\xi,k)\in B(r_i)$, then this is valid by condition (2.4). If $(\xi,k)\notin B(r_i)$, then by definition $a(\cdot)=0$ and this factor is $c(\xi,k)=e^{-(\xi^2+k^2)(\tau-t_{s_i})}/\widehat{\lambda}_{s_i}$. However, by the definition of r_i, the fact that $(\xi,k)\notin B(r_i)$ is equivalent to the inequality $c(\xi,k)\le 1$.

Thus, we see that the expression on the right-hand side of (3.12) does not exceed

$$\widehat{\lambda}_{s_i}\frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}|z_i(\xi,k)|^2\,d\xi+\widehat{\lambda}_{s_{i+1}}\frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}|z_{i+1}(\xi,k)|^2\,d\xi.$$

The function $z_j(\cdot)$ is the Fourier transform of the function $u(\cdot,t_{s_j},f)-g_{s_j}(\cdot)$, whose norm in $L_2(M)$ does not exceed δ_j, $j=i,i+1$, according to (3.11). Hence, by the Plancherel theorem, the norm of the last expression does not exceed $\widehat{\lambda}_{s_i}\delta_{s_i}^2+\widehat{\lambda}_{s_{i+1}}\delta_{s_{i+1}}^2$.

Thus, the error of any method of the form (2.5) does not exceed $\sqrt{\widehat{\lambda}_{s_i}\delta_{s_i}^2+\widehat{\lambda}_{s_{i+1}}\delta_{s_{i+1}}^2}$. Together with estimate (3.10), this means that all such methods are optimal and $E_\tau(\overline{g}(\cdot),\overline{\delta})=e^{-\theta(\tau)}$.

By completing the square, one can easily show that inequality (2.4) is equivalent to the inequality

$$\left|a(\xi,k)-\frac{\widehat{\lambda}_{s_{i+1}}e^{-(\xi^2+k^2)(\tau-t_{s_i})}}{\widehat{\lambda}_{s_i}e^{(\xi^2+k^2)(t_{s_{i+1}}-t_{s_i})}+\widehat{\lambda}_{s_{i+1}}e^{-(\xi^2+k^2)(t_{s_{i+1}}-t_{s_i})}}\right|$$
$$\le\frac{\sqrt{\widehat{\lambda}_{s_i}\widehat{\lambda}_{s_{i+1}}}\,e^{(\xi^2+k^2)t_{s_{i+1}}}}{\widehat{\lambda}_{s_i}e^{(\xi^2+k^2)(t_{s_{i+1}}-t_{s_i})}+\widehat{\lambda}_{s_{i+1}}e^{-(\xi^2+k^2)(t_{s_{i+1}}-t_{s_i})}}\sqrt{h(\xi,k)},$$

where

$$h(\xi,k)=-e^{-2(\xi^2+k^2)\tau}+\widehat{\lambda}_{s_i}e^{-2(\xi^2+k^2)t_{s_i}}+\widehat{\lambda}_{s_{i+1}}e^{-2(\xi^2+k^2)t_{s_{i+1}}},$$

and this function is nonnegative, as established above.

The inequality obtained obviously implies that the set of functions $a(\cdot)$ satisfying the hypotheses of the theorem is nonempty, and hence the theorem is proved in the case when $\tau\in(t_1,t_{s_k})$. The situations with $0<\tau<t_1$ and $\tau>t_{s_k}$ can be treated in a similar but technically much simpler manner; therefore, we will not dwell on them.

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